Computational Complex Analysis

Frank Jones 2018

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Preface

Think about the difference quotient definition of the derivative of a function from the real number field to itself. Now change the word "real" to "complex." Use the very same difference quotient definition for derivative. This turns out to be an amazing definition indeed. The functions which are differentiable in this complex sense are dramatically called *holomorphic functions*.

This book initiates a basic study of such functions. That is all I can do in a book at this level, for the study of holomorphic functions has been a serious field of research for centuries. In fact, there's a famous unsolved problem, The Riemann Hypothesis, which is still being studied to this day; it's one of the Millennium Problems of the Clay Mathematics Institute. Solve it and win a million dollars! The date of the Riemann Hypothesis is 1859. The Clay Prize was announced in 2000.

I've entitled this book *Computational Complex Analysis*. The adjective *Computational* does not refer to doing difficult numerical computations in the field of complex analysis; instead, it refers to the fact that (essentially pencil-and-paper) computations are discussed in great detail.

A beautiful thing happens in this regard: we'll be able to give proofs of almost all the techniques we use, and these proofs are interesting in themselves. It's quite impressive that the only background required for this study is a good understanding of basic real calculus on two-dimensional space! Our use of these techniques will produce all the basic theorems of beginning complex analysis, and at the same time I think will solidify our understanding of two-dimensional real calculus.

This brings up the fact that two-dimensional real space is equivalent in a very definite sense to one-dimensional complex space!

Many students at the beginning of their study of this material are a little scared of *complex* numbers and *imaginary* numbers. Part of this fear is the unfortunate historical choice of the adjectives *complex* and *immaginary*. A bonus of studying this subject is that students become quite at home with the complex number field. It's hard to be afraid of our friends, the complex numbers.

Here's another observation about adjectives. There are two related subjects:

real analysis and *complex* analysis. Normal language would seem to indicate that the latter subject would be more difficult than the former. Exactly the reverse is what is true! The main reason is that our *holomorphic* functions are such beautifully behaved objects!

Chapter 1

Introduction

Section A: Complex Numbers

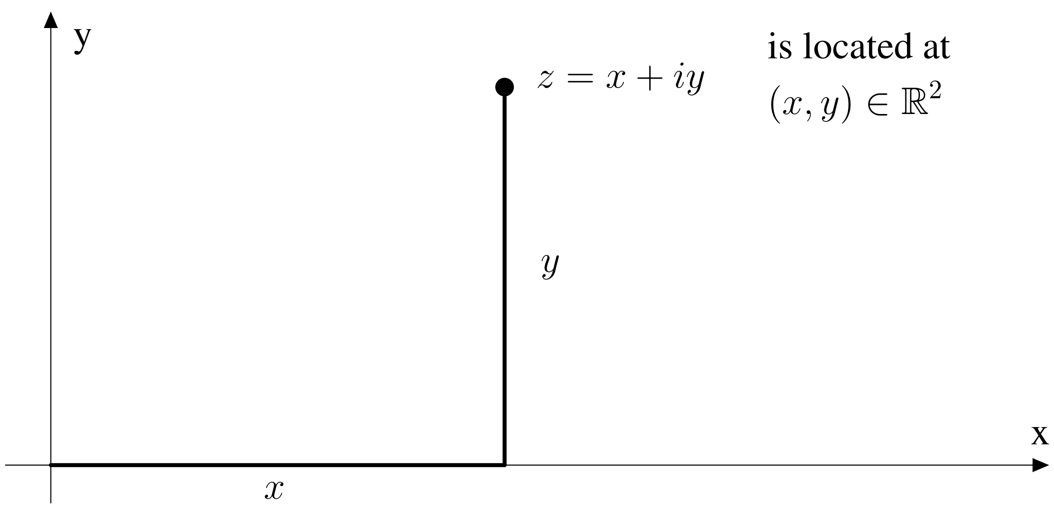
$\mathbb{C}$, the field of *complex numbers*, is the set of all expressions of the form $x + iy$, where

- $x, y \in \mathbb{R}$
- i is a special number
- addition and multiplication: the usual rules, except
- $\boxed{i^2 = -1}$

