Jan 16  definition of $f'(z)$
the Cauchy–Riemann equation

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complex analytic functions
Differntiation

Today we began a thrilling introduction to complex analysis. It all starts with a seemingly innocent and reasonable definition of derivative, using complex numbers instead of real numbers. But we shall learn very soon what an enormous step this really is!

**Definition:** Let \( f \) be a complex valued function defined on some neighborhood of a point \( z \in \mathbb{C} \). We say that \( f \) is complex differentiable at \( z \) if

\[
\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}
\]

exists in \( \mathbb{C} \) — crucial!

In case this limit exists, it's called the complex derivative of \( f \) at \( z \), and it denoted either \( f'(z) \) or \( \frac{df}{dz} \).

This truly seems naive, as it's completely similar to the beginning definition in calculus. But we shall see that the properties of \( f \) which follow from this definition are astonishing!

What makes this all so powerful is that in the difference quotient, the denominator \( h \in \mathbb{C} \) must be allowed simply to tend to 0, no restrictions on "how" or particular directions; merely \( |h| \to 0 \).
BASIC PROPERTIES

- \( f'(z) \) exists \( \iff \) \( f \) is continuous at \( z \).

  For if \( \frac{f(z+h) - f(z)}{h} \) has a limit, then since \( h \to 0 \),
  the numerator must also have limit 0, so that
  \[
  \lim_{{h \to 0}} \frac{f(z+h) - f(z)}{h} = f'(z).
  \]

- \( f + g \) differentiable \( \iff \) \( f + g \) is too, and \( (f + g)' = f' + g' \).

- PRODUCT RULE: also \( fg \) is differentiable, and
  \[
  (fg)' = f'g + fg'.
  \]

  Proof:
  \[
  \frac{f(z+h)g(z+h) - f(z)g(z)}{h} = \frac{f(z+h)g(z+h) - g(z)g(z) + g(z)g(z) - f(z)g(z)}{h} \\
  \quad = \frac{g(z)}{h} \cdot \frac{f(z+h) - f(z)}{h} + \frac{f(z) - f(z)}{h} \cdot g(z) \\
  \quad \downarrow \quad \downarrow \quad \downarrow \\
  \quad g'(z) \quad f'(z) \quad g(z)
  \]
  by continuity

- \( \frac{dz}{dz} = 1 \) and then we prove by induction that for \( n=1,2,3,\ldots \)
  \[
  \frac{d}{dz} z^n = nz^{n-1}
  \]

- QUOTIENT RULE \( \left( \frac{f}{g} \right)' = \frac{gf' - fg'}{g^2} \) provided that \( g \neq 0 \).

- CHAIN RULE: if \( g \) is differentiable at \( z \) and \( f \) is
  differentiable at \( g(z) \), then \( f \circ g \) is
  differentiable at \( z \), and
  \[
  (f \circ g)'(z) = f'(g(z))g'(z).
  \]

All these properties are proved just as in "real" calculus, so I've not
bothered to write out detailed proofs of them all.
EXAMPLES

- Möbius transformations — directly from the quotient rule

\[
\left( \frac{a\xi + b}{c\xi + d} \right)' = \frac{ad - bc}{(c\xi + d)^2}
\]

- Exponential function

First for \( h \to 0 \) we have

\[
\frac{e^{h} - 1}{h} = \lim_{n \to \infty} \frac{h^{n-1}}{n!} = 1 + \frac{h}{2} + \frac{h^2}{6} + \cdots
\]

has limit 1 as \( h \to 0 \). Thus

\[
\frac{e^{z+h} - e^z}{h} = e^z \frac{e^{h} - 1}{h} \to e^z.
\]

Conclusion:

\[
\frac{de^z}{dz} = e^z.
\]

- Trigonometric \& hyperbolic functions (follow immediately from \( \exp \))

\[
\frac{d\sinh z}{dz} = \cosh z, \quad \frac{d\cosh z}{dz} = \sinh z
\]

\[
\frac{d\sin z}{dz} = \cos z, \quad \frac{d\cos z}{dz} = -\sin z
\]

- \( \frac{d}{dz} \) and \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \)

By an audacious — but useful — abuse of notation we write

\[
f(z) = f(x + iy) \overset{!}{=} f(x, y).
\]

This sets up a correspondence between a function defined on \( \mathbb{C} \) and a function defined on \( \mathbb{R}^2 \), but we use the same name for these functions!
Now suppose $f'(z)$ exists. In the definition we then restrict $h$ to be real so the limit still exists, of course, and we compute

$$f'(z) = \lim_{h \to 0} \frac{f(z + hi) - f(z)}{hi} = \lim_{h \to 0} \frac{f(x+ih, y) - f(x, y)}{h}$$

$$= \frac{\partial f}{\partial x}(x, y).$$

Likewise, let $h = it$:

$$f'(z) = \lim_{t \to 0} \frac{f(z + it) - f(z)}{it} = \lim_{t \to 0} \frac{f(x, y + it) - f(x, y)}{it}$$

$$= \frac{1}{i} \frac{\partial f}{\partial y}(x, y).$$

We then conclude that

$$f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

This second equality is a famous relationship, called

**THE CAUCHY-RIEMANN EQUATION:**

$$\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}$$

**WARNING** — everyone else calls this the Cauchy–Riemann equations. After expressing $f$ in terms of its real and imaginary parts as $f = u + iv$, then we indeed get 2 equations:

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}
\end{align*}
\]

In a very precise sense, the converse is also valid, as we now discuss.
We suppose that $f$ is differentiable at $(x, y)$ in the multivariable calculus sense. This means that not only do the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at $(x, y)$, but also they provide the coefficients for a good linear approximation to $f(z+h) - f(z)$ for small $|h|$: 

$$\lim_{h \to 0, \ h \in \mathbb{C}} \frac{f(z+h) - f(z) - \frac{\partial f}{\partial x}(z)h_1 - \frac{\partial f}{\partial y}(z)h_2}{|h|} = 0.$$ 

(Remember: $z = x + iy$ is fixed.) We've denoted $h = h_1 + i h_2$.

