# MATH 465/565: The Tensor Product

## The tensor product

Let $V$ and $W$ be vector spaces over a field $k$. Then the **tensor product** $V \otimes_k W$ of $V$ and $W$ over $k$ is the vector space spanned by all the symbols $v \otimes w$ for $v \in V$ and $w \in W$, subject to the relations[1]:

(1) $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$ for $v_1, v_2 \in V$ and $w \in W$,
(2) $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$ for $v \in V$ and $w_1, w_2 \in W$, and
(3) $(cv) \otimes w = v \otimes (cw) = c(v \otimes w)$ for $c \in k$, $v \in V$, and $w \in W$.

**Proposition 1.** *If $\{e_\alpha\colon\ \alpha \in A\}$ is a basis for $V$ over $k$ and $\{f_\beta\colon\ \beta \in B\}$ is a basis for $W$, then*

$$\{e_\alpha \otimes f_\beta\colon\ (\alpha, \beta) \in A \times B\}$$

*is a basis for $V \otimes_k W$.*

In particular, if $V \cong k^n$ and $W \cong k^m$ are finite-dimensional, then

$$\dim(V \otimes_k W) = (\dim V)(\dim W),$$

so we can identify $V \otimes_k W$ with the vector space of $n \times m$ matrices, so that the tensor product operation $\otimes$ on $V \times W$ corresponds to matrix multiplication of column vectors in $k^n$ by row vectors in $k^m$.

## Bilinear maps

Let $U$ be another $k$-vector space. A $k$-**bilinear map** $\phi\colon V \times W \to U$ is a function with the properties that:

(1) $\phi(v_1 + v_2, w) = \phi(v_1, w) + \phi(v_2, w)$ for $v_1, v_2 \in V$ and $w \in W$,
(2) $\phi(v, w_1 + w_2) = \phi(v, w_1) + \phi(v, w_2)$ for $v \in V$ and $w_1, w_2 \in W$, and
(3) $\phi(cv, w) = \phi(v, cw) = c\phi(v, w)$ for $c \in k$, $v \in V$, and $w \in W$.

Then the tensor product of elements $(v, w) \mapsto v \otimes w$ is a bilinear map $V \times W \to V \otimes_k W$, and we can characterize the tensor product by the following "universal property" (which is helpful in proving the above proposition and remarks):

**Proposition 2.** *Suppose $\phi\colon V \times W \to U$ is a $k$-bilinear map. Then there is a unique $k$-linear map $\widetilde{\phi}\colon V \otimes_k W \to U$ so that $\widetilde{\phi}(v \otimes w) = \phi(v, w)$ for all $v \in V$ and $w \in W$.*

## Tensor products of algebras

If $V$ and $W$ are both $k$-algebras (rather than just $k$-vector spaces), then $V \otimes_k W$ has a unique structure of a $k$-algebra for which the multiplication is defined by

$$(v_1 \otimes w_1) \cdot (v_2 \otimes w_2) = (v_1 \cdot v_2) \otimes (w_1 \cdot w_2),$$

on the generators of $V \otimes_k W$.

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[1] More precisely, we define $V \otimes_k W$ to be the quotient of a vector space whose basis $\{e_{v,w}\}$ is in bijection with $V \times W$ by the linear subspace spanned by the elements of the forms $e_{v,(w_1+w_2)} - e_{v,w_1} - e_{v,w_2}$, $e_{(v_1+v_2),w} - e_{v_1,w} - e_{v_2,w}$, $e_{cv,w} - ce_{v,w}$, and $e_{v,cw} - ce_{v,w}$.