MATH 465/565: Homework 9

Due Friday, April 8, 2011

We work over an algebraically closed field k.

1. Let $H \subseteq \mathbb{P}^n$ be a hyperplane, $X \subseteq H$ an irreducible closed subset, and y be any point in $\mathbb{P}^n \backslash H$. Let $Y \subseteq \mathbb{P}^n$ be the union of all the lines connecting y with some point $x \in X$. (We call Y the *cone over X with vertex y*.) Show that Y is an irreducible projective variety and $\dim Y = \dim X + 1$. [Hint: pick a convenient coordinate system.]

2. (a) Prove that any finite set $S \subseteq \mathbb{A}^2$ is the zero locus of two polynomials. [Hint: We can take the first polynomial to be a polynomial in x alone.]

   (b) Prove that any finite set $S \subseteq \mathbb{P}^2$ is the zero locus of two homogeneous polynomials.

   (c) Find an example of a finite set $S \subseteq \mathbb{P}^2$ so that the homogeneous ideal $I(S)$ of S can not be generated by two homogeneous polynomials.[1]

3. Let $X \subseteq \mathbb{P}^n$ be closed, $I = I(X)$ its homogeneous ideal. Let $R_d = k[X_0, \dots, X_n]_d$ be the vector space of degree d homogeneous polynomials in $n+1$ variables, and let $I_d = I \cap R_d$. The *Hilbert function* of X in $\mathbb{P}^n$ is
$$h_X(d) = \dim(R_d/I_d) = \dim R_d - \dim I_d.$$

   (a) Determine h_X when $X \subseteq \mathbb{P}^2$ consists of three points.[2]

   (b) Determine h_X when $X \subseteq \mathbb{P}^2$ consists of four points.[2]

4. Given a ring R, an *ascending chain of prime ideals of length n* consists of $n+1$ prime ideals
$$p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_n$$
of R, totally ordered by inclusion. The *Krull dimension* of R is the supremum of the lengths of all the ascending chains of prime ideals in R.

   (a) If X is an affine variety, show that the Krull dimension of its coordinate ring $k[X]$ is equal to the dimension of X.

   (b) Compute the Krull dimension of the ring $\mathbb{Z}[x]$.

5. Show that the linear groups $GL_n(k)$, $SL_n(k)$, and $PGL_n(k)$ have natural structures as quasiprojective varieties and compute their dimensions.

[1] A projective variety $X \subseteq \mathbb{P}^n$ of dimension m is a *complete intersection* if its homogenous ideal $I(X)$ is generated by exactly $n-m$ elements; X is a *set-theoretic complete intersection* if X is the zero locus of $n-m$ homogeneous polynomials. This S provides an example of a set-theoretic complete intersection which is not a complete intersection.

[2] There may be several different cases with different answers depending on how the points are arranged. You should find h_X for each case.

6. Given $n+1$ homogeneous polynomials $F_0, \ldots, F_n \in k[X_0, \ldots, X_n]$ of degrees $d_0, \ldots, d_n$ in $n+1$ variables, we may consider their common zero locus. Let $R_d = k[X_0, \ldots, X_n]_d$, the vector space of homogeneous polynomials in $n+1$ variables, and set

$$\Gamma = \{([F_0], \ldots, [F_n], x) \in \mathbb{P}(R_{d_0}) \times \cdots \times \mathbb{P}(R_{d_n}) \times \mathbb{P}^n : \quad F_0(x) = F_1(x) = \cdots = F_n(x) = 0\}.$$

(a) Show that Γ is a closed subset of $\mathbb{P}(R_{d_0}) \times \cdots \times \mathbb{P}(R_{d_n}) \times \mathbb{P}^n$.

(b) By considering the projection map $\psi\colon \Gamma \to \mathbb{P}^n$, show that

$$\dim \Gamma = n + \sum_i \left(\dim \mathbb{P}\left(R_{d_i}\right) - 1\right).$$

(c) By considering the projection map $\varphi\colon \Gamma \to \prod_i \mathbb{P}(R_{d_i})$, show that

$$\dim \varphi(\Gamma) = \dim \Gamma = \dim \prod_i \mathbb{P}(R_{d_i}) - 1.$$

(d) Show that there exists a multihomogeneous polynomial R in the coefficients of the F_i such that $R = 0$ if and only if the system $F_0(x) = F_1(x) = \cdots = F_n(x) = 0$ has a non-zero solution.

(e) What is the polynomial R in the case $d_0 = \cdots = d_n = 1$?