That definition actually extends to $\mathbb{R}^n$ just as well as $\mathbb{R}^2$. But in $\mathbb{R}^2$ we have an advantage in that we can replace the denominator $|h|$ by the complex number $h$ without disturbing the fact that the limit is 0:

$$\lim_{h \to 0, \ h \in \mathbb{C}} \frac{f(z+h) - f(z) - \frac{\partial f}{\partial x}(z)h_1 - \frac{\partial f}{\partial y}(z)h_2}{h} = 0.$$ 

Now assume that the Cauchy-Riemann equation is satisfied. Then we may replace $\frac{\partial f}{\partial y}$ by $i \frac{\partial f}{\partial x}$ and thus conclude that

$$\lim_{h \to 0, \ h \in \mathbb{C}} \frac{f(z+h) - f(z) - \frac{\partial f}{\partial x}(z)(h_1 + i h_2)}{h} = 0.$$ 

i.e.,

$$\lim_{h \to 0, \ h \in \mathbb{C}} \frac{f(z+h)-f(z)}{h} = \frac{\partial f}{\partial x}(z).$$

Therefore, we conclude that $f'(z)$ exists, so $f$ is differentiable in the complex sense!
Cauchy–Riemann equation in polar coordinates

We employ the usual polar coordinates

\[ \begin{align*} x &= r \cos \theta \\ y &= r \sin \theta \end{align*} \quad (z = re^{i\theta}) \quad (r > 0 \text{ of course}) \]

and then abuse notation by writing \( f = f(x, y) \) as

\[ f = f(r \cos \theta, r \sin \theta) \]

and then computing the \( r \) and \( \theta \) partial derivatives of this composite function and designating them as \( \frac{\partial f}{\partial r} \) and \( \frac{\partial f}{\partial \theta} \) (terrible!). Then the chain rule gives

\[ \begin{align*} \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta, \\
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x} (-r \sin \theta) + \frac{\partial f}{\partial y} (r \cos \theta). \end{align*} \]

Now suppose \( f \) satisfies the Cauchy–Riemann equation and substitute

\[ \frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}; \]

\[ \begin{align*} \frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x} (\cos \theta + i \sin \theta), \\
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x} (-r \sin \theta + i r \cos \theta). \end{align*} \]

Thus

\[ \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} e^{i \theta}, \]

\[ \frac{1}{ir} \frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} e^{i \theta}. \]

We conclude that

\[ \frac{\partial f}{\partial r} = \frac{1}{ir} \frac{\partial f}{\partial \theta}. \]
Our calculations show that since \( \frac{\partial f}{\partial z} = f' \),
\[
    f'(z) = e^{-i\theta} \frac{\partial f}{\partial \theta} = \frac{1}{i} e^{-i\theta} \frac{\partial f}{\partial \theta}.
\]

**EXERCISE**  Prove that \( \frac{\partial f}{\partial r} = \frac{1}{i r} \frac{\partial f}{\partial \theta} \) implies the original Cauchy–Riemann equation.

**Complex logarithm.**

We've derived the defining equation
\[
    \log z = \log |z| + i \arg z.
\]

In terms of polar coordinates,
\[
    \log z = \log r + i \theta.
\]

We pause to discuss an easy but crucial idea. When we are faced with the necessity of using \( \log r \) or \( \arg z \), we almost-always work in a certain region of \( \mathbb{C} \sim \{0\} \) in which it is possible to define \( \arg z \) in a continuous manner. A typical situation might be the following: exclude the nonnegative real axis and define \( \arg z \) so that
\[
    0 < \arg z < 2\pi.
\]

Then we would have e.g.
\[
    \log(-1) = \pi i, \quad \log(e^{\pi i}) = 1 + i\pi,
\]

In such a situation \( \log z \) is also a well defined function of \( z \), and the polar form of the Cauchy–Riemann equation applies immediately:
\[
    \frac{\partial}{\partial r} \log z = \frac{\partial}{\partial r} (\log r + i\theta) = \frac{1}{r},
\]
\[
    \frac{\partial}{\partial \theta} \log z = \frac{\partial}{\partial \theta} (\log r + i\theta) = \frac{1}{r} \frac{\partial}{\partial r} \log r - \frac{1}{r^2}.
\]

Thus,
\[
    \frac{\partial}{\partial r} \log z = \frac{1}{r} \frac{\partial}{\partial \theta} \log z.
\]
Thus $\log z$ has a complex derivative, which equals $e^{i\theta} \frac{1}{z} = \frac{1}{r}e^{i\theta} = \frac{1}{z}$. We've thus obtained the expected formula

$$\frac{d}{dz} \log z = \frac{1}{z}.$$ 

(\text{Be sure to notice that although } \log z \text{ is ambiguous, the ambiguity is of the form of an additive constant } 2\pi i n, \text{ so } \frac{d}{dz} \log z \text{ annihilates that constant.})

\underline{Now an extremely important definition will be given and discussed:}

**DEFINITION** Let $D \subseteq \mathbb{C}$ be an open set, and assume that $D \overset{f}{\rightarrow} \mathbb{C}$ is a function which is of class $C^1$. That is, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are defined at each point of $D$ and are themselves continuous functions on $D$.

Suppose also that $f'(z)$ exists at every point $z \in D$.

Then we say that $f$ is a **holomorphic function** on $D$.