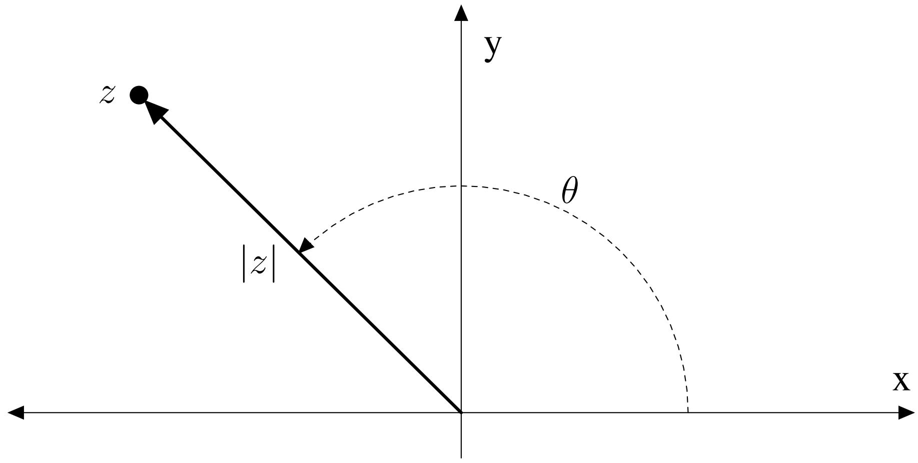
The complex number 0 is simply $0 + i0$. $\mathbb{C}$ is a field, since every complex number other than 0 has a multiplicative inverse:

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}.$$

CARTESIAN REPRESENTATION:



<u>POLAR REPRESENTATION</u>:



$|z| = \sqrt{x^2 + y^2} =$ the *modulus* of z.
The usual polar angle θ is called "the" *argument* of z: $\boxed{\arg z}$
All the usual care must be taken with $\arg z$, as there is not a unique determiantion of it. For instance:

$$\arg(1+i) = \frac{\pi}{4} \text{ or } \frac{9\pi}{4} \text{ or } \frac{-7\pi}{4} \text{ or } \frac{201\pi}{4} \cdots .$$

THE EXPONENTIAL FUNCTION is the function from $\mathbb{C}$ to $\mathbb{C}$ given by the power series

$$\begin{aligned}\exp(z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots .\end{aligned}$$

We shall soon discuss power series in detail and will see immediately that the above series converges absolutely. We will use the notation

$$e^z \text{ for } \exp(z).$$

<u>PROPERTIES</u>:

- $\boxed{e^{z+w} = e^z e^w}$ (known as the *functional equation* for $\exp$)
- If $z \in \mathbb{R}$, then e^z is the usual calculus function.
- If $t \in \mathbb{R}$, then we have Euler's formula

$$\boxed{e^{it} = \cos t + i \sin t}$$

We can easily give a sort of proof of the functional equation. If we ignore the convergence issues, the proof goes like this:

$$\begin{aligned}
e^z e^w &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) && \\
&= \left(\sum_{m=0}^{\infty} \frac{z^m}{m!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) && \text{change dummy} \\
&= \sum_{m,n=0}^{\infty} \frac{z^m}{m!}\frac{w^n}{n!} && \text{multiply the series} \\
&= \sum_{\ell=0}^{\infty} \sum_{m+n=\ell} \frac{z^m w^n}{m!n!} && \text{diagonal summation} \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{m=0}^{\ell} \frac{\ell!}{m!(\ell-m)!} z^m w^{\ell-m} && n = \ell - m \\
&= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{m=0}^{\ell} \binom{\ell}{m} z^m w^{\ell-m} && \text{binomial coefficient} \\
&= \sum_{\ell=0}^{\infty} \frac{(z+w)^\ell}{\ell!} && \text{binomial formula} \\
&= e^{z+w}. && \text{definition}
\end{aligned}$$

What a lovely proof! The crucial functional equation for $\exp$ essentially follows from the binomial formula! (We will eventually see that the manipulations we did are legitimate.)

The polar representation of any nonzero complex number can now be written as

$$z = |z|e^{i \arg z}.$$

Geometric description of complex multiplication: The polar form helps us here. Suppose z and w are two nonzero complex numbers, and write

$$\begin{aligned}
z &= |z|e^{i\theta} && (\theta = \arg z); \\
w &= |w|e^{i\varphi} && (\varphi = \arg w).
\end{aligned}$$

Then we have immediately that

$$zw = |z||w|e^{i(\theta+\varphi)}.$$

We may thus conclude that the product zw has the polar coordinate data

$$|zw| = |z||w|,$$

$$\arg(zw) = \arg(z) + \arg(w).$$

Thus, for a fixed $w \neq 0$, the operation of mapping z to zw

- multiplies the modulus by $|w|$,
- adds the quantity $\arg w$ to $\arg z$.

In other words, zw results from z by

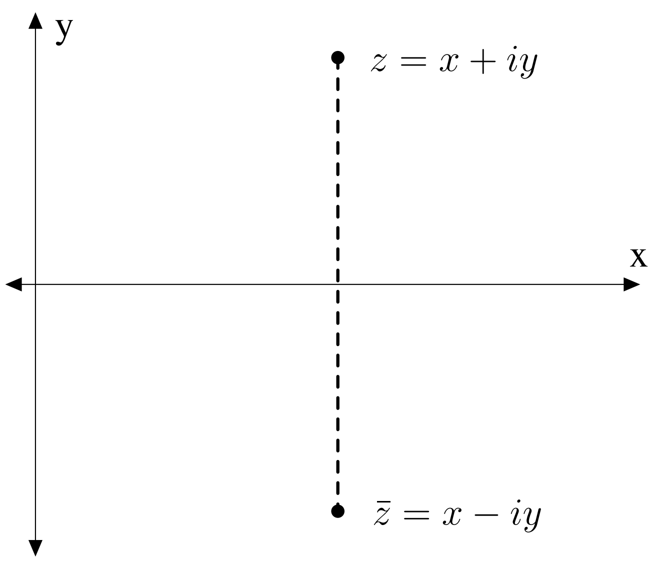
- stretching by the factor $|w|$, and
- rotating by the angle $\arg w$.

