complex analysis

chapter iv
residues (part 1)

feb 15 residues

feb 18 evaluation of definite integrals

feb 20 evaluating, continued

feb 22 evaluations, continued

mar 4 principal value integrals

mar 6 integrals involving exponentials (and trig functions)

mar 8 trig functions, continued

hw 14 f

hw 15 w

hw 16 f

exam 2

hw 17 w
RESIDUES

We are preparing to do some truly amazing things with our theory, but first we need an important definition. This is all in the context of a holomorphic function with an isolated singularity at $z_0$. Let $f$ be such a function. We are going to define a complex number based on this situation, but we do it in 3 separate ways, and we'll observe that these 3 ways yield the same number.

**Definition 1:** Let $\gamma$ be any small loop surrounding $z_0$ in the CCW sense; then our number equals

\[
\frac{1}{2\pi i} \oint_{\gamma} f(z) \, dz.
\]

This number does not depend on $\gamma$, thanks to Cauchy's integral theorem. For surround $z_0$ with a very small circle $\tilde{\gamma}$ in the region between $\gamma$ and this circle we have from Cauchy's theorem

\[
\oint_{\gamma} f(z) \, dz = \oint_{\tilde{\gamma}} f(z) \, dz.
\]

Thus

\[
\oint_{\gamma} f(z) \, dz = \oint_{\text{CCW circle}} f(z) \, dz,
\]

so the left side does not depend on $\gamma$.

**Definition 2:** Convergent Laurent expansion of $f$ near $z_0$:

\[
f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n.
\]
Then our number equals

\[ C_{-1}. \]

That is, we focus our attention on the Laurent expansion:

\[ f(z) = \cdots + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \cdots \]

and if \( (C_{-1}) \) we use.

This agrees with the first definition since

\[ \int f(z)\,dz = \sum_{n=-\infty}^{\infty} c_n \int (z-z_0)^n\,dz = C_{-1} \int \frac{dz}{z-z_0} \]

\[ -2\pi i C_{-1}. \]

**Definition 3:** This definition relies on trying to integrate \( f(z) \) as an "indefinite integral" near \( z_0 \). The trouble is precisely with the term \( \frac{c_{-1}}{z-z_0} \). For

\[ \int f(z)\,dz = \sum_{n=-1}^{\infty} c_n (z-z_0)^{n+1} + \left( \int \frac{C_{-1}}{z-z_0}\,dz \right). \]

Thus the number we want is the unique \( a \in \mathbb{C} \) such that

\[ f(z) - \frac{a}{z-z_0} = \text{the derivative of a holomorphic function} \]

in the region \( 0 < |z-z_0| < r \),

some \( r > 0 \).
Definition: The residue of $f$ at $z_0$ is the number defined in all 3 of the above definitions. We denote it as $\text{Res}(f, z_0)$.

We now list a number of properties and examples:

1. If $f$ is holomorphic at $z_0$ (i.e., $z_0$ is a removable singularity of $f$), then $\text{Res}(f, z_0) = 0$.

2. $\text{Res}\left(\frac{1}{z-z_0}, z_0\right) = 1$. (Most basic case.)

3. $\text{Res}(z-z_0)^n, z_0) = 0$ if $n \in \mathbb{Z}$, $n \neq -1$.

4. $\text{Res}\left(e^{\frac{1}{z}}, 0\right) = 1$.

5. $\text{Res}\left(\sin\frac{1}{z}, 0\right) = 1$.

6. $\text{Res}\left(\cos\frac{1}{z}, 0\right) = 0$.

7. If $f$ is an even function, $\text{Res}(f, 0) = 0$.

8. Suppose $f$ has a simple pole (i.e., pole of order 1) at $z_0$. Then we have $f(z) = \frac{c_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n(z-z_0)^n$. 

Feb 15(3)
so that

$$(z-z_0) f(z) = c_1 + (z-z_0) \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

so that the left side has limit $c_1$ as $z \to z_0$:

**Simple Pole**

$$\text{Res}(f, z_0) = \lim_{z \to z_0} (z-z_0) f(z),$$

9. A corollary is now the following, which is the handiest special case!

Suppose we know that

$$f(z) = \frac{a(z)}{b(z)},$$

where both numerator and denominator are holomorphic at $z_0$, and $b(z_0) = 0$, $b'(z_0) \neq 0$. Then $z_0$ is a simple pole for $f$, and our previous result yields

$$\text{Res}(f, z_0) = \lim_{z \to z_0} \frac{(z-z_0) a(z)}{b(z)}$$

$$= \lim_{z \to z_0} \frac{a(z)}{b(z)} \frac{z-z_0}{z-z_0} \to 0$$

$$= \frac{a(z_0)}{b'(z_0)}.$$

For the record:

$$\text{Res}\left(\frac{a(z)}{b(z)}, z_0\right) = \frac{a(z_0)}{b'(z_0)} \quad \text{if} \quad b(z_0) = 0, \quad b'(z_0) \neq 0.$$

10. $\text{Res}(\cot z, 0) = 1$ since $\cot z = \frac{\cos z}{\sin^2 z}$ and at 0 $\cos 0 = 1$, $\sin 0 = 0$, $\sin(0) = 1$.

More generally,

$$\text{Res}\left(\cot z, n\pi\right) = 1.$$ (Caustic)

11. $\text{Res}(\csc z, n\pi) = (-1)^n$. 

12. \[ \text{Res} \left( \frac{1}{e^{z+1}}, 0 \right) = 1. \]

13. \[ \text{Res} \left( \frac{z}{e^{z}-1}, 0 \right) = 0. \]

14. Let's compute the residue at 0 of \( \frac{1}{a-z} e^{\frac{1}{2}z} \), where \( a \neq 0 \).

None of our easy examples apply, so we resort to series:

\[ \frac{1}{a-z} = \frac{1}{a} \frac{1}{1-\frac{z}{a}} = \sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}}. \]

\[ e^{\frac{1}{2}z} = \sum_{n=0}^{\infty} \frac{z^n}{n! 2^n}. \]

Multiply the series and look for the \( \frac{1}{2}z \) terms:

\[ \frac{1}{a} \frac{1}{1} + \frac{1}{a^2} \frac{1}{2} + \frac{1}{a^3} \frac{1}{3} + \ldots. \]

This equals \(-1 + \sum_{k=0}^{\infty} \frac{1}{a^{k+1} k!} \), so

\[ \text{Res} \left( \frac{1}{a-z} e^{\frac{1}{2}z}, 0 \right) = e^{\frac{1}{2a}} - 1. \]

Incidentally,

\[ \text{Res} \left( \frac{1}{a-z} e^{\frac{1}{2}z}, a \right) = -e^{\frac{1}{2a}}. \]
Now we come to a major theorem. Before stating it, let us be sure we completely understand the context.

As in the Cauchy integral theorem, we deal with a "nice" bounded open set $D \subset \mathbb{C}$, whose boundary $\partial D$ consists of finitely many curves. We also assume that $f$ is holomorphic on an open set containing $D \cup \partial D$ except for finitely many isolated singularities $z_1, \ldots, z_n$, all contained in the open set $D$.

Then we have the

**Residue Theorem**

$$\frac{1}{2\pi i} \oint_{\partial D} f(z) \, dz = \sum_{k=1}^{n} \text{Res}(f, z_k).$$

The proof is an easy application of the Cauchy integral theorem if we first remove from $D$ small discs $E_k$ centered at the $z_k$'s, and the resulting open set be denoted $D' = D \setminus \bigcup_{k=1}^{n} E_k$.

Since $f$ is holomorphic on $D'$, Cauchy's theorem yields

$$0 = \frac{1}{2\pi i} \oint_{\partial D'} f(z) \, dz.$$

But $\partial D' = \partial D \cup \bigcup_{k=1}^{n} \partial E_k$, so we obtain

$$0 = \frac{1}{2\pi i} \oint_{\partial D} f(z) \, dz + \sum_{k=1}^{n} \frac{1}{2\pi i} \oint_{\partial E_k} f(z) \, dz.$$
But
\[ \frac{1}{2\pi i} \oint_{\Gamma_k} f(z) \, dz = -\frac{1}{2\pi i} \oint_{\Gamma_k^c} f(z) \, dz \]
clockwise \hspace{1cm} \text{counterclockwise}
\[ = -\text{Res} \left( \frac{f}{z^n} \right). \] \hspace{1cm} \text{(def. of residue)}

This proves the theorem. \hspace{2cm} \text{Q.E.D.}

**HW 13 M SOLUTION**

\[(1+z)^{\alpha + \beta} = (1+z)^\alpha (1+z)^\beta\]

is a valid identity if \(|z| < 1\) and we use the principal value of \(\text{arg} \ (1+z)\) in all 3 places.