So of course we have at our disposal quite an array of holomorphic functions:

- $\exp$
- and such, $\cosh$, $\sinh$, $\cosh$,
- all Möbius transformations
- $\log$
- all polynomials in $z$: $f(z) = a_0 + a_1 z + \ldots + a_n z^n$
- all rational functions in $z$: $\frac{\text{polynomial}}{\text{polynomial}}$
We had quite a long and interesting discussion about this concept, and here are the major points.

1. We don't actually need to say that \( D \) is an open set! The very existence of \( f'(z) \) is that
   \[
   f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}
   \]
   and this requires \( f(z+h) \) to be defined for all sufficiently small \( |h| \), and thus that \( f \) is defined in some neighborhood of \( z \).

2. The assumption that \( f \in C^1 \) can be dispensed with, as a fairly profound theorem implies that it follows from just the assumption that \( f'(z) \) exists for every \( z \in D \).

3. "Holomorphic" is not a word you'll see in most basic books on complex analysis. Usually those books use the word "analytic."
   However, I want us to use "analytic" function to refer to a function which in a neighborhood of each \( z_0 \) in its domain can be represented as a power series
   \[
   \sum_{n=0}^{\infty} a_n (z-z_0)^n
   \]
   with a positive radius of convergence.

    - It's pretty easy to prove (and we shall do so) that every analytic function is holomorphic.
    - A much more profound theorem will also be proved — that every holomorphic function is analytic.
1. Start with the product decomposition of $z^n - 1$. Replace $z$ by $\frac{e^z}{W}$, assuming $W \neq 0$:

$$\frac{z^n}{W^n} - 1 = \prod_{k=0}^{n-1} \left( \frac{z}{W} - e^{\frac{2\pi ik}{n}} \right);$$

multiply both sides by $W^n$:

$$z^n - W^n = \prod_{k=0}^{n-1} (z - W e^{\frac{2\pi ik}{n}}).$$

Finally, if $W = 0$, then the equation is trivial.

2. Express the $n^{th}$ roots of unity in clockwise order to get the result. Or conjugate the equation from 1 to get:

$$\frac{z^n}{W^n} - 1 = \prod_{k=0}^{n-1} \left( \frac{z}{W} - e^{\frac{2\pi ik}{n}} \right);$$

and then replace $z$ with $\frac{1}{W}$ by their conjugates.

3. We have from 2:

$$\frac{z^n}{W^n} = \prod_{k=0}^{n-1} e^{\frac{-2\pi ik}{n}} \left( e^{\frac{2\pi ik}{n}} - e^{\frac{-2\pi ik}{n}} \right);$$

$$= \prod_{k=0}^{n-1} e^{\frac{-2\pi ik}{n}} \cdot \prod_{k=0}^{n-1} \left( e^{\frac{2\pi ik}{n}} - e^{\frac{-2\pi ik}{n}} \right).$$

This product equals $e^{-\frac{n-1}{2} \cdot \frac{2\pi in}{n}} \cdot n \cdot \frac{n-1}{2}$

$$= e^{-\frac{n-1}{2} (n-1)} = \left(e^{-\frac{\pi i}{n}}\right)^{n-1} = (-1)^{n-1}.$$

4. Replace $z$ by $e^{iz}$ and $W$ by $e^{-iz}$:

$$e^{inz} - e^{-inz} = (-i)^{n-1} \prod_{k=0}^{n-1} \left( e^{i\left(\frac{2\pi ik}{n}\right)} - e^{-i\left(\frac{2\pi ik}{n}\right)} \right).$$
That is,

\[ 2i \sin nz = (-i)^{n-1} \sum_{k=0}^{n-1} \frac{2^k}{k!} \sin \left( \frac{\pi k}{n} \right). \]

\[ \sin nz = (-i)^{n-1} \sum_{k=0}^{n-1} \frac{2^k}{k!} \sin \left( \frac{\pi k}{n} \right). \]

\[ (-i)^{n-1} = 2^{n-1}. \]

\[ \frac{\sin nz}{\sin z} = 2^{n-1} \sum_{k=1}^{n} \frac{1}{k!} \sin \left( \frac{\pi k}{n} \right). \]

Now the right side is continuous even at \( z = 0 \). So we conclude that

\[ \lim_{z \to 0} \frac{\sin nz}{\sin z} = 2^{n-1} \sum_{k=1}^{n} \frac{1}{k!}. \]

There are several ways to evaluate this limit directly... including L'Hôpital's rule if you are careful. (More about this later.) Another is to use the power series for \( \sin z \) to show that

\[ \lim_{z \to 0} \frac{\sin z}{z} = 1. \]

Therefore

\[ \frac{\sin nz}{\sin z} = \frac{\sin nz}{nz} \cdot \frac{nz}{\sin z} \cdot n \to 1 \cdot 1 \cdot n = n. \]

6. \( \cos z = \cos w \)

\[ \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iw} + e^{-iw}}{2}. \]

\[ \frac{e^{iz} - e^{-iz}}{2} = \frac{e^{iw} - e^{-iw}}{2}. \]

\[ e^{iz} - e^{iw} = e^{i(w+z)} - e^{i w}. \]

\[ e^{iw} - e^{i(z-w)} = 0. \]

\[ i(z-w) - 1 = 0 \quad \text{or} \quad e^{-i(w+z)} = 1. \]

\[ z-w = 2k\pi \quad \text{or} \quad w+z = 2k\pi. \]

7. \( \sin z = \cos \left( \frac{\pi}{2} - z \right) \), so \( \sin z = \sin w \)

\[ \frac{\pi}{2} - z \to \frac{\pi}{2} - w = 2k\pi \]

\[ \frac{\pi}{2} - z + w = 2k\pi. \]

\[ \frac{\pi}{2} - z \to \frac{\pi}{2} - w = 2k\pi. \]

\[ \frac{\pi}{2} - z + w = 2k\pi. \]
HW 5M due January 28, 2013

Let $D$ be the open half plane

$$D = \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \}.$$  

Let $f$ be the function defined on $D$ by $f(z) = z^2$. Of course, $f$ is holomorphic.