PROBLEM 1-1. Let a, b, c be three distinct complex numbers. Prove that these numbers are the vertices of an equilateral triangle $\Leftrightarrow$

$$\boxed{a^2 + b^2 + c^2 = ab + bc + ca}$$

(Suggestion: first show that the translation of a, b, c does not change the equilateral triangle's nature (clear), and also does not change the algebraic relation. Then show the same for multiplication by a fixed non-zero complex number.)

More $\mathbb{C}$ notation:



The complex *conjugate* of $z = x + iy$ is $\bar{z} = x - iy$.

The *real part* of z, denoted $\mathrm{Re}(z)$, is equal to x; the *imaginary part* of z, denoted $\mathrm{Im}(z)$, is equal to y. Notice that both $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are real numbers.

The choice of the adjective *imaginary* is perhaps unfortunate, but it is what everyone uses. In fact, if $\operatorname{Re}(z) = 0$, we say that z is a *pure* number. It has the form $z = iy$ for $y \in \mathbb{R}$. We also have:

- $z + \overline{z} = 2\operatorname{Re}(z)$.
- $z - \overline{z} = 2i\operatorname{Im}(z)$.
- $\overline{zw} = \overline{z}\,\overline{w}$.
- $|z|^2 = z\overline{z}$.

We can therefore observe that the important formula for $|zw|$ follows purely algebraically:

$$|zw|^2 = (zw)(\overline{zw}) = z\,w\,\overline{z}\,\overline{w} = z\overline{z}w\overline{w} = |z|^2|w|^2.$$

PROBLEM 1-2. Now let a, b, c be three distinct complex numbers each with modulus 1. Prove that these numbers are the vertices of an equilateral triangle $\Leftrightarrow$

$$\boxed{a + b + c = 0}$$

(Suggestion: $0 = (a + b + c)^2 = \cdots$; use Problem 1-1.)

Remark: The *centroid* of a triangle with vertices a, b, c is the complex number

$$\frac{a + b + c}{3}.$$

The situation of Problem 1-2 concerns a triangle with centroid 0 and the same triangle *inscribed* in the unit circle. The latter statement means that the circumcenter of the triangle is 0.

PROBLEM 1-3. Let a, b, c, d be four distinct complex numbers each with modulus 1. Prove that these numbers are vertices of a rectangle $\Leftrightarrow$

$$\boxed{a + b + c + d = 0}$$

PROBLEM 1-4. Suppose the centroid and circumcenter of a triangle are equal. Prove that the triangle is equilateral.

PROBLEM 1-5. Suppose the centroid and the incenter of a triangle are equal. Prove that the triangle is equilateral.

PROBLEM 1-6. Suppose the incenter and the circumcenter of a triangle are equal. Prove that the triangle is equilateral.

More about the exponential function: In the power series for $\exp(z)$, split the terms into even and odd terms:

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0,2,\ldots}^{\infty} \frac{z^n}{n!} + \sum_{n=1,3,\ldots}^{\infty} \frac{z^n}{n!} \\ &=: \cosh z + \sinh z. \end{aligned}$$

In other words,

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.$$

Hyperbolic Cosine Hyperbolic Sine

It is simple algebra to derive the corresponding addition properties, just using $e^{z+w} = e^z e^w$. For instance,

$$\begin{aligned} 2\sinh(z+w) &= e^{z+w} - e^{-z-w} \\ &= e^z e^w - e^{-z} e^{-w} \\ &= (\cosh z + \sinh z)(\cosh w + \sinh w) \\ &\quad - (\cosh z - \sinh z)(\cosh w - \sinh w) \\ &\overset{\text{algebra}}{=} \cosh z \cosh w + \cosh z \sinh w + \sinh z \cosh w + \sinh z \sinh w \\ &\quad - \cosh z \cosh w + \cosh z \sinh w + \sinh z \cosh w - \sinh z \sinh w \\ &= 2 \sinh z \cosh w + 2 \cosh z \sinh w. \end{aligned}$$

Thus,

- $\sinh(z+w) = \sinh z \cosh w + \cosh z \sinh w$,

and likewise

- $\cosh(z+w) = \cosh z \cosh w + \sinh z \sinh w$.