In terms of MacLaurin series,
\[ \sum_{n=0}^{\infty} \frac{(\alpha + \beta)_n}{n!} z^n = \sum_{j=0}^{\infty} \binom{\alpha}{j} z^j \sum_{k=0}^{\infty} \binom{\beta}{k} z^k. \]

So equal the coefficients of \(z^n\) on the two sides of this equation:

\[ \binom{\alpha + \beta}{n} = \sum_{j+k=n} \binom{\alpha}{j} \binom{\beta}{k}. \]

\[ \text{i.e.,} \quad \binom{\alpha + \beta}{n} = \sum_{j=0}^{n} \binom{\alpha}{j} \binom{\beta}{n-j}. \] \hspace{2cm} \text{Q.E.D.}
HW 14 F  due Feb 22, 2013

1. For any positive integer n, find all the singularities of 
the function \( \frac{1}{z^n + 1} \) and calculate all the corresponding 
residues. Also, compute the sum of all the residues.

2. Suppose that \( f \) has a pole at \( z_0 \) of order \( \leq N \). 
The function \( (z - z_0)^N f(z) \) has a removable singularity 
at \( z_0 \). Prove that
\[
\text{Res}(f, z_0) = \frac{1}{\Gamma(N-1)} \left. \frac{d}{dz} \right|_{z=z_0} \left[ (z - z_0)^N f(z) \right]
\]

3. Use the principal determination of \( \log \), calculate the residues 
of \( \frac{\log z}{(z^2 + 1)^2} \) at each of its singularities.

4. For any nonnegative integer \( n \), calculate the residues of 
\( (z^2 + 1)^{-n-1} \) at each of its poles. Present your answer 
with the binomial coefficient \( \binom{2n}{n} \) displayed prominently.

5. Find the residues of \( \csc^n z \) at \( z = 0 \) for 
\( n = 1, 2, 3, 4, 5 \). 
(That is, \( \frac{1}{\sin^n z} \)).
REMARK ABOUT THE RESIDUE THEOREM: It contains the Cauchy integral formula. (Of course, we actually used the Cauchy integral theorem in its proof.) For consider the usual scene for the Cauchy formula:

\[ \oint_D \frac{f(z)}{z - \zeta} \, dz \]

We write the expected integral

\[ \frac{1}{2\pi i} \oint_D \frac{f(s)}{s - \zeta} \, ds \]

and notice that for fixed \( \zeta \in D \) the function \( f(s)/s - \zeta \) is a holomorphic function of \( s \) with one isolated singularity, \( \zeta \). And it's the easy case

\[ \text{Res} \left( \frac{f(s)}{s - \zeta}, \zeta \right) = \frac{f(\zeta)}{1} = f(\zeta) = \frac{d}{ds} \left( \frac{f(s)}{s - \zeta} \right) |_{s=\zeta}. \]

So we use the residue theorem \( \Rightarrow \)

\[ \frac{1}{2\pi i} \oint_D \frac{f(s)}{s - \zeta} \, ds = f(\zeta). \]

WHAT'S AHEAD FOR US: The residue theorem is an amazing tool for accomplishing all sorts of things in complex analysis. It can produce wonderful theoretical results and also astonishing computations. We could present these in either order. However, I prefer the computational aspects first because these techniques will give us lots of practice in dealing with our new concept of residues, and I think will also give us a nice change of pace in the middle of the semester.

So here we go!
EVALUATION OF SOME DEFINITE INTEGRALS

1. An example for babies

A quite elementary integral in basic calculus is \( \int_{-\infty}^{\infty} \frac{1}{x^2+1} \, dx = \pi \).

Now we approach it in an entirely different way, much more complicated than actually necessary for such a problem ... but the technique will lead the way for more interesting situations.

- Define the holomorphic function \( f(z) = \frac{1}{z^2+1} \). This function has isolated singularities at \( i \) and \( -i \).

- Devise a clever path. Here it is:

  for large \( R \) we have to approximate the desired integral, and we have a pole of \( f \) inside.

  The residue theorem gives immediately

  \[
  \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z^2+1} \, dz = \text{Res} \left( \frac{1}{z^2+1}, i \right) = \frac{1}{2i} \left|_{z=i} \right. = \frac{1}{2i}.
  \]

  Thus

  \[
  \int_{\gamma} \frac{dz}{z^2+1} = \pi.
  \]

- Let \( R \to \infty \). We have

  \[
  \int_{-R}^{R} \frac{dx}{x^2+1} + \int_{\text{semicircle}} \frac{dz}{z^2+1} = \pi.
  \]
The real integral is just what we want. We do not want to evaluate the semicircular integral but instead to show that it has limit 0 as $R \to \infty$. So we employ the basic estimate for path integrals:

$$\left| \int_{\text{semicircle}} \frac{dz}{z^2+1} \right| \leq \max \left| \frac{1}{z^2+1} \right| \cdot \text{length of curve}.$$  

(see Feb 6(2))

The length of the curve is $\pi R$. And for $|z|=R$ we have from the triangle inequality:

$$|z^2+1| \geq |z^2|-1 = |z|^2-1 = R^2-1.$$  

Thus we achieve the estimate

$$\frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty.$$  

- Final result:

$$\lim_{R \to \infty} \int_{-R}^{R} \frac{dx}{x^2+1} = \pi.$$  

2. A more challenging example

Use the above template for $f(z) = \frac{1}{z^4+1}$.

The denominator has a (simple) zero at the fourth root of $-1 = e^{\pi i}$:

$$e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}.$$  

And the residue of $f$ at each one equal:

$$\frac{1}{4z^3} = \frac{z}{4z^4} = -\frac{z}{4}.$$  

The residue theorem \( \Rightarrow \)

$$\frac{1}{2\pi i} \int \frac{dz}{z^4+1} = \text{sum of residues at } e^{\pi i/4} \text{ and } e^{3\pi i/4}$$

$$= -\frac{e^{\pi i/4}}{4} - \frac{e^{3\pi i/4}}{4}.$$
\[
= - \frac{\frac{1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}}}{4} = -\frac{\sqrt{2}}{4}.
\]

Thus,
\[
\int_{-\infty}^{\infty} \frac{dz}{z^4+1} = 2\pi i \left( -\frac{\sqrt{2}}{4} \right) = \frac{\pi}{\sqrt{2}}.
\]

Again, the integral on the semi-circle has modulus bounded by
\[
\frac{\pi R}{R^4-1} \to 0.
\]

Conclusion:
\[
\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.
\]

3. Another

Use \( f(z) = \frac{1}{z^4+1} \).

Six poles this time:
\( e^{i\pi/6}, i, e^{5i\pi/6} \), etc.

Everything works the same way.

The sum of the three residues is
\[
\sum_{n=1}^{3} \frac{1}{6z^5} = \sum_{n=1}^{3} \frac{z^{n-1}}{6z^6} = -\frac{1}{6} \sum_{n=1}^{3} z^{n-1} = -\frac{1}{6} \left[ e^{i\pi/6} + i + e^{5i\pi/6} \right] = -\frac{1}{6} \left[ \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} \right] = -\frac{1}{6} \cdot 2\sqrt{3} = -\sqrt{3}/3.
\]

So the integral we obtain is \( 2\pi i (-\sqrt{3}/3) = \frac{2\pi}{3} \):
\[
\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}.
\]
Another, let's try \( f(z) = \frac{1}{z^3 - i} \).

The three poles are roots of \( z^3 = i = e^{\frac{i\pi}{4}} = e^{\frac{5i\pi}{4}} = e^{\frac{9i\pi}{4}} \), so we obtain \( z = e^{\frac{i\pi}{4}}, e^{\frac{5i\pi}{4}}, e^{\frac{9i\pi}{4}} \).

This time we save a small amount of work by using the lower semicircle, so we deal with one residue only. The residue at \(-i\) equals \( \frac{1}{5i^2} = \frac{1}{3(-1)} = \frac{1}{3} \).

So the residue theorem gives \( \int_{\gamma} \frac{dz}{z^3 - i} = \frac{2\pi i}{3} \).

Again the integral on the semicircle tends to 0 as \( R \to \infty \), so our limiting equation is \( \int_{-\infty}^{\infty} \frac{dx}{x^3 + i} = \frac{2\pi i}{3} \).