1. Prove that $f$ is a bijection of $D$ onto a set $D' \subset \mathbb{C}$.

2. What is $D'$?

3. The inverse function $f^{-1}$ maps $D'$ onto $D$. We’ll actually prove a general theorem asserting that inverses of holomorphic functions are always holomorphic. But in this problem I want you to prove directly that $f^{-1}$ is holomorphic.

4. For every real number $0 < a < \infty$ let $L_a$ be the straight line

$$L_a = \{ z \in \mathbb{C} \mid \text{Re}(z) = a \}.$$  

Prove that the images $f(L_a)$ are parabolas.

5. Prove that the focus of each parabola $f(L_a)$ is the origin.

6. For each real number $b$ let $M_b$ be the ray

$$M_b = \{ z \in \mathbb{C} \mid \text{Im}(z) = b \}.$$  

Since $f$ is conformal, the sets $f(M_b)$ and the parabolas $f(L_a)$ are orthogonal to one another.

Describe the sets $f(M_b)$.  

CONFORMAL TRANSFORMATIONS

Roughly speaking, the adjective conformal refers to the preservation of angles. More specifically, consider a situation in which a function $F$ from one type of region to another is differentiable in the vector calculus sense. And consider a point $p$ and its image $F(p)$. Calculus then enables us to move tangent vectors at $p$ to tangent vectors at $F(p)$... some sort of notation like this is frequently used:

$$h \text{ (tangent vector at } p) \rightarrow DF(p)h.$$

Here $DF(p)$ is often the Jacobian matrix of $F$ at $p$ and the symbol $DF(p)h$ refers to multiplication of a matrix and a vector.

Then if $h_1$ and $h_2$ are tangent vectors at $p$, they have a certain angle $\theta$ between them:

We are interested in the angle between the images under $F$ of these tangent vectors:

$$\Delta \theta \rightarrow DF(p)h_1 \rightarrow DF(p)h_2$$

If this angle is also $\theta$ and this happens at every $p$ and for all tangent vectors, we say that $F$ is a conformal transformation.

Tersely,

conformal means angle-preserving.

Examples: Mercator projections of the earth; stereographic projections.
Now we specify this for holomorphic functions. So assume that $f$ is holomorphic and that at a fixed point $z = 0$ we know that $f'(z) \neq 0$. Let the polar form of this number be

$$f'(z) = Ae^{i\alpha} \quad \text{(where } A > 0, \alpha \in \mathbb{R}).$$

By definition

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

Rewrite this relationship as

$$f(z+h) = f(z) + f'(z)h \quad \text{approximately}.$$

This means that $f$ transforms a tangent vector $h$ at $z$ to the vector at $f(z)$ given by

$$f'(z)h.$$

In other words, directions $h$ at $z$ are transformed to directions $f'(z)h$ at $f(z)$:

This action does two things to $h = (1)$ multiplies its modulus by $A$ and $(2)$ rotates it by the angle $\alpha$.

We conclude immediately that $f$ preserves angles:

The moduli of all the infinitesimal vectors at $z$ are multiplied by

the same positive number $A$. 
SUMMARY: Every holomorphic function $f$ is conformal at every $z$ with $f'(z)\neq 0$. Infinitesimal vectors at $z$ are magnified by the positive number $|f'(z)|$.

EXAMPLE: $f(z) = z^3$.

$z = i \rightarrow f'(i) = -3$

$\text{Arg} f(i) = -i$

But notice that $f'(0) = 0$ and $f$ does not preserve angles at $0$—instead, it multiplies them by 3.
(COMPLEX) POWER SERIES

1. Infinite series of complex numbers.

We shall need to discuss \( \sum_{n=0}^{\infty} a_n \), where \( a_n \in \mathbb{C} \).

Convergence of such a series is no mystery at all. We form the sequence of partial sums

\[ S_n = a_0 + \ldots + a_n, \]

and just demand that

\[ \lim_{n \to \infty} S_n = L \quad \text{exists.} \]

Then we say

\[ \sum_{n=0}^{\infty} a_n = L \quad \text{is convergent.} \]

Equivalently, we could reduce everything to two real series, require that they converge, and then

\[ \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \text{Re}(a_n) + i \sum_{n=0}^{\infty} \text{Im}(a_n). \]

Necessarily, if a series converges, then \( \lim_{n \to \infty} a_n = 0. \) (For \( a_n = S_n - S_{n-1} \to L - L = 0. \))

Converge is of course false: the "harmonic series" \( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \) diverges.

Absolute convergence is what we'll usually see. We say that \( \sum_{n=0}^{\infty} |a_n| \) converges absolutely if \( \sum_{n=0}^{\infty} |a_n| \) converges. Then there's an important

Theorem If a series converges absolutely, then it converges.

(The proof relies on the completeness of \( \mathbb{R} \).)
2. Most important example of a power series - the Geometric Series

\[ \sum_{n=0}^{\infty} z^n, \text{ where } z \in S. \]

By our necessity condition, if this series converges, then \( z^n \to 0 \). That is, \( |z|^n = (|z|^n) \to 0 \). That is, \( |z| < 1 \).

Conversely, suppose \( |z| < 1 \). Then

\[ S_N = 1 + z + \ldots + z^N = \frac{1 - z^{N+1}}{1 - z} \]

\[ = \frac{1}{1 - z} - \frac{z^{N+1}}{1 - z}. \]

Now simply note that

\[ \left| \frac{z^{N+1}}{1 - z} \right| = \frac{|z|^{N+1}}{|1 - z|} \to 0 \] because \( |z| < 1 \).

**SUMMARY:**

\( \sum_{n=0}^{\infty} z^n \) converges if and only if \( |z| < 1 \).