Trigonometric Functions: By definition, for all $z \in \mathbb{C}$ we have

$$\cos z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!},$$

and

$$\sin z := \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

(The known Maclaurin series for real z lead to this definition for complex z.)

There is a simple relation between the hyperbolic functions and the trigonometric ones:

$$\boxed{\begin{aligned} \cosh(iz) &= \cos z \\ \sinh(iz) &= i \sin z \end{aligned}}$$

Conversely,

$$\boxed{\begin{aligned} \cos(iz) &= \cosh z \\ \sin(iz) &= i \sinh z \end{aligned}}$$

The definitions of $\cos$ and $\sin$ can also be expressed this way:

$$\boxed{\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \end{aligned}}$$

We also immediately derive

- $\sin(z+w) = \sin z \cos w + \cos z \sin w$,
- $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

Notice that

$$e^{iz} = \cos z + i \sin z.$$

PROBLEM 1-7. Show that

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y.$$

Likewise, show that
$$|\cosh z|^2 = (?)^2 + (?)^2.$$

More geometrical aspects of $\mathbb{C}$: We shall frequently need to deal with the modulus of a sum, and here is some easy algebra

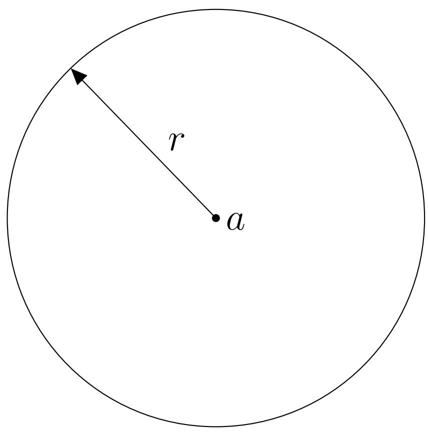
$$\begin{aligned}
|z+w|^2 &= (z+w)(\overline{z+w}) \\
&= (z+w)(\overline{z}+\overline{w}) \\
&= z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w} \\
&= |z|^2 + 2\operatorname{Re}(z\overline{w}) + |w|^2.
\end{aligned}$$

I will call this the

LAW OF COSINES: $\boxed{|z+w|^2 = |z|^2 + 2\operatorname{Re}(\overline{z}w) + |w|^2}$

As an illustration let us write down the equation of a circle in $\mathbb{C}$. Suppose the circle has a center $a \in \mathbb{C}$ and radius $r > 0$. Then z is on the circle $\Leftrightarrow$ $|z-a| = r$. That is, according to the above formula:

$$|z|^2 - 2\operatorname{Re}(z\overline{a}) + |a|^2 = r^2.$$



Dot product formula relating $\mathbb{R}^2$ and $\mathbb{C}$:

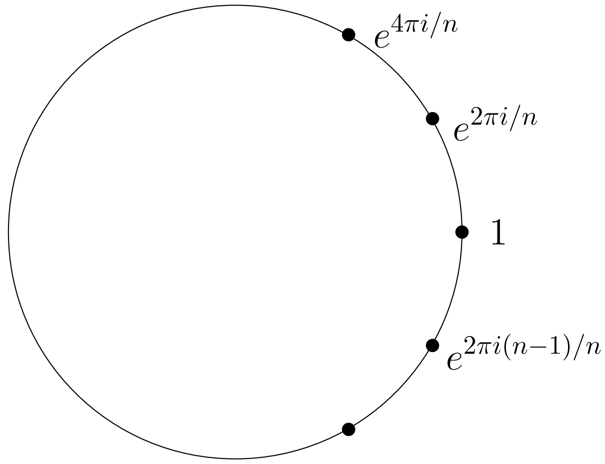
$$z \cdot w = \operatorname{Re}(z\overline{w})$$

$\mathbb{R}^2$ dot product

ROOTS OF UNITY This is about the solutions of the equation $z^n = 1$, where n is a fixed positive integer. We find n distinct roots, essentially by inspection:

$$z = e^{2\pi ik/n} \quad \text{for } k = 0, 1, \ldots, n-1.$$

These are, of course, equally spaced points on the unit circle.



Simple considerations of basic polynomial algebra show that the polynomial $z^n - 1$ is exactly divisble by each factor $z - e^{2\pi ik/n}$. Therefore,

$$\boxed{z^n - 1 = \prod_{k=0}^{n-1} \left(z - e^{2\pi ik/n}\right)}$$

is an identity for the polynomial $z^n - 1$.

COMPLEX LOGARITHM This is about an inverse "function" for $\exp$. In other words, we want to solve the equation $e^w = z$ for w. Of course, $z = 0$ is not allowed.