Reverse the direction:

\[
\int_{-\infty}^{\infty} \frac{dx}{x^3 - i} = \frac{2\pi i}{3}.
\]

(P.S. We could have solved it this way:

\[
\int_{-\infty}^{\infty} \frac{dx}{x^3 - i} = \int_{-\infty}^{\infty} \frac{x^3 + i}{x^6 + 1} \cdot dx = 0 + i \int_{-\infty}^{\infty} \frac{dx}{x^6 + 1} = \frac{2\pi i}{3}
\]

from the preceding example.)
5. Long detailed discussion of another example: \[ \int_0^\infty \frac{x^{\alpha-1}}{1+x} \, dx, \quad x \in \mathbb{R}. \]

a. Convergence issues:

Near \( \infty \) the integrand is approximately \( x^{\alpha-2} \), so we require \( \alpha < 1 \).

Near \( 0 \) the integrand is approximately \( x^{-1} \), so we require \( \alpha > 0 \).

Then, \( 0 < \alpha < 1 \).

b. Choose a holomorphic function:

Let \( f(z) = \frac{z^{\alpha-1}}{1+z} \) but we realize we'll have to cope with the ambiguity in

\[ z^\alpha = \exp(\alpha \log z) = \exp(\alpha (\log |z| + \text{arg} z)) \]

\[ = |z|^\alpha \exp(i\alpha \text{arg} z). \]

c. Residues:

\( z=0 \) is a branch point, not an isolated singularity. There is no isolated singularity at \( z = -1 \), and the residue is easy:

\[ \text{Res} (f, -1) = \frac{(-1)^{\alpha-1}}{1} = -(-1)^\alpha. \]

\[ = -\exp(i\alpha \text{arg}(-1)). \]

d. Path of integration:

Rather tricky! We need to have the positive real axis as part of the path, we need to surround \(-1\) and we need \( 0 \) to be outside!

Here is what we do:
\[ y \text{ is the boundary of the shaded region.} \]

In this region, we should use the choice of \( \arg z \) so that \( 0 \leq \arg z < 2\pi \).

Then we have

\[ \text{Re}(f - 1) = -\exp(i\pi) = -e^{i\pi}. \]

E. Residue theorem yields immediately that

\[ \int_{\gamma} \frac{e^{-z}}{z^2 + 1} \, dz = -2\pi i \cdot e^{i\pi}. \]

f. \( R \to \infty \)

The integral on the circle is bounded by

\[ \max_{|z|=R} |f(z)| \cdot 2\pi R = \max_{|z|=R} \frac{R^{\alpha - 1}}{(1 + |z|)^{\alpha + 1}} \cdot 2\pi R \]

\[ \leq \frac{R^{\alpha - 1}}{R^2} \cdot 2\pi R = \frac{2\pi}{1 - \alpha} \approx 2\pi R^{\alpha - 1}. \]

Since \( \alpha < 1 \), this tends to 0 as \( R \to \infty \).

g. \( \varepsilon \to 0 \)

The integral on the circle is bounded by

\[ \max_{|z|=\varepsilon} \frac{e^{-z}}{|z|^2 + 1} \cdot 2\pi \varepsilon \]

\[ \leq \frac{\varepsilon^{\alpha + 1}}{1 - \varepsilon} \cdot 2\pi \varepsilon \approx 2\pi \varepsilon\alpha. \]
Since \( a > 0 \) this tends to 0 as \( \varepsilon \to 0 \).

REMARK: our criteria for convergence of the desired integral matches perfectly with what is needed in the path integral as \( R \to \infty, \varepsilon \to 0 \).

h. We have left two integrals along the positive real axis.

In the upper one \( z = x \) with argument 0, so it becomes

\[
\int_0^\infty \frac{x^{a-1}}{x+1} \, dx = \int_0^\infty \frac{x^{a-1}}{x+1} \, dx = \frac{\varepsilon}{\varepsilon + 1} \int_0^\infty \frac{x^{a-1}}{x+1} \, dx = \frac{\varepsilon}{\varepsilon + 1} I.
\]

But in the "lower" one \( z = x \) with argument \( \pi \), so it becomes

\[
-\int_0^\infty \frac{x^{a-1}}{x+1} \, dx = -\int_0^\infty \frac{e^{\pi i a} x^{a-1}}{x+1} \, dx = -e^{i\pi a} \frac{\varepsilon}{\varepsilon + 1} I.
\]

i. Summary: the equation in \( e \) becomes under limit

\[
I - e^{i\pi a} \frac{\varepsilon}{\varepsilon + 1} I = -2\pi i e^{i\pi a}.
\]

Solve for \( I = -2\pi i e^{i\pi a} \):

\[
I = \frac{-2\pi i e^{i\pi a}}{1 - e^{i\pi a}} = \frac{2\pi i}{e^{i\pi a} - e^{i\pi a}} = \frac{\pi \text{Im}}{\text{sh} \pi a}.
\]

CONCLUSION:

\[
\int_0^\infty \frac{x^{a-1}}{x+1} \, dx = \frac{\pi \text{Im}}{\text{sh} \pi a} \quad \text{for} \quad 0 < x < 1.
\]

This is just about the easiest example of this type of analysis, but I have gone to great detail to justify all the reasoning. After some practice this should become almost routine for you.
REMARK: #5 on HW (4F): to find the residues

\[ \text{Res}(\cos^n z, 0) \quad \text{for } n = 1, 2, \ldots \]

The easy cases are:

- \( n = 1 \): simple pole, residue = 1.
- \( n = 2, 4, 6, \ldots \): even function, residue = 0.

For \( n = 3 \) this is a beautiful and elegant technique, which I choose to call integration by parts. It's based on the fact that if \( f \) is a holomorphic function defined on a closed path \( \gamma \), then:

\[ \oint_{\gamma} f(z) \, dz = 0. \]

This is a FTC fact, see Fehr (1). Now apply this to a product of holomorphic functions:

\[ 0 = \oint_{\gamma} (fg)' \, dz = \oint_{\gamma} f'g \, dz + \oint_{\gamma} fg' \, dz, \]

or

\[ \oint_{\gamma} fg' \, dz = -\oint_{\gamma} f'g \, dz. \]

In the case of functions with isolated singularities at \( z = 0 \), when we use a small circle \( \Gamma \) surrounding \( z_0 \), we obtain

\[ \text{Res}(f'g, z_0) = -\text{Res}(fg', z_0). \]

Now we try this on \( \csc^3 z \) near \( z_0 = 0 \). That

\[ \sin^{-3} z = \left( \frac{\sin^2 z}{-2} \right) \sec z, \]

so
\[
\text{Feb 20(5)}
\]
\[
\text{\text{Re} } \left( \sin^{-2} z, 0 \right) = \text{Re} \left( \frac{\sin^{-2} z}{2} \sec z, 0 \right) \quad \text{IBP}
\]
\[
= \text{Re} \left( \frac{\sin^{-2} z}{2} \sec z + \tan z, 0 \right)
\]
\[
= \text{Re} \left( \frac{\sin^{-2} z}{2} \sec z, 0 \right).
\]

Look what just happened! We started with a pole of order 3, and now we have a pole of order 1! So we're in the easy case,
\[
\text{Re} \left( \frac{\sec^2 z}{2\sin^2 z}, 0 \right) = \frac{1}{2\cos 0} = \frac{1}{2}.
\]

REMARK: "Integration by parts" is somewhat of a misnomer. For Definition 3, given the result immediately that a derivative of a holomorphic function has no residue, since on page Feb 15(2) we simply take \( a = 0 \). Nevertheless, I like the IBP name for this principle, as the result is so reminiscent of such a procedure.
For any positive integer $n$, find all the singularities of the function $\frac{1}{z^n + 1}$ and calculate all the corresponding residues. Also, compute the sum of all the residues.