And then it converges absolutely, and

\[ \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}. \]

3. **Definition.** A power series centered at \( z_0 \) is an infinite series of the form

\[ \sum_{n=0}^{\infty} a_n (z - z_0)^n, \]

where the coefficients \( a_n \) are complex numbers.

A. Usually in developing the properties of such series we'll work with the center \( z_0 = 0 \).
Simple warning: the first term in this series is not really 
\[ a_0 (z - z_0)^0, \]
it is actually a lazy way of writing the constant \( a_0 \). A more legitimate expression would be 
\[ a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n \] ... none ever bottoms.

THEOREM (easy but crucial!) If a power series 
\[ \sum_{n=0}^{\infty} a_n z^n \]
converges when \( z = z_1 \), and if \( |z_2| < |z_1| \), then it converges absolutely when \( z = z_2 \).

(easy) PROOF \( \sum_{n=0}^{\infty} a_n z_1^n \) converges \( \Rightarrow \lim_{n \to \infty} a_n z_1^n = 0 \)
\[ \Rightarrow |a_n z_1^n| \leq \text{a constant } C \text{ for all } n \geq 0. \]

Therefore,
\[ |a_n z_2^n| \leq C |z_1|^{-n} |z_2|^n = C \left( \frac{|z_2|}{|z_1|} \right)^n. \]

Since \( \frac{|z_2|}{|z_1|} < 1 \), the geometric series \( \sum_{n=0}^{\infty} \left( \frac{|z_2|}{|z_1|} \right)^n \). Therefore,
\[ \sum_{n=0}^{\infty} |a_n z_2^n| \] converges.

That is,
\[ \sum_{n=0}^{\infty} a_n z_2^n \] converges absolutely.

\( \Box \)
RADIUS OF CONVERGENCE

It's an easy but extremely important fact that every power series has associated with it a unique \( 0 \leq R \leq \infty \) such that

\[
\begin{cases}
|z| < R & \Rightarrow \text{the power series converges absolutely at } z,
|z| > R & \Rightarrow \text{the power series diverges at } z.
\end{cases}
\]

This is a quick result from what we have just proved.

There is actually a formula for \( R \) in general, but it won't be needed by us. Just to be complete, here's that formula:

\[
R = \frac{1}{\limsup_{n \to \infty} |a_n|^{1/n}}.
\]

Useful observation: suppose \( |z| < R \), where \( R \) is the radius of convergence of \( \sum_{n=0}^{\infty} a_n z^n \). Choose any \( \eta \) such that \( |z| < |\eta| < R \).

Then from the preceding lecture we have an estimate

\[
|a_n| \leq C |\eta|^n.
\]

Now consider the quantity \( n a_n z^n \);

notice \[
|n a_n z^n| \leq C n \left(\frac{|\eta|}{|z|}\right)^n.
\]

Since \( \frac{|\eta|}{|z|} < 1 \), the real series

\[
\sum_{n=1}^{\infty} n \left(\frac{|\eta|}{|z|}\right)^n
\]

converges. We can actually appeal to the basic calculus ratio test to check this. Therefore,

\[
\sum_{n=0}^{\infty} |n a_n z^n| < \infty.
\]
Thus, not only does \( \sum_0^\infty a_n z^n \) converge absolutely, but the series with larger coefficients \( \sum_1^\infty a_n \) also converges absolutely...remembar, \( |z| < R \).

**CONCLUSION:** multiplying the coefficients \( a_n \) of a power series by \( n \) does not change the radius of convergence.

**RATIO TEST** We just mentioned this result of basic calculus, namely, suppose that a series of positive numbers \( \sum_0^\infty a_n \) has the property that

\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l \quad \text{exists}.
\]

Then,

\[
\begin{cases}
  l < 1 & \Rightarrow \text{the series converges}, \\
  l > 1 & \Rightarrow \text{the series diverges}.
\end{cases}
\]

\( l = 1 \) : no conclusion, in general.

And now we apply this to power series \( \sum_0^\infty a_n z^n \) with the property that

\[
\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = l \quad \text{exists}.
\]

Then we can apply the ratio test to the series \( \sum_0^\infty |a_n z^n| \), since

\[
\lim_{n \to \infty} \frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} = l |z|^1.
\]

Thus,

\[
\begin{cases}
  |z| < 1 & \Rightarrow \text{convergence}, \\
  |z| > 1 & \Rightarrow \text{divergence}.
\end{cases}
\]

That is, the radius of convergence of the power series equals

\[
R = \frac{1}{l}.
\]
EXAMPLES

- \( \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad R = \infty \)

- \( \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad R = 1 \)

- \( \sum_{n=0}^{\infty} \frac{z^n}{n!^2} \quad R = 0 \)

Also, convergence for \( |z| = R \) can happen variously:

- \( \sum_{n=0}^{N} \frac{z^n}{n!^2} \) diverges for all \( |z| = 1 \)

- \( \sum_{n=0}^{\infty} \frac{z^n}{n!^2} \) converges for all \( |z| = 1 \)

- \( \sum_{n=0}^{\infty} \frac{z^n}{n!} \) diverges for \( z = 1 \), converges for all other \( (z| = 1 \).

we don't actually know this at the present time in this course.

SIMPLE PROPERTIES OF POWER SERIES

Let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) have radius of convergence \( R_1 \),

\( g(z) = \sum_{n=0}^{\infty} b_n z^n \) have radius of convergence \( R_2 \).

SUM

\[ f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n \] has radius of convergence \( \geq \min(R_1, R_2) \).

PRODUCT

\[ f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n \] has radius of convergence \( \geq \min(R_1, R_2) \),

where

\[ c_n = \sum_{k=0}^{n} a_k b_{n-k} \].
DERIVATIVE
For $1 < |z| < R_1$, the function $f$ has a complex derivative, and

$$f'(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$$

... notice some radius of convergence.