Quite easy: represent $w = u + iv$ in Cartesian form and $z = re^{i\theta}$ in polar form. Then we need

$$\begin{aligned} e^{u+iv} &= re^{i\theta}; \\ e^u e^{iv} &= re^{i\theta}; \end{aligned}$$

this equation is true $\Leftrightarrow$ $e^u = r$ and $e^{iv} = e^{i\theta}$.
As $r > 0$, we have $\underline{u = \ln r}$. Then $\underline{v = \theta + 2\pi \cdot \text{integer}}$.
As $\theta = \arg z$, we thus have the formula $w = \ln r + i(\theta + 2\pi n)$, and we write

$$\boxed{\log z = \ln |z| + i \arg z}$$

Here, of course, $\ln$ is the usual natural logarithm for positive real numbers. Thus, $\log z$ and $\arg z$ *share the same sort of ambiguity*.

Properties:

- $e^{\log z} = z$ (no ambiguity)
- $\log e^z = z$ (ambiguity of $2\pi ni$)
- $\log(zw) = \log z + \log w$ (with ambiguity)
- $\log(z^n) = n \log z$ (with ambiguity).

E.g.

$$\log(1+i\sqrt{3}) = \ln 2 + i\frac{\pi}{3},$$
$$\log(-6) = \ln 6 + i\pi,$$
$$\log(re^{i\theta}) = \ln r + i\theta.$$

$\boxed{\textbf{MÖBIUS TRANSFORMATIONS}}$ This will be only a provisional definition, so that we will become accustomed to the basic manipulations.

We want to deal with functions of the form

$$f(z) = \frac{az+b}{cz+d},$$

where a, b, c, d are complex constants. We do not want to include cases where f is constant, meaning that $az+b$ is proporitional to $cz+d$. I.e. meaning that the vectors (a, b) and (c, d) in $\mathbb{R}^2$ are linearly dependent. A convenient way to state this restriction is to require that $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$. This we shall always require.

Easy calculation: if $g(z) = \dfrac{a'z+b'}{c'z+d'}$, then the composition $f \circ g$ [i.e. $(f \circ g)(z) = f(g(z))$] corresponds to the matrix product

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}$ (with $\lambda \neq 0$), then these two matrices give the same transformation.

$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the *extended* complex plane, and we then also define

$$\begin{aligned} f(-d/c) &= \infty, \\ f(\infty) &= a/c. \end{aligned}$$

(we will have much more to say about these formulas later.)

The functions we have defined in this way are called *Möbius transformations*. Each of them gives a bijection of $\hat{\mathbb{C}}$ onto $\hat{\mathbb{C}}$. And each of them has a unique inverse:

$$f(z) = \frac{az+b}{cz+d} \quad \Longrightarrow \quad f^{-1} = \frac{dz-b}{-cz+a}.$$

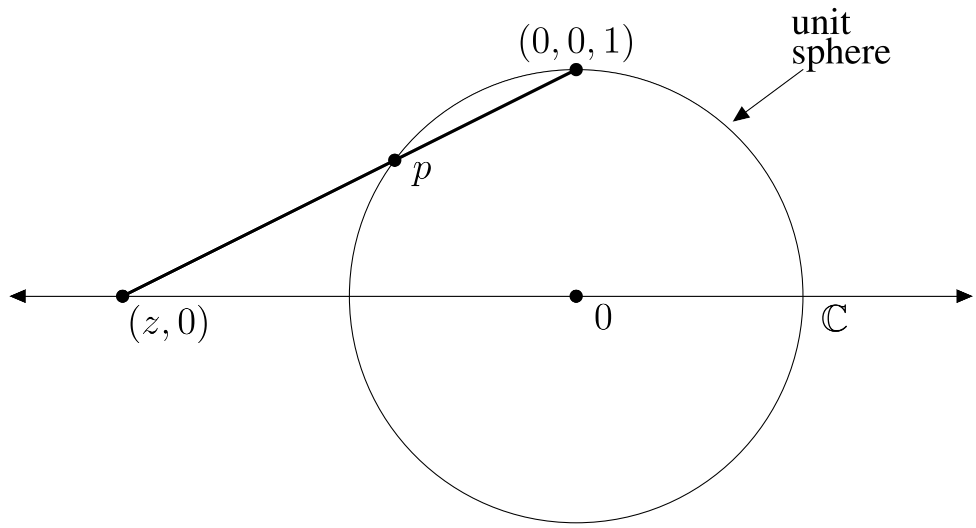
PROBLEM 1-8. Let C be the circle in $\mathbb{C}$ with center $a \in \mathbb{C}$ and radius $r > 0$. (From page 8 we know that $z \in C \Leftrightarrow |z|^2 - 2\,\mathrm{Re}(z\overline{a}) + |a|^2 = r^2$.)

We want to investigate the outcome of forming $\dfrac{1}{z}$ for all $z \in C$.

1. If $0 \notin C$, define
$$D = \left\{ \frac{1}{z} \middle| z \in C \right\}.$$
Prove that D is also a circle, and calculate its center and radius:
$$\begin{aligned} &\text{center} = ? \\ &\text{radius} = ? \end{aligned}$$

2. If $0 \in C$, then instead define
$$D = \left\{ \frac{1}{z} \middle| z \in C, z \neq 0 \right\}.$$
What geometric set is D? Prove it.