The singularities occur when $z^n = -1$. There are $n$ distinct roots, which can be calculated this way:

$$z^n = -1 = e^{\pi i} = e^{\frac{3\pi i}{n}} = \cdots = e^{\frac{(2n-1)\pi i}{n}}$$

so the roots are

$$e^{\frac{\pi i}{n}}, e^{\frac{3\pi i}{n}}, \ldots, e^{\frac{(2n-1)\pi i}{n}}$$

If $z = e^{\frac{(2k+1)\pi i}{n}}$, then

$$\text{Res} \left( \frac{1}{z^n + 1}, z \right) = \frac{1}{nz^{n-1}} = \frac{z}{nz^n} = -\frac{z}{n}$$

The sum of these residues is

$$-\frac{1}{n} \sum_{k=0}^{n-1} e^{\frac{2k\pi i}{n}}$$

which, if $n \geq 2$, equals

$$-\frac{1}{n} e^{\frac{\pi i}{n}} \frac{e^{\frac{2\pi i}{n} n} - 1}{e^{\frac{2\pi i}{n}} - 1} = 0.$$
2. Suppose that $f$ has a pole at $z_0$ of order $\leq N$. The function $(z-z_0)^{-N}f(z)$ has a removable singularity at $z_0$. Prove that

$$\text{Res}(f, z_0) = \left. \frac{d^{N-1}}{dz^{N-1}} \left[ (z-z_0)^{-N}f(z) \right] \right|_{z=z_0}$$

Sufficient to do this for $z_0$. Then the Laurent expansion of $f$ is

$$f(z) = \sum_{n=-N}^{\infty} C_n (z-z_0)^n$$

and

$$z^{-N}f(z) = \sum_{n=-N}^{\infty} C_n z^{n-N} = C_N + C_{N+1} z^{-1} + \cdots + C_{-1} z^{-N+1} + C_0 z^{-N} + \cdots.$$

Then

$$\left. \frac{d^{N-1}}{dz^{N-1}} \right|_{z=z_0} z^{-N}f(z) = C_{-1} (N-1)! + \text{terms as } z^{-k} \text{ for } k \geq N.$$

Evaluate at $0$ to get

$$C_{-1} (N-1)! = (N-1)! \text{Res}(f, 0).$$

QED

3. Using the principal determination of $\log z$, calculate the residues of $\frac{\log z}{(z^3+1)^2}$ at each of its singularities.

Use the above result. As $i$ and $-i$ are poles of order $2$,

$$\text{Residue at } \pm i = \frac{d}{dz} \left[ \frac{(z \mp i)^2 \log z}{(z^3+1)^2} \right]_{\pm i}$$

$$= \frac{d}{dz} \left[ \frac{\log z}{(z \mp i)^3} \right]_{\pm i}$$

$$= \left. \frac{1}{(z \mp i)^2} - \frac{2 \log z}{(z \pm i)^3} \right|_{\pm i}$$

(by product rule)

$$= \frac{1}{(2i)^2} - \frac{2 \log(2i)}{(2 \pm i)^3}$$

$$= \frac{1}{4i^2} - \frac{2 \log(2i)}{(2 \pm i)^3}$$

$$= \frac{i}{4} - \frac{2 \log(2i)}{8(\pm i^3)}$$

$$= \frac{i}{4} + \frac{\pi i}{8(\pm i^3)} = \frac{\pi i}{8} \pm \frac{i}{4}.$$
For any nonnegative integer $n$, calculate the residue of $(z^2 + 1)^{-n-1}$ at each of its poles. Present your answer with the binomial coefficient $\binom{2n}{n}$ displayed prominently.

Very similar result. Pole of order $n+1$ at $i$, so we form

$$\left(\frac{z-i}{z+i}\right)^n = \binom{-1}{n} \frac{1}{(z+i)^{n+1}}$$

Calculate the $n+1$ derivative:

$$(-n)(-n-1)\cdots(-2n)(z+i)^{-2n-1}$$

$$= (-1)^n (2n)(2n+1)\cdots(n+1)(z+i)^{-2n-1}.$$ 

Evaluate at $i$:

$$(-1)^n (2n)\cdots(n+1) 2^{-2n+1} i^{-2n+1}$$

$$= (2n)\cdots(n+1) 2^{-2n+1} i^{-1}.$$ 

Divide by $n!$:

$$\binom{2n}{n} 2^{-2n+1} i^{-1}.$$ 

Residue at $i = \frac{\binom{2n}{n} i}{2^{2n+1}}$.

Likewise,

Residue at $-i = \frac{\binom{2n}{n} i}{2^{2n+1}}$.

There are several approaches to the next problem, and I want you to see one you might not think of. I think it's the best approach in this particular case. It's based on Definition 3 on page Feb 15(2). Here it is:

If $f$ has an isolated singularity at $z_0$, then

$$\text{Res}(f', z_0) = 0.$$
As a corollary, if \( f \) and \( g \) have isolated singularities at \( z_0 \), then the residue of \((fg)'\) at \( z_0 \) is 0; since this is \( f'g + fg' \), we obtain a sort of "integration by parts" result:

\[
\text{Res} (f'g, z_0) = -\text{Res} (fg', z_0).
\]

5. Find the residues of \( \csc^n z \) at \( z = 0 \) for

\[
\text{Res} (\csc^n z, 0) = \frac{1}{2\cos^2 0} = \frac{1}{2}.
\]

- \( \csc^3 z = \frac{1}{\sin^3 z} = \left(\frac{-\sin^{-2} z}{2}\right)' \sec z \) and this has the same residue at 0 as \( \frac{-\sin^{-2} z (\sec z)'}{2} = \frac{-\sin^{-2} z \sec z \tan z}{2} = \frac{-\sin^{-1} z \sec^2 z}{2} \), and this has a simple pole at 0. Thus the residue equals

\[
\text{Res} \left( \frac{\sec^2 z}{2\sin^2 z}, 0 \right) = \frac{1}{2\cos 0} = \frac{1}{2}.
\]

- \( \csc^5 z = \frac{1}{\sin^5 z} = \left(\frac{-\sin^{-4} z}{4}\right)' \sec z \) has same residue as

\[
\frac{-\sin^{-4} z (\sec z)'}{4} = \frac{-\sin^{-4} z \sec z \tan z}{4} = \left(\frac{-\sin^{-3} z}{8}\right)' \sec^3 z
\]

same residue as

\[
\frac{\sin^{-2} z (\sec^3 z)}{8} = \frac{2\sin^{-2} z \sec^3 z \tan z}{8} = \frac{3}{8} \sin^{-2} z \sec^4 z \ldots \text{simple pole}
\]

The desired residue is \( \frac{3/8}{1} = \frac{3}{8} \).

7. Given residue \( z \), all even \( n \) give residue 0.
6. Integrals of a certain form:

\[ \int_{0}^{\infty} \frac{a(x)}{b(x)} \, dx \text{ where:} \]

- \(a, b\) are polynomials
- degree \(b \geq\) degree \(a + 2\)
- \(b \neq 0\) on \([0, \infty) \subset \mathbb{R}\)

Such situations can always be handled with residue theory, so we'll derive a general result and then apply it to a particular example.

The approach is quite clever! We define:

\[ f(z) = \frac{a(z)}{b(z)} \log z \]

and use this type of path:

![Diagram of a path with a small circle inside a large circle with radius \(\varepsilon\) and \(R\).]

Just as in the preceding lecture, we use

\[ 0 < \arg z < 2\pi. \]

The integrals on the two circles tend to 0 in the limit as \(\varepsilon \to 0\) and \(R \to \infty\).

The extra factor of \(\log z\) is of very little concern, since

\[ |\log z| = |\log |z| + i\arg z| \leq |\log |z| + 2\pi i|, \]

so when \(|z| = R\) we have

\[ |\log z| \leq |\log + 2\pi i| \leq 2 \log R \]

and for \(|z| = \varepsilon\)

\[ |\log z| \leq |\log + 2\pi i| \leq 2 \log \frac{1}{\varepsilon} \]

as \(R \to \infty\) and \(\varepsilon \to 0\). Thus on the circle \(|z| = R\) we have
\[ \int_{|z|=R} f(z)dz \leq \frac{\text{constant}}{R^2} \cdot \log R \cdot 2\pi R \leq \frac{\text{constant} \cdot \log R}{R} \]

which tends to 0 as \( R \to \infty \) ... for \( \log R \to \infty \) much slower than \( R \).

And on the circle \( |z|=\epsilon \) we have a similar estimate
\[ \int_{|z|=\epsilon} f(z)dz \leq \text{constant} \cdot \log \frac{1}{\epsilon} \cdot 2\pi \epsilon \leq \text{constant} \cdot \epsilon \log \frac{1}{\epsilon}. \]

Again this tends to 0 as \( \epsilon \to 0 \).

So we apply the residue theorem and then let \( R \to \infty, \epsilon \to 0 \), and obtain in the limit
\[ \int_0^\infty \frac{\cos \theta}{\cos \phi} \log x \, dx = \int_0^\infty \frac{\cos \theta}{\cos \phi} \left[ \log x + 2\pi i \right] \, dx \]

\[ = 2\pi i \times \text{sum of the residues of } f(z). \]

Notice that on the path \((0,\infty)\) above the path \( \log z = \log x \ (\arg z = 0) \),
but on the path \((\infty,0)\) below the axis, \( \log z = \log |z| + 2\pi i \),
because \( \arg z = 2\pi \).