We'll soon be able to prove the fact about products and the fact about $f'(z)$ with very little effort, almost no calculation involved. But I want to show you a direct proof for $f'(z)$. So let $1 < |z| < R_1$ be fixed, and $h \in \mathbb{C}$ with small modulus, so that in particular $|z| + |h| < R_1$.

Then we compute

$$f(z+h) - f(z) - h \sum_{n=1}^{\infty} a_n z^{n-1} = \sum_{n=1}^{\infty} \left[ a_n (z+h)^n - a_n z^n - n a_n z^{n-1} h \right]$$

$$= \sum_{n=2}^{\infty} a_n \left[ (z+h)^n - z^n - n z^{n-1} h \right]$$

by binomial formula

$$= \sum_{n=2}^{\infty} a_n \left[ \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^k - z^n - n z^{n-1} h \right]$$

$$= \sum_{n=2}^{\infty} a_n \left[ \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^k \right]$$

$$= h^2 \sum_{n=2}^{\infty} a_n \left[ \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^k \right]$$

Divide by $h$:

$$\frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} a_n z^{n-1} = h \sum_{n=2}^{\infty} a_n \left[ \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^{k-2} \right]$$

It follows easily that $f'(z)$ exists and equals $\sum_{n=1}^{\infty} a_n z^{n-1}$.

Therefore, every power series is holomorphic on its open disc of convergence.
Let $D$ be the open half plane

$$D = \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \}.$$

Let $f$ be the function defined on $D$ by $f(z) = z^2$. Of course, $f$ is holomorphic.

1. Prove that $f$ is a bijection of $D$ onto a set $D' \subset \mathbb{C}$.

2. What is $D'$?

I'll use polar coordinates for $D$: $z = re^{i\theta}$, $0 < r < \infty$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Then

$$z^2 = r^2 e^{2i\theta}, \quad -\pi < 2\theta < \pi.$$

Thus we read off that

$$D' = \mathbb{C} \sim \text{nonpositive real axis}$$

and each $w \in D'$ can be written

$$w = re^{i\phi}, \quad \phi \in (-\pi, \pi) \text{ and } r > 0.$$

The corresponding $z$ is then

$$z = \sqrt{r} \, e^{i\phi/2}.$$

**BIJECTION**

3. The inverse function $f^{-1}$ maps $D'$ onto $D$. We'll actually prove a general theorem asserting that inverses of holomorphic functions are always holomorphic. But in this problem I want you to prove directly that $f^{-1}$ is holomorphic.

We've just seen that

$$f^{-1}(re^{i\phi}) = \sqrt{r} \, e^{i\phi/2}.$$

Thus

$$\frac{\partial f^{-1}}{\partial r} = \frac{1}{2\sqrt{r}} \, e^{i\phi/2}, \quad \text{and} \quad \frac{\partial f^{-1}}{\partial \phi} = \frac{i}{2\sqrt{r}} \, e^{i\phi/2}.$$

So we find

$$\frac{\partial f^{-1}}{\partial \phi} = \frac{1}{i\sqrt{r}} \frac{\partial f^{-1}}{\partial r}.$$

(As a bonus,

$$f^{-1}(w) = e^{i\phi} \frac{\partial f^{-1}}{\partial \phi} = \frac{1}{2\sqrt{r}} \, e^{i\phi/2}.$$ 

i.e.

$$\left. \frac{d\sqrt{w}}{dw} \right|_{w=r} = \frac{1}{2\sqrt{r}}.$$

)
4. For every real number \(0 < a < \infty\) let \(L_a\) be the straight line
\[ L_a = \{z \in \mathbb{C} \mid \text{Re}(z) = a\}. \]
Prove that the images \(f(L_a)\) are parabolas.

Now I'll use cartesian coordinates:
\[ z = x + iy, \quad w = z^2 = u + iv. \]

1. e. ,
\[ \begin{cases} u = x^2 - y^2 \\ v = 2xy. \end{cases} \]

\((L_a): x = a \Rightarrow u = a^2 - y^2 \text{ and } v = 2ay. \) Thus \( y = \frac{v}{2a} \) and we have
\[ u = a^2 - \frac{v^2}{4a^2}, \]
the cartesian eqn. of a parabola.

5. Prove that the focus of each parabola \(f(L_a)\) is the origin.

\[ f(L_a) \]
\[ w = (u, v) \]
\[ f(2a^2) \]
directrix
\[ 1u + \sqrt{1} = 1w = 1z = 1z = x + iy = x^2 + y^2 = a^2 + y^2 = 2a^2 - u, \]
so that
distance from \(w\) to \(O\) = distance from \(w\) to directrix.
This shows that \(O\) is the focus.

6. For each real number \(b\) let \(M_b\) be the ray
\[ M_b = \{z \in \mathbb{C} \mid \text{Im}(z) = b\}. \]
Since \(f\) is conformal, the sets \(f(M_b)\) and the parabolas \(f(L_a)\) are orthogonal to one another.

Describe the sets \(f(M_b)\).

\((M_b): y = b, \) so \( u = x^2 - b^2 \) and \( v = 2bx. \)
\(b = 0: \) \( w = x^2 \) gives the positive real axis.
\(b \neq 0: \) \( u = \frac{v^2}{4b^2} - b^2 \text{ and } \text{sign}(v) = \text{sign}(b); \)
half a parabola.
A power series centered at 0 is often called a **Maclaurin series**.

In the following exercises, simplify your answers as much as possible.