More about the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:
This enjoys a beautiful geometric depiction as the unit sphere in $\mathbb{R}^3$, by means of *stereographic projection*, which we now describe. There are several useful ways of defining this projection, but I choose the following:



SIDE VIEW

Let $\mathbb{R}^3$ be given Cartesian coordinates (x, y, t), where $z = x + iy$.

Project the unit sphere onto $\mathbb{C}$ from the north pole $(0, 0, 1)$.

Straight lines through the north pole which are not horizontal intersect the plane $t = 0$ and the unit sphere and set up a bijection between $\mathbb{C}$ and the unit sphere minus $(0, 0, 1)$, as shown in the figure.

When $z \to \infty$, the projection $p \to (0, 0, 1)$. Thus, by decreeing that the north pole corresponds to some point, we are led to adjoining ∞ to $\mathbb{C}$.

Thus $\hat{\mathbb{C}}$ is "equivalent" to the unit sphere in $\mathbb{R}^3$. So $\hat{\mathbb{C}}$ is often called the ***Riemann Sphere***.

<u>More about Möbius transformations</u>:

- Baby case: given 3 distinct complex numbers u, v, w, it is easy to find a Möbius f such that
$$\begin{cases} f(u) = 0 \\ f(v) = \infty \\ f(w) = 1 \end{cases}$$
In fact, f is uniquely determined, and we <u>must</u> have
$$f(z) = \frac{z-u}{z-v}\frac{w-v}{w-u}. \qquad (\star)$$

- Embellishment: we can even allow u or v or w to be ∞, and again there is a unique Möbius f. Here are the results:

$f(\infty)=0$ $f(v)=\infty : f(z)=\dfrac{w-v}{z-v}$ $f(w)=1$	$f(u)=0$ $f(\infty)=\infty : f(z)=\dfrac{z-u}{w-u}$ $f(w)=1$	$f(u)=0$ $f(v)=\infty : f(z)=\dfrac{z-u}{z-v}$ $f(\infty)=1$

(Remark: each case results from ($\star$) by replacing u, v, w by ∞ formally.)

- General case: given 3 distinct points $u, v, w \in \mathbb{C}$ and also 3 distinct point $u', v', w' \in \mathbb{C}$, then there is a unique Möbius f such that

$$\begin{cases} f(u)=u' \\ f(v)=v' \\ f(w)=w'. \end{cases}$$

Proof: Use the previous case twice:

$$\begin{array}{ccc} \begin{pmatrix} u \\ v \\ w \end{pmatrix} & \xrightarrow{f} & \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} \\ g\Big\downarrow & & h\Big\downarrow \\ \begin{pmatrix} 0 \\ \infty \\ 1 \end{pmatrix} & & \begin{pmatrix} 0 \\ \infty \\ 1 \end{pmatrix} \end{array}$$

Then $f = h^{-1} \circ g$.

QED

Möbius transformations and circles:
According to Problem 1-8 the image of a circle under the action of $z \mapsto \dfrac{1}{z}$ is another circle (or straight line). *The same is true if instead of* $\dfrac{1}{z}$*, we use any Möbius transformation*. Let

$$f(z) = \frac{az+b}{cz+d}.$$

Case 1 ($c=0$): Then we may as well write $f(z) = az+b$. This transformation involves multiplication by $|a|$, rotation by $\arg a$, and translation by b. Thus, circles are preserved by f.

Case 2 ($c \neq 0$): Then we may as well write $f(z) = \dfrac{az+b}{1z+d}$, where $ad - b \neq 0$. But then

$$f(z) = \frac{a(z+d)}{z+d} + \frac{b-ad}{z+d} = a + \frac{b-ad}{z+d},$$

so f is given by translation, then reciprocation, then multiplication, then translation. All operations preserve "circles" if we include straight lines.

PROBLEM 1-9. Start from the result we obtained on page 8: if $n \geq 2$ is an integer, then

$$z^n - 1 = \prod_{k=0}^{n-1} \left(z - e^{2\pi ik/n}\right).$$

1. Prove that for any $z, w \in \mathbb{C}$

$$z^n - w^n = \prod_{k=0}^{n-1} \left(z - we^{2\pi ik/n}\right).$$

2. Prove that

$$z^n - w^n = \prod_{k=0}^{n-1} \left(z - we^{-2\pi ik/n}\right).$$

3. Prove that

$$z^n - w^n = (-i)^{n-1} \prod_{k=0}^{n-1} \left(e^{\pi ik/n}z - e^{-\pi ik/n}w\right).$$

4. Replace z by e^{iz} and w by e^{-iz} and show that

$$\boxed{\sin nz = 2^{n-1} \prod_{k=0}^{n-1} \sin\left(z + \frac{\pi k}{n}\right)}$$

5. Show that

$$\boxed{\prod_{k=1}^{n-1} \sin\frac{\pi k}{n} = \frac{n}{2^{n-1}}}$$

6. Prove that $\cos z = \cos w \Leftrightarrow \left\{ \begin{array}{c} z - w = 2k\pi \\ \text{or} \\ z + w = 2k\pi \end{array} \right\}$ for some $k \in \mathbb{Z}$.