So when we subtract the integrals, \( \log x \) disappears, and we're left with
\[ -2\pi i \int_0^\infty \frac{\cos \theta}{\cos \phi} \, dx. \]

Divide by \(-2\pi i\) to achieve the formula
\[ \int_0^\infty \frac{\cos \theta}{\cos \phi} \, dx = -\text{the sum of all residues of } \frac{a(z)}{b(z)} \frac{1}{\log z}. \]

(Here \( 0 < \arg z < 2\pi \).

(Though it doesn't really.)
EXAMPLE: Let \( a > 0 \) and consider

\[
I = \int_{0}^{\infty} \frac{x}{(x+a)(x^2+1)} \, dx.
\]

All our requirements are met. The poles are located at \(-a\), \(i\), and \(-i\), and they are all simple! When the residues are computed we obtain

\[
\text{Res} \left( \frac{e^{-i \theta} \log z}{(z+a)(z^2+1)}, \, -a \right) = \frac{-a \log(a)}{1 \cdot (a^2+1)}
\]

\[
= \frac{-a \log(a) + i \pi}{a^2 + 1},
\]

at \( i \) we have the residue

\[
\frac{\log i}{(2i)(2i)} = \frac{\log 2}{4i} = \frac{\log (a - i)}{4(a^2 + 1)}.
\]

And at \(-i\)

\[
\frac{-i \log(-i)}{(-2i)(2i)} = \frac{3\pi i}{4(a - i)} = \frac{3\pi (a + i)}{4(a^2 + 1)}.
\]

\[
\frac{1}{a^2 + 1} \left\{ -a \log a - \frac{3\pi i}{4} \right\} + \frac{i \pi a + \pi}{4} + \frac{3\pi a - 3\pi}{4}
\]

\[
= \frac{1}{a^2 + 1} \left\{ -a \log a - \frac{\pi}{2} \right\}.
\]

Therefore we obtain from the formula the result that for \( a > 0 \)

\[
\int_{0}^{\infty} \frac{x}{(x+a)(x^2+1)} \, dx = \frac{a \log a + \frac{\pi}{2}}{a^2 + 1}.
\]

REMARK: Many examples we demonstrate can actually be done with

single contour calculus. This is a good example, as are 1, 2, 3, 4.

We could even find the definite integral first. But even so, these

techniques are exceedingly beautiful both in such cases!
HW 15W due Mar 6, 2013

For any integer \( n \geq 2 \), compute the integral

\[
\int_0^\infty \frac{dx}{x^n + x^{n-1} + \ldots + x + 1}.
\]

**DISCUSSION:** This fits what we have just done, with \( a(z) = 1 \) and \( b(z) = z^n + z^{n-1} + \ldots + z + 1 \). Here's an approach using lots of calculations:

- **Poles:**
  \[
b(z) = \frac{z^{n+1}}{z-1} \quad \text{so poles occur when } z = z_k = (\cos \frac{2\pi k}{n+1}) + i \sin \frac{2\pi k}{n+1} \quad (k = 1, \ldots, n).
\]

  I suggest defining

  \[
  \omega = e^{i \frac{2\pi}{n+1}}
  \]

  so that the poles are \( a_k \omega^k \) for \( 1 \leq k \leq n \).

- **Residues of** \( \frac{\log z}{b(z)} \) **are**

  \[
  \frac{\log a_k}{b'(a_k)} = \frac{k \log \omega}{b'(\omega^k)} = \frac{2\pi i}{n+1} \frac{k}{b'(\omega^k)}.
  \]

  For \( z = \omega^k \) we have

  \[
  b'(z) = \frac{-z^{n+1} - 1}{(z-1)^2} = \frac{z^{n+1} - 1}{z-1} = \frac{a^{n+1}}{z^{n+1}} \quad (WHY?)
  \]

  - **Formula of page 2** \( \Rightarrow \) our integral equals

    \[
    \frac{1}{n+1} \sum_{k=1}^{n} 2\pi i k a_k \omega^k = \frac{2\pi i}{n+1} \left( \sum_{k=1}^{n} k \omega^k - \sum_{k=1}^{n} k \omega^{2k} \right).
    \]

  - **Algebra:** **SHOW THAT** the term \( \ldots \) on the preceding line equals

    \[
    \frac{1}{a_n} \frac{a_0}{\omega^2} = \frac{a_n}{(\omega^2 - \omega)} \frac{\omega}{\omega^2 - \omega - 2}\frac{\omega^2}{\omega^2 - \omega^2}.
    \]

  - **Finish:** Greatly simplify to get the final answer in the form

    \[
    \frac{\pi}{(n+1) \sin \frac{2\pi}{n+1}}.
    \]
7. Principal value integrals refer to reasonable attempts to define a sort of integral even when the integrand is not actually integrable. There are two different situations where this may occur. Here are illustrations of these types:

**Type 1** \[\int_{-1}^{1} \frac{1}{x} \, dx\] doesn't actually exist, since the two sub-integrals do not exist:

\[\int_{-\infty}^{0} \frac{1}{x} \, dx = -\infty \quad \text{and} \quad \int_{0}^{\infty} \frac{1}{x} \, dx = +\infty.\]

So what we may do is first delete a symmetric interval about 0 and then perform a limit:

\[\int_{-\varepsilon}^{-1} \frac{1}{x} \, dx + \int_{1}^{\varepsilon} \frac{1}{x} \, dx = \log |x| \bigg|_{-\varepsilon}^{-1} + \log |x| \bigg|_{1}^{\varepsilon}\]

\[= (\log \varepsilon - 0) + (1 - \log \varepsilon)\]

\[= 1.\]

So the limit as \(\varepsilon \to 0\) does exist, and it called the principal value integral. Notation:

\[\text{PV} \int_{-1}^{1} \frac{1}{x} \, dx = 1.\]

**Type 2** \[\int_{-\infty}^{\infty} \frac{x}{x^2 + 1} \, dx\] again does not exist, for

\[\int_{-\infty}^{0} \frac{x}{x^2 + 1} \, dx = -\infty \quad \text{and} \quad \int_{0}^{\infty} \frac{x}{x^2 + 1} \, dx = +\infty.\]

But we can define a principal value by integrating from \(-R+R\) and then letting \(R \to \infty\). Notation:

\[\text{PV} \int_{-\infty}^{\infty} \frac{x}{x^2 + 1} \, dx = 0.\]
The residue theorem can often be of use in calculating such integrals. I'll give as an example a typical situation, and we'll see why.

So assume \( f(z) \) is a rational function of \( z \) for which

\[
\text{degree of denominator} > 2 + \text{degree of numerator},
\]

just as on page 292.1.

Furthermore, if \( f \) has any real poles, assume they are simple.

We shall then apply the residue theorem when our path integral has this form:

We are of course familiar with the large semi-circle. The new twist is that we have semi-circles of small radius centered at the real (simple) poles of \( f \).

\( R \) is so large that all the real poles of \( f \) are between \(-R\) and \( R \), and all the poles with imaginary part greater than 0 satisfy \( |z| > R \).

The residue theorem then gives immediately the equation.
\[ \int_{\gamma} f(z) \, dz = 2\pi i (Q) \]

where we have defined
\[ Q = \sum_{\text{Im}(z) > 0} \text{Res} (f, z). \]

Because of the restriction on degrees of denominator and numerator of \( f \), we can let \( R \to \infty \) and obtain
\[ \int_{\gamma_{\varepsilon}} f(z) \, dz = 2\pi i R. \]

where \( \gamma_{\varepsilon} \) represents the real axis with semicircular arcs of radius \( \varepsilon \) situated about the real poles of \( f \).