1. Find the Maclaurin series for \( \frac{1}{(1-x)^3} \).

2. Find the Maclaurin series for \( \left( \frac{x}{3-x} \right)^2 \).

3. Find the Maclaurin series for \( e^x \sin x \).

4. Let \( \omega = e^{2\pi i/3} \). (1 + \omega + \omega^2 = 0)
   
   Find the Maclaurin series for \( \frac{e^x + e^{\omega x} + e^{\omega^2 x}}{3} \).

5. Find explicitly \( \sum_{n=0}^{\infty} \frac{(-1)^n (x-\pi i)^n}{n!} \).

6. Find explicitly \( \sum_{n=0}^{\infty} \frac{5^n}{n!} \).
More basic results about power series

**THEOREM**: Suppose that \( f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \) is a power series with a positive radius of convergence. And suppose that \( f(z) = 0 \) for an infinite sequence of points \( z \) converging to \( z_0 \).

Then \( f = 0 \). In other words, \( a_n = 0 \) for all \( n \).

**PROOF**: We assume \( z_0 = 0 \) with no loss of generality. Our proof is by contradiction, so we suppose that not all \( a_n = 0 \).

Then we have \( a_N \neq 0 \) for a smallest \( N \), so that

\[
 f(z) = \sum_{n=0}^{\infty} a_n z^n
 = z^N \sum_{n=0}^{\infty} a_n z^{n-N}
 = z^N g(z),
\]

where \( g(z) \) is the power series

\[
 g(z) = \sum_{k=0}^{\infty} a_{N+k} z^k
 = a_N + a_{N+1} z + \cdots.
\]

Then \( f(z) = 0 \) and \( z \neq 0 \) \( \Rightarrow \) \( g(z) = 0 \). Therefore our hypothesis implies that \( g(z) = 0 \) for an infinite sequence of points \( z \) converging to \( 0 \).

But \( \lim_{z \to 0} g(z) = g(0) = a_N \). Thus \( a_N = 0 \). Contradiction. QED
Taylor series

Again we suppose that \( f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \) is a power series with positive radius of convergence. Then we observe

\[
\begin{align*}
    f(z_0) &= a_0, \\
    f'(z) &= \sum_{n=1}^{\infty} na_n (z-z_0)^{n-1}, \text{ so } f'(z_0) &= a_1, \\
    f''(z) &= \sum_{n=2}^{\infty} n(n-1)a_n (z-z_0)^{n-2}, \text{ so } f''(z_0) &= 2a_2.
\end{align*}
\]

In this manner we find

\[
f^{(k)}(z_0) = k! a_k.
\]

Therefore,

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n.
\]

The right side of this equation is called the Taylor series of \( f \) centered at \( z_0 \).

(If \( z_0 = 0 \), it's called the Maclaurin series of \( f \).)

Changing center of power series

First, a couple of examples:

Example 1 \( f(z) = \sum_{n=0}^{\infty} z^n \) for \( |z| < R = 1 \), the geometric series.

Let's see about an expansion of \( f(z) \) centered instead at \( \frac{1}{2} \).

There we write

\[
f(z) = \frac{1}{1-z} \quad \text{(sum of geometric series)}
\]

\[
= \frac{1}{\frac{1}{2} - (z+\frac{1}{2})} \\
= \frac{1}{\frac{1}{2} - \frac{z+\frac{1}{2}}{\frac{1}{2}}} \\
= \frac{2}{3} \cdot \frac{1}{1 - \frac{z+\frac{1}{2}}{\frac{1}{2}}} \\
= \frac{2}{3} \cdot \sum_{n=0}^{\infty} \left( \frac{z+\frac{1}{2}}{\frac{1}{2}} \right)^n \quad \text{(a different geometric series)}
\]
and this series converges in the disk $|z + \frac{1}{2}| < \frac{3}{2}$.

$$f(z) = \sum_{n=0}^{\infty} \frac{(z + \frac{1}{2})^n}{(\frac{3}{2})^n}.$$  

Example 2. $f(z) = \frac{1}{z}$, and we want to express this in a power series centered at $z_0 = 0$. Then as in the preceding example, we write:

$$f(z) = \frac{1}{z_0} + (z-z_0).$$

$$= \frac{1}{z_0} \frac{1}{1+\frac{z-z_0}{z_0}}.$$  

$$= \frac{1}{z_0} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-z_0}{z_0}\right)^n \quad \text{(geometric series)}.$$  

$$= \sum_{n=0}^{\infty} \left(\frac{-1}{z_0}\right)^n \left(\frac{z-z_0}{z_0}\right)^n.$$  

A Taylor series with radius of convergence $|z_0|$:  

A very general theorem:

Let $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ be a power series with radius of convergence $R$, and assume $|z-z_0| < R$. Then $f(z) = \sum_{n=0}^{\infty} b_n (z-z_1)^n$, and the radius of convergence of this new series is

$$R - |z_1-z_0|.$$  

Although it is easy enough to prove this theorem with basic manipulations, we already know, such a proof is tedious and boring. Well soon be able to prove this theorem and many others with almost no effort at all!
These ideas lead us to an important definition:

Suppose \( f \) is a \( \mathbb{C} \)-valued function defined on an open subset \( D \subseteq \mathbb{C} \), and suppose that for every \( z_0 \in D \) we are able to write

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for all } |z - z_0| < R(z_0)
\]

where \( R(z_0) \) is some positive number. Then we say that \( f \) is (complex) analytic on \( D \).

It's then quite clear that every analytic function is holomorphic.