7. Prove that $\sin z = \sin w \Leftrightarrow \left\{ \begin{array}{c} z - w = 2k\pi \\ \text{or} \\ z + w = ? \end{array} \right\}$ for some $k \in \mathbb{Z}$.

Section B: Linear Functions on $\mathbb{R}^2$

An extermely important part of the subject of *linear* algebra is the discussion of *linear* functions. By definition, a *linear* function from one vector space to another is a function f which satisfies the two conditions

$$\begin{aligned} f(p+q) &= f(p) + f(q) \\ f(ap) &= af(p). \end{aligned}$$

These equations have to hold for all p and q and for all scalars a.

For example, the linear functions from $\mathbb{R}$ to $\mathbb{R}$ are these:

$$f(t) = mt,$$

where $m \in \mathbb{R}$. Notice that $mt + b$ is <u>not</u> a linear function of t unless $b = 0$. Such a function is said to be an *affine* function of t.

Out focus in this section is linear functions from $\mathbb{R}^2$ to $\mathbb{R}^2$. From multivariable calculus, we know that linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be described economically in terms of *matrix* operations, the key ingredient being $m \times n$ matrices. Where $m = n = 2$ (our case), these operations produce a unique representation of any linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ in the form

$$\boxed{f(x, y) = (ax + by, cx + dy)}$$

Moreover, this linear function has an inverse $\Leftrightarrow$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

I.e. $\Leftrightarrow$

$$ad - bc \neq 0.$$

This determinant is also called the determinant of the linear function f, and written $\mathbf{det}\, \boldsymbol{f}$.

It's a useful and easy exercise to phrase all this in complex notation. This is easily done, because

$$x = \frac{z+\overline{z}}{2} \text{ and } y = \frac{z-\overline{z}}{2i}.$$

The simple result is

$$f(z) = Az + B\overline{z}. \tag{$\star$}$$

where A and B are complex numbers.

We need to see the condition for f to have an inverse:

PROBLEM 1-10. f as defined by ($\star$) has an inverse $\Leftrightarrow$

$$|A| \neq |B|.$$

In fact, prove that $\det f = |A|^2 - |B|^2$.

In fact, complex algebra enables us to calculate the inverse of f easily: just imagine solving the equation

$$w = f(z)$$

for z as a function of w. Here's how:

$$\begin{aligned}
&Az + B\overline{z} = w; \\
&\overline{B}z + \overline{A}\overline{z} = \overline{w}; && \text{conjugate} \\
&\overline{A}(Az + B\overline{z}) - B(\overline{B}z + \overline{A}\overline{z}) = \overline{A}w - B\overline{w}. && \text{eliminate } \overline{z}
\end{aligned}$$

This becomes

$$\left(|A|^2 - |B|^2\right) z = \overline{A}w - \overline{B}w.$$

Thus,

$$z = \frac{\overline{A}}{|A|^2 - |B|^2} w - \frac{B}{|A|^2 - |B|^2} \overline{w},$$

and this expresses f^{-1} as a linear function in complex notation.

CRUCIAL REMARK: It's elementary but exteremely important to distinguish these two concepts:

- linear functions from $\mathbb{R}^2$ to $\mathbb{R}^2$.
- linear functions from $\mathbb{C}$ to $\mathbb{C}$.

For in terms of our complex notation, f is a linear function from $\mathbb{R}^2$ to $\mathbb{R}^2$ since

$$f(tz) = tf(z) \text{ for all } \underline{\text{real}}\ t.$$

In contrast, f is a linear function from $\mathbb{C}$ to $\mathbb{C}$ $\Leftrightarrow$

$$f(tz) = tf(z) \text{ for all } \underline{\text{complex}}\ t.$$

($\mathbb{R}^2$ is a <u>real</u> vector space of dimension 2, but $\mathbb{C}$ is a <u>complex</u> vector space of dimension $1 \ldots$ in other words, $\mathbb{C}$ is a <u>field</u>.) This agrees with the definition of linear function, which contains the condition $f(ap) = af(p)$. Here a is any scalar: for $\mathbb{R}^2$, a is real, but for $\mathbb{C}$, a is complex.