It is fascinating to see what happens when we let \( \varepsilon \to 0 \). We can deal with each real pole individually. So look at a pole at \( x_0 \in \mathbb{R} \) and the portion of \( \gamma_{\varepsilon} \) from \( a \) to \( b \), where \( a < x_0 < b \):

\[ a \rightarrow \circ \rightarrow b \]

Parametrize the semicircle as \( z = x_0 + \varepsilon e^{i\Theta} \), where \( \Theta \) travels from \( \pi \) to 0. We obtain
\[ \int_{a}^{x_0 - \varepsilon} f(x) \, dx + \int_{0}^{\pi} f(x_0 + \varepsilon e^{i\Theta}) \varepsilon e^{i\Theta} \, d\Theta + \int_{x_0 + \varepsilon}^{b} f(x) \, dx. \]

The first and third of these integrals will have the limit as \( \varepsilon \to 0 \):
\[ \text{PV} \int_{a}^{b} f(x) \, dx \quad \text{provided the limit exists!} \]

This limit does indeed exist, as we see from analyzing the second integral.
From page Feb 15(4) we have

$$\text{Res}(f, x_0) = \lim_{z \to x_0} (z - x_0) f(z)$$

since $x_0$ is a simple pole. Thus

$$\text{Res}(f, x_0) = \lim_{\epsilon \to 0} \epsilon e^{i \theta} f(x_0 + \epsilon e^{i \theta}) \quad (\text{uniformly with } \theta)$$

and we therefore obtain

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \int_0^\pi f(x_0 + \epsilon e^{i \theta}) \epsilon e^{i \theta} d\theta = \int_0^\pi i \text{Res}(f, x_0) d\theta$$

$$= -i \pi \text{ Res } f(x_0).$$

Doing this for each pole then yields the formula

$$\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{C_\epsilon} f(z) dz = \text{PV} \int_{-\infty}^{\infty} f(x) dx - i \pi \bigcap R^*,$$

where

$$\bigcap R^* = \sum_{x \in R} \text{Res}(f, x).$$

Thus we have derived the result that under the given restriction on $f$

$$\text{PV} \int_{-\infty}^{\infty} f(x) dx = 2\pi i \sum_{\text{Im}(z) > 0} \text{Res}(f, z) + i \pi \sum_{x \in R} \text{Res}(f, x).$$

\[\text{Example: } f(z) = \frac{1}{z^2 + 1}. \text{ The real pole is at } -1, \text{ and the real pole is at } -\frac{i}{\sqrt{3}}. \text{ The other pole to consider is } e^{i\pi/3}. \text{ with residue } \frac{1}{3z^2} = \frac{1}{3} e^{-2i\pi/3} = \frac{1}{3} \left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right). \text{ Thus we have to calculate } 2\pi i \left(-\frac{1}{6} - \frac{\sqrt{3}i}{2}\right) + \frac{\pi i}{3} = \frac{\pi i}{\sqrt{3}} \text{ so we have }\]
\[ \text{PV} \int_{-\infty}^{\infty} \frac{dx}{x^3+1} = \frac{\pi}{\sqrt{3}}. \]

HW 16F due Mar 8, 2013

For \(0 < \alpha < 1\) calculate
\[ R \int_{0}^{\infty} \frac{x^{\alpha-1}}{1-x} \, dx. \]

**RECOMMENDATION:**

\[ f(z) = \frac{z^{\alpha-1}}{1-z} \]

with
\[ 0 < \arg z < 2\pi. \]
From page Feb 22(4) we have

- **Poles:** \( \omega = e^{\frac{2\pi i}{n+1}} \); poles \( \omega^k \) for \( 1 \leq k \leq n \).

- **Residues:** give \( b'(z) = - \frac{z^{n+1} - 1}{(z-1)^2} + \frac{(n+1)z^n}{z-1} = \frac{(n+1)z^n}{z^2 - z} \).

  \( \Rightarrow 0 \) because \( z^{n+1} = 1 \).

- **Formula of page Feb 22(2):** gives the integral equal to

  \[ - \sum_{k=1}^{n} \text{Res} \left( \frac{\log z}{k(z)}, \omega^k \right) = - \sum_{k=1}^{n} \frac{2\pi i k}{(n+1) b'(\omega^k)} \]

  \[ = - \sum_{k=1}^{n} \frac{2\pi i k}{n+1} \frac{\omega^k - \omega^k}{\omega^k - \omega^{-k}} \]

  \[ = \frac{2\pi i}{(n+1)^2} \left( \sum_{k=1}^{n} k \omega^k - \sum_{k=1}^{n} k \omega^{-k} \right). \]

But \( b(z) = \sum_{k=0}^{n} z^k \Rightarrow zb'(z) = \sum_{k=1}^{n} k z^k. \)

Thus our integral equals

\[ \frac{2\pi i}{(n+1)^2} \left( \omega b'(\omega) - \omega^2 b'(\omega^2) \right) \]

**Finals:** This is

\[ \frac{2\pi i}{n+1} \left( \frac{1}{\omega - 1} - \frac{1}{\omega^2 - 1} \right) = \frac{2\pi i}{n+1} \frac{(\omega+1)-1}{\omega^2 - 1} = \frac{2\pi i}{n+1} \frac{\omega}{\omega^2 - 1} \]

\[ = \frac{2\pi i}{n+1} \frac{1}{\omega - \frac{1}{\omega}} = \frac{2\pi i}{n+1} \frac{1}{e^{\frac{2\pi i}{n+1}} - e^{-\frac{2\pi i}{n+1}}} \]

\[ = \frac{\pi}{n+1} \frac{1}{\sin \frac{2\pi}{n+1}}. \]

\[ \int_{0}^{\infty} \frac{dx}{x^n + x^{n-1} + \ldots + x + 1} = \frac{\pi}{(n+1) \sin \frac{2\pi}{n+1}}. \]
8. Integrals involving exponentials (and trig functions).

This is an extremely important type of definite integrals, and we'll gain extra proficiency in our techniques as well as in our use of Exp. Today we'll look at two rather typical examples.

Example 1: \( \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x^2 + 1} \, dx \).

First, recall a simple equation \( |e^w| = e^{\text{Re} w} \). Therefore, \( |e^{i\alpha x}| = e^{-\text{Im}(\alpha x)} = e^{-\alpha \text{Im}(x)} \). So if \( \text{Im}(\alpha) \neq 0 \), the integrand blows up exponentially as \( x \to \infty \) or \( x \to -\infty \), and we have no existing integral. Therefore, we definitely must assume that \( \alpha \in \mathbb{R} \).

We are led to define \( f(z) = \frac{e^{i\alpha z}}{z^2 + 1} \), a holomorphic function with isolated simple poles at \( \pm i \).

Let's try our familiar semicircle:

Knowing we'll need to worry about the path \( |z| = R \), we examine

\[
|f(z)| = \frac{|e^{i\alpha z}|}{|z^2 + 1|} = \frac{e^{-\alpha \text{Im}(z)}}{|z^2 + 1|} < \frac{e^{-\alpha \text{Im}(\Re z)}}{|\Re z^2 - 1|} = \frac{e^{-\alpha \text{Im}(z)}}{R^2 - 1}.
\]

We'll be in bad trouble if \( \alpha < 0 \) ! (Since \( \text{Im}(z) > 0 \).) Therefore, we also assume \( \alpha > 0 \). Then we have for \( |z| = R \):

\[
|f(z)| \leq \frac{1}{R^2 - 1}.
\]
and we conclude that since the length of the semicircle is $\pi R$,
the path integral
\[ \left| \oint_{|z|=R, \text{Im}(z)>0} f(z) \, dz \right| \leq \frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty. \]

Thus, the residue theorem implies (after letting $R \to \infty$) that
\[ \int_{-\infty}^{\infty} f(x) \, dx = 2\pi i \text{ Res}(f(z), i) = 2\pi i \frac{e^{ia}}{2i} = \pi e^{-a}. \]

Thus, we obtain
\[ \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} \, dx = \pi e^{-a} \quad \text{for } a > 0. \]

Finally, if $a < 0$ we obtain immediately by conjugation the result $\pi e^{a}$.

Therefore, we have in general
\[ \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} \, dx = \pi e^{-|a|} \quad \text{for all } a \in \mathbb{R}. \]

What a terrific result! If $a = 0$, this is a very elementary integral since certain has derivative $\frac{1}{x+1}$. But for $a \neq 0$ there's no convenient indefinite integral.