After we obtain Cauchy's integral formula, we'll see that the exact converse is valid:

\[\star\] every holomorphic function is analytic!
HW 6W SOLUTIONS

1. \[ \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \; \text{ differentiation } \Rightarrow \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}. \]
   
   Once again, \[ \frac{2}{(1-z)^3} = \sum_{n=2}^{\infty} n (n-1) z^{n-2}. \] Thus
   \[ \frac{1}{(1-z)^3} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2} z^{n-2}. \]

2. \[ \frac{1}{3} \frac{1}{1-\frac{z}{3}} = \sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}} \; \text{ (geometric series).} \]
   
   Differentiation \Rightarrow \[ \frac{1}{(3-z)^2} = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{3^{n-1}}. \]

   Multiply by \( z^2 \):
   \[ \left( \frac{z^2}{3-z} \right)^2 = \sum_{n=1}^{\infty} \frac{n z^{n+1}}{3^{n-1}}. \]

3. \[ e^{z \sin z} = e^{z} \frac{e^{iz} - e^{-iz}}{2i} \]
   
   \[ = \frac{1}{2i} \left( (z+i)^n - (1-i)^n \right) \]
   
   \[ = \frac{1}{2i} \sum_{n=0}^{\infty} \frac{(z+i)^n - (1-i)^n}{n!} \]
   
   \[ = \frac{1}{2i} \sum_{n=0}^{\infty} \frac{(\sqrt{2} e^{i \pi/4})^n - (\sqrt{2} e^{-i \pi/4})^n}{n!} \]
   
   \[ = \frac{1}{2i} \sum_{n=0}^{\infty} \frac{2^{n/2} e^{i n \pi/4} - e^{-i n \pi/4}}{n!} \]
   
   \[ = \sum_{n=0}^{\infty} \frac{2^{n/2} \sin \frac{n \pi}{4}}{n!} \frac{z^n}{n!}. \]
4. $\frac{e^z + \omega^2z + \omega^4z}{1 + \omega^n + \omega^{2n}}$ has $z^n$ coefficient equal to
   \[ \frac{1}{n!} \]
   If $n \equiv 0 \pmod{3}$, then $\omega = \omega^2 = \omega^4 = 1$ since $\omega^3 = 1$.
   If $n \equiv 1 \pmod{3}$, it is $\frac{1 + \omega + \omega^2}{n!} = 0$.
   If $n \equiv 2 \pmod{3}$, it is $\frac{1 + \omega^2 + \omega^4}{n!} = 0$.

   Thus,
   \[ \frac{e^z + \omega^2z + \omega^4z}{3} = \sum_{n \equiv 0 (\pmod{3})} \frac{z^n}{n!} \]
   \[ = \sum_{k=0}^{\infty} \frac{z^{3k}}{(3k)!} \]

5. \[ \sum_{n=0}^{\infty} \frac{(-1)^n (z - \pi i)^n}{n!} = e^{-(z - \pi i)} = e^z e^{-\pi i} = -e^{-\pi i} \]
   for all $z \in \mathbb{C}$

6. Geometric:
   \[ \sum_{n=0}^{\infty} \frac{(z/5)^n}{n!} = \sum_{n=0}^{\infty} \left( \frac{z}{5} \right)^n \]
   \[ = \frac{1}{1 - \frac{z}{5}} \text{ for } |z| < 5/5. \]
Although we didn't discuss this Monday, it fits in that subject matter.

The principle involved here is based on simple single-variable calculus:

Lemma Suppose \( f = f(z) \) has partial derivatives of first order which satisfy

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \quad \text{on a rectangle} \quad (x_0, x_1) \times (y_0, y_1).
\]

Then \( f \) is constant on that rectangle.

Theorem Suppose \( D \subseteq \mathbb{C} \) is an open connected set and \( D \not\supseteq \mathbb{C} \) has partial derivatives of first order which satisfy

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \quad \text{on} \quad D.
\]

Then \( f \) is constant on \( D \).

Proof By the lemma, \( f \) is constant on rectangular contained in \( D \).
Since \( D \) is connected, \( f \) is constant on \( D \).

QED

Corollary Suppose \( D \subseteq \mathbb{C} \) is an open connected set and \( D \not\supseteq \mathbb{C} \) is holomorphic on \( D \) with \( f'(z) = 0 \) for all \( z \in D \).

Then \( f \) is constant.

Illustration: For \( |z| < 1 \) the number \( 1 - z \) can be chosen to have

\[-\pi < \arg (1 - z) < \pi. \]

Then

\[
\frac{d}{dz} \log (1 - z) = \frac{1}{1 - z} = -\sum_{n=0}^{\infty} z^n = -\frac{\sum_{n=0}^{\infty} z^n}{1 - z}.
\]

Thus \( \log (1 - z) + \sum_{n=1}^{\infty} \frac{z^n}{n} \) satisfies the hypothesis of the corollary for \( |z| < 1 \), and is thus constant.

At \( z = 0 \) it equals 0. Therefore,

\[
\log (1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{for} \quad |z| < 1.
\]
EXAM 1  Due February 1, 2013

1. Find the unique Möbius transformation $f$ which satisfies

\[
\begin{align*}
    f(1) & = 1, \\
    f(-1) & = -1, \\
    f(3) & = 2.
\end{align*}
\]

2. Let $f$ and $g$ be two power series centered at 0, with radii of convergence $R_1$ and $R_2$, respectively.

   a. Prove that if $R_1 < R_2$, then the power series $f + g$ has radius of convergence equal to $R_1$.

   b. Give an example of $f$ and $g$ with $R_1 = R_2 = 1$, for which $f + g$ has radius of convergence 3.

3. Define the Joukowski function $J$ by the equation

\[
    J(z) = \frac{z + \frac{1}{z}}{2}.
\]

   a. Prove that $J$ is a bijection of the region \( \{ z \in \mathbb{C} \mid |z| > 1 \} \) onto a region of the complex plane, and state exactly what that region is.

   b. Let $C_r$ be the circle with center 0 and radius $r > 1$. Show that $J$ transforms this circle onto an ellipse whose foci are 1 and -1.

4. Find explicitly the sum of the power series

\[
    \sum_{n=1}^{\infty} n^2 z^n.
\]

HONOR PLEDGE