**Example 2**
\[ \int_{-\infty}^{\infty} \frac{x e^{iax}}{x^2+1} \, dx. \]

Again, we must assume that $a \in \mathbb{R}$. We'll also first deal with the case $a > 0$. (The case $a = 0$ is quite different, as
\[ \int_{-\infty}^{\infty} \frac{x}{x^2+1} \, dx \]
exists only in the principal value sense --- and clearly is $0$.)
Thus we define
\[ f(z) = \frac{ze^iz}{z^2+1}. \]

The residue at \( z = i \) is
\[ \frac{i e^{i(1)}}{2i} = \frac{1}{2} e^{-a}. \]

We employ the same semicircular path, and we first try to estimate the path integral along \( |z| = R \):
\[
\left| \int_{|z|=R} f(z) \, dz \right| \leq \int_{|z|=R} \frac{1}{|z|^2} e^{-a|z|} \, |dz|.
\]

\[
= \frac{R}{R^2-1} \int_0^{\pi} e^{-aR \sin \theta} \, R \, d\theta.
\]

\[
= \frac{R^2}{R^2-1} \int_0^{\pi} e^{-aR \sin \theta} \, d\theta.
\]

Ugh! We can no longer simply use the estimate \( e^{-aR \sin \theta} \leq 1 \), so we have to be cleverer. Not knowing how to integrate \( e^{-aR \sin \theta} \), we employ a useful estimate. First, we can integrate from 0 to \( \pi/2 \) only and double the answer to get the estimate
\[
\frac{2R^2}{R^2-1} \int_0^{\pi/2} e^{-aR \sin \theta} \, d\theta < \frac{2R^2}{R^2-1} \int_0^{\pi/2} e^{-aR \sin \theta} \, d\theta.
\]

For large \( R \). Then we estimate \( \sin \theta \) from below by observing its graph:
\[
\sin \theta \geq \frac{2\theta}{\pi} \text{ for } 0 \leq \theta \leq \pi/2.
\]

Therefore we find
\[
\int_0^{\pi/2} e^{-aR \sin \theta} \, d\theta < \int_0^{\pi/2} e^{-aR \frac{2\theta}{\pi}} \, d\theta
\]
\[
= \int_0^{\pi/2} e^{-aR \frac{2\theta}{\pi}} \, d\theta.
\]

\[
= \int_0^{\pi} e^{-aR \frac{\theta}{\pi}} \, d\theta = \frac{\pi}{2aR}.
\]
Conclusion: the path integral of \( f(z) \) along the semi-circle has modulus no bigger than \( \frac{R^2}{R^2-1} \frac{3\pi}{2aR} \to 0 \) as \( R \to \infty \).

Thus, we again obtain from the residue theorem

\[
\int_{-\infty}^{\infty} f(x)dx = 2\pi i \text{Res}(f(z), \epsilon) = 2\pi i \frac{\epsilon^3}{2} = \pi \epsilon e^{-\epsilon}.
\]

Here is the result for all \( \epsilon \in \mathbb{R} \):

\[
\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+1} \, dx = \begin{cases} 
\pi e^{-\epsilon} & \text{for } \epsilon > 0, \\
0 & \text{for } \epsilon = 0 \text{ (PV integral)}, \\
-\pi ie^\epsilon & \text{for } \epsilon < 0.
\end{cases}
\]

**SUMMARY:**

Using Euler's formula \( e^{ix} = \cos ax + i\sin ax \), we see that the symmetry of the integrand gives the two results in this form:

\[
\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} \, dx = \pi e^{-\epsilon} \quad \text{for } \epsilon \in \mathbb{R};
\]

\[
\int_{-\infty}^{\infty} \frac{x \sin ax}{x^2+1} \, dx = \begin{cases} 
\pi e^{-\epsilon} & \text{for } \epsilon > 0, \\
0 & \text{for } \epsilon = 0, \\
-\pi e^\epsilon & \text{for } \epsilon < 0.
\end{cases}
\]

**REMARK:** The integrals \( \int_{-\infty}^{\infty} \frac{x \sin ax}{x^2+1} \, dx \) are not principal value integrals, as the integrand is an even function of \( x \). However, they are improper integrals as they are not absolutely integrable:

\[
\int_{-\infty}^{\infty} \left| \frac{x \sin ax}{x^2+1} \right| \, dx = \infty \quad \text{(for } \epsilon > 0).\]
HW IGF SOLUTION

\[ f(z) = \frac{z^{\alpha-1}}{1-z} \quad \text{with} \quad 0 < \arg z < 2\pi. \]

1. The path integral around a small circle centered at 0 tends to 0 as the radius tends to 0 because \( \alpha > 0 \).

2. The path integral around a large circle centered at 0 tends to 0 as the radius tends to \( \infty \) because \( \alpha < 1 \).

3. The remaining figure is this:

\[ \text{Diagram of a path integral} \]

4. As \( f \) has no poles inside the region, the residue theorem yields the remaining path integral = 0. Along the top part of the path we have

\[ \int_{0}^{1+\epsilon} \frac{x^{\alpha-1}}{1-x} \, dx + \int_{1+\epsilon}^{\infty} \frac{x^{\alpha-1}}{1-x} \, dx + \int \frac{x^{\alpha-1}}{1-z} \, dz \quad (0 \leq \epsilon < 1) \]

and along the bottom part

\[ \int_{\infty}^{1+\epsilon} \frac{x^{\alpha-1} e^{2\pi i (k-1)}}{1-x} \, dx + \int_{1+\epsilon}^{0} \frac{x^{\alpha-1} e^{2\pi i (k-1)}}{1-x} \, dx + \int \frac{x^{\alpha-1}}{1-z} \, dz \quad (0 < \epsilon < \infty). \]

5. Top part as \( \epsilon \to 0 \) has limit

\[ \text{PV} \int_{0}^{\infty} \frac{x^{\alpha-1}}{1-x} \, dx = -\pi \text{ Re}(\frac{z^{\alpha-1}}{1-z}) \]

and the bottom part has limit

\[ \text{PV} \int_{0}^{\infty} \frac{x^{\alpha-1} e^{2\pi i k}}{1-x} \, dx = -\pi i \text{ Re}(\frac{z^{\alpha-1} e^{-2\pi i}}{1-z}) \).

6. Thus our situation is now that their sum is 0:

\[ \text{PV} \int_{0}^{\infty} \frac{x^{\alpha-1}}{1-x} \, dx \left[ 1 - e^{2\pi i k} \right] = \pi i \text{ Re}(\frac{z^{\alpha-1}}{1-z}) + \pi i \text{ Re}(\frac{z^{\alpha-1}}{1-z}) \]

\[ \text{top} \quad \text{bottom} \]

\[ = 0 \]
7. These residues are different since at the point 1 on the top:

\[ x - 1 = 1 \Rightarrow \]

\[ x = e^{2\pi i (x-1)} = e^{2\pi i}. \]

Whereas "on the bottom:

\[ x - 1 = 1 \Rightarrow \]

\[ x = e^{2\pi i (x-1)} = e^{2\pi i}. \]

Therefore the sum on the right side is

\[ \pi i \left[ -1 - e^{2\pi i} \right]. \]

8. Now solve for the desired integral:

\[ \text{PV} \int_0^\infty \frac{x-1}{x-1} \, dx = \pi i \frac{e^{2\pi i}}{e^{2\pi i}} \]

\[ = \pi i \frac{e^{\pi i} + e^{-\pi i}}{e^{\pi i} - e^{-\pi i}} \]

\[ = \pi i \frac{2 \cos \pi i}{2i \sin \pi i}. \]

9. Final result:

\[ \text{PV} \int_0^\infty \frac{x-1}{x-1} \, dx = \pi i \cot \pi i. \]
MATH 382

EXAM 2  Due March 11, 2013

1. Suppose that \( f \) and \( g \) are entire holomorphic functions such that
\[
|f(z)| \leq |g(z)| \quad \text{for all } z \in \mathbb{C}.
\]
Prove that there exists a constant \( c \) such that
\[
f(z) = c g(z) \quad \text{for all } z \in \mathbb{C}.
\]

2. The tangent function is of course defined by
\[
\tan z = \frac{\sin z}{\cos z}.
\]

   a. Denote the Maclaurin series of \( \tan z \) by
   \[
   \tan z = \sum_{n=0}^{\infty} c_n z^n.
   \]

   What is the radius of convergence of this series?

   b. Prove that \( c_n = 0 \) for all even \( n \).

   c. Prove that \( c_n > 0 \) for all odd \( n \).

3. Let \( a(z) \) and \( b(z) \) be polynomials such that
   \[
   \text{degree of } b \geq 2 + \text{degree of } a.
   \]
   Find the sum of all the residues of the function \( \frac{a(z)}{b(z)} \).

4. For any complex number \( a \) calculate
\[
\int_{-\infty}^{\infty} \frac{1}{(x-a)(x-i)} \, dx
\]
(if \( a \in \mathbb{R} \) this is a principal value integral).

HONOR PLEDGE
Two more examples involving trigonometric functions

- The first can be found in almost every textbook on complex analysis. It's

  \[ \int_0^\infty \frac{\sin x}{x} \, dx \quad \text{where as usual } a \in \mathbb{R}. \]

  This integral is of course 0 if \( a = 0 \). Otherwise, it is an improper
  integral, since
  \[ \int_0^\infty \left| \frac{\sin x}{x} \right| \, dx = \infty. \quad \text{(Not hard to show.)} \]

  Thus, it has to be interpreted as
  \[ \lim_{R \to \infty} \int_0^R \frac{\sin ax}{x} \, dx. \]

The choice of holomorphic function is crucial! We must not choose \( \frac{\sin az}{z} \)
because of its large modulus when \( \text{Im} z = 0 \). Therefore, the logical
choice is \( f(z) = \frac{e^{i a z}}{z} \).

(Thi actually introduces a pole at 0.) Prior experience leads us to
assume at first that \( a > 0 \) and to choose a path like this:

\[ \int_{-\infty}^{-\epsilon} f(x) \, dx + \int_{-\epsilon}^{\epsilon} f(x) \, dx + \int_{\epsilon}^\infty f(x) \, dx = 0. \]

Our earlier estimates on
page Mar. 6 (3) show
that the integral along
\( l \left( = R \text{ tends to } 0 \right) \)
since \( a > 0 \). Then, the
residue theorem gives

\[ \int_{-\infty}^{-\epsilon} f(x) \, dx + \int_{-\epsilon}^{\epsilon} f(x) \, dx + \int_{\epsilon}^\infty f(x) \, dx = 0. \]

Thus our work on March 4 yields in the limit as \( \epsilon \to 0 \)
\[ \text{Mar-8(2)} \]
\[
\text{PV} \int_{-\infty}^{\infty} f(x)dx - \pi \text{Res}(f,0) = 0.
\]

The residue is 1, so our result is
\[
\text{PV} \int_{-\infty}^{\infty} \frac{e^{iax}}{x} \, dx = \pi i \text{ for } a > 0.
\]

I.e.,
\[
\text{PV} \int_{-\infty}^{\infty} \frac{\sin ax + i \sinh ax}{x} \, dx = \pi i.
\]

All that survives from this equation is
\[
\int_{-\infty}^{\infty} \frac{\sin ax}{x} \, dx = \pi \text{ for } a > 0.
\]

Note: no PV is left, so that goes only with the cosine term. Or we could have taken the imaginary part of each side.

Since the integrand is even as a function of \( x \), we obtain
\[
\int_{-\infty}^{\infty} \frac{\sin ax}{x} \, dx = \frac{\pi}{2} \text{ for } a > 0.
\]

Finally, since \( \sin ax \) is an odd function of \( a \), our final result is
\[
\int_{-\infty}^{\infty} \frac{\sin ax}{x} \, dx = \begin{cases} 
\frac{\pi}{2} & \text{if } a > 0 \\
0 & \text{if } a = 0 \\
-\frac{\pi}{2} & \text{if } a < 0.
\end{cases}
\]

- Our second example is the integral
\[
\int_{-\infty}^{\infty} \frac{1 - \cos ax}{x^2} \, dx.
\]

(This is a proper integral, thanks to the boundedness of the integrand as \( x \to 0 \) and its decay at \( \infty \) like \( x^{-2} \).)
Our experience leads us to assume $a \geq 0$ and to choose
\[ f(z) = \frac{1 - e^{iaz}}{z^2}. \]

This function has a simple pole at $z = 0$, with
\[ \text{Res}(f, 0) = \text{Res}(\frac{-iaz + \cdots}{z^2}) = 0. \]

Using the same path as above and letting $R \to \infty$, $\varepsilon \to 0$, we obtain
\[ \text{PV} \int_{-\infty}^{\infty} \frac{1 - e^{iax}}{x^2} \, dx = \pi i (-ia) = \pi a. \]

And then Euler's formula gives
\[ \int_{-\infty}^{\infty} \frac{1 - e^{iax}}{x^2} \, dx = \pi a \text{ for } a > 0. \]

Final result:
\[ \int_{0}^{\infty} \frac{1 - e^{iax}}{x^2} \, dx = \frac{\pi |a|}{2} \text{ for all } a \in \mathbb{R}. \]

(A standard trig identity $\Rightarrow$
\[ \int_{0}^{\infty} \frac{2 \sin^2 \frac{ax}{2}}{x^2} \, dx = \frac{\pi |a|}{2} \]
and then $a \to 2a$
\[ \int_{0}^{\infty} \frac{\sin^2 ax}{x^2} \, dx = \frac{\pi |a|}{2}. \]

HW 17W due Mar 13, 2013
Calculate
\[ \text{PV} \int_{0}^{\infty} \frac{\cos ax}{x^2 - 1} \, dx \text{ for } a \in \mathbb{R}. \]
1. Suppose that $f$ and $g$ are entire holomorphic functions such that $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a constant $c$ such that $f(z) = cg(z)$ for all $z \in \mathbb{C}$.

Trivial case: $g(z) = 0$ for all $z$. Otherwise, the zeros of $g$ are isolated. Define $h(z) = \frac{f(z)}{g(z)}$ at all points where $g(z) \neq 0$. Then $h$ is holomorphic and $|h(z)| \leq 1$. The zeros of $g$ are isolated singularities of $h$, and the fact that $h$ is bounded implies they singularities are removable (Riemann's theorem). Remove these singularities so that $h$ becomes an entire holomorphic function. Then Liouville's theorem $\Rightarrow h$ is constant (since it's bounded). Call the constant $c$. Then $f/g = c$ so $f = cg$. QED

2. The tangent function is of course defined by $\tan z = \frac{\sin z}{\cos z}$.

a. Denote the Macaulay series of $\tan z$ by $\tan z = \sum_{n=0}^{\infty} c_n z^n$.

What is the radius of convergence of this series?

The function $\tan$ is holomorphic except for (simple) poles at the zeros of $\cos$. We know them to be $\frac{\pi}{2} + n\pi$. Thus the poles of $\tan$ closest to 0 are $\pm \frac{\pi}{2}$.

a. $R = \frac{\pi}{2}$.

b. Prove that $c_n = 0$ for all even $n$.

c. Prove that $c_n > 0$ for all odd $n$.

b. As $\tan(-z) = -\tan z$, all even terms $c_n = 0$.

c. Repeat the argument on page Feb 6 (6) or else use that result: $\tan' = \sec^2 = (\sum_{n=0}^{\infty} b_n z^{2n})^2$, all $b_n > 0$.

Do the squarings: $\tan' = \sum_{n=0}^{\infty} b_n z^{2n+2}$, all $b_n > 0$.

Integrate: $\tan z = \sum_{n=0}^{\infty} b_n \frac{z^{2n+1}}{2n+1}$, so $c_n = \frac{b_{n+1}}{2n+1} > 0$. QED
3. Let $a(z)$ and $b(z)$ be polynomials such that 
\[
\text{degree of } b \geq 2 + \text{degree of } a.
\]
Find the sum of all the residues of the function $\frac{a(z)}{b(z)}$.

For sufficiently large $R$, all the poles of $\frac{a(z)}{b(z)}$ satisfy $|z| < R$. The residue theorem implies that
\[
\int_{|z|=R} \frac{a(z)}{b(z)} \, dz = 2\pi i \sum \text{residues}.
\]

But for sufficiently large $R$ we have $|\frac{a(z)}{b(z)}| \leq \frac{\text{const}}{R^2}$. Thus
\[
2\pi i \sum \text{residues} \leq \left| \int_{|z|=R} \frac{a(z)}{b(z)} \, dz \right| \leq \frac{\text{const}}{R^2} \cdot 2\pi R \to 0 \quad \text{as} \quad R \to \infty.
\]

Since LHS does not depend on $R$, it must be 0. QED

4. For any complex number $a$ calculate
\[
\int_{-\infty}^{\infty} \frac{1}{(x-a)(x-i)} \, dx
\]
(if $a \in \mathbb{R}$ this is a principal value integral).

For #4 give me your answer only.

Case I: $\text{Im} \, a > 0$

Let $R \to \infty$ to get
\[
\int_{-\infty}^{\infty} f(x) \, dx = 0.
\]

So the answer is 0.
Case 2 \( \text{Im} \sigma < 0 \)

We get

\[
\int_{-\infty}^{\infty} f(x) dx = 2\pi i \text{Re}(f,a)
\]

\[
= \frac{2\pi i}{a-i}, \text{ so the answer is } \frac{2\pi i}{i-a}
\]

Case 3 \( \text{Im} \sigma = 0 \)

We get

\[
\text{PV} \int_{-\infty}^{\infty} \frac{f(x) dx}{(x-a)(x-i)} = \frac{\pi i}{a-i}, \text{ so the answer is } \frac{\pi i}{i-a}
\]

**Summary:**

\[
\int_{-\infty}^{\infty} \frac{dx}{(x-a)(x-i)} = \begin{cases} 
0 & \text{if } \text{Im} \sigma > 0 \\
\frac{\pi i}{i-a} & \text{if } \text{Im} \sigma = 0 \\
\frac{2\pi i}{i-a} & \text{if } \text{Im} \sigma < 0 
\end{cases}
\]