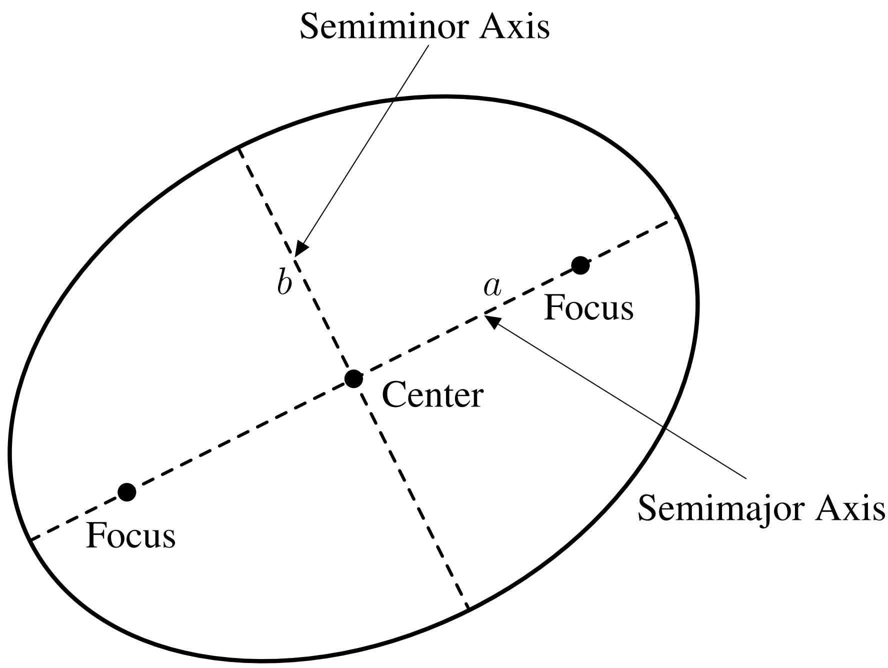
Thus, the linear function $f(z) = Az + B\overline{z}$ is a linear function from $\mathbb{C}$ to $\mathbb{C}$ $\Leftrightarrow$ $B = 0$.

REMARK: f preserves the orientation of $\mathbb{R}^2$ $\Leftrightarrow$ $\det f > 0$ $\Leftrightarrow$ $|A| > |B|$. Loosely speaking, this condition requires f to have more of z than $\overline{z}$.

Section C: Complex Description of Ellipses

This material wil not be used further in this text, but I've included it to provide an example of using complex numbers in an interesting situation.

You are familiar with the basic definition and properties of an ellipse contained in $\mathbb{R}^2$:



We're assuming $0 < b < a$. Recall the distance from the center of the ellipse to each focus is $\sqrt{a^2 - b^2}$.

A standard model for such an ellipse is given by the defining equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Parametrically, this ellipse can also be described as

$$\begin{cases} x = a\cos\theta, \\ y = b\sin\theta. \end{cases}$$

Let's convert this parametric description to complex notation:

$$\begin{aligned} x + iy &= a\cos\theta + ib\sin\theta \\ &= a\frac{e^{i\theta} + e^{-i\theta}}{2} + b\frac{e^{i\theta} - e^{-i\theta}}{2} && \text{Euler's equation} \\ &= \frac{a+b}{2}e^{i\theta} + \frac{a-b}{2}e^{-i\theta}. \end{aligned}$$

This formula represents the ellipse as the image of the unit circle under the action of the linear function:

$$f(z) = \frac{a+b}{2}z + \frac{a-b}{2}\overline{z}.$$

That ellipse is of course oriented along the coordinate axes. It's quite interesting to generalize this. So, we let f be any invertible linear function from $\mathbb{R}^2$ to $\mathbb{R}^2$, and use complex notation to write

$$f(z) = Az + B\overline{z},$$

where A and B are complex numbers with $|A| \neq |B|$ (see Section B). Then we obtain an ellipse (or a circle) as the set

$$\left\{ Ae^{i\theta} + Be^{-i\theta} \,|\, \theta \in \mathbb{R} \right\}.$$

This ellipse is centered at the origin.

Now we give a geometric description of this ellipse. First, write the polar representation of A and B:

$$A = |A|e^{i\alpha},$$
$$B = |B|e^{i\beta}.$$

Then,

$$f(e^{i\theta}) = |A|e^{i(\alpha+\theta)} + |B|e^{i(\beta-\theta)}.$$

The modulus of $f(e^{i\theta})$ is largest when the unit complex numbers satisfy

$$e^{i(\alpha+\theta)} = e^{i(\beta-\theta)}.$$

That is, when

$$\alpha + \theta = \beta - \theta \mod 2\pi;$$

or, rewritten, when

$$\theta = \frac{\beta - \alpha}{2} \mod \pi.$$

For such θ, we have

$$f(e^{i\theta}) = \pm\left(|A| + |B|\right) e^{i\frac{\alpha+\beta}{2}}.$$

In the same way, the modulus of $f(e^{i\theta})$ is smallest when

$$\begin{aligned} e^{i(\alpha+\theta)} &= -e^{i(\beta-\theta)} \\ &= e^{i(\beta+\pi-\theta)}. \end{aligned}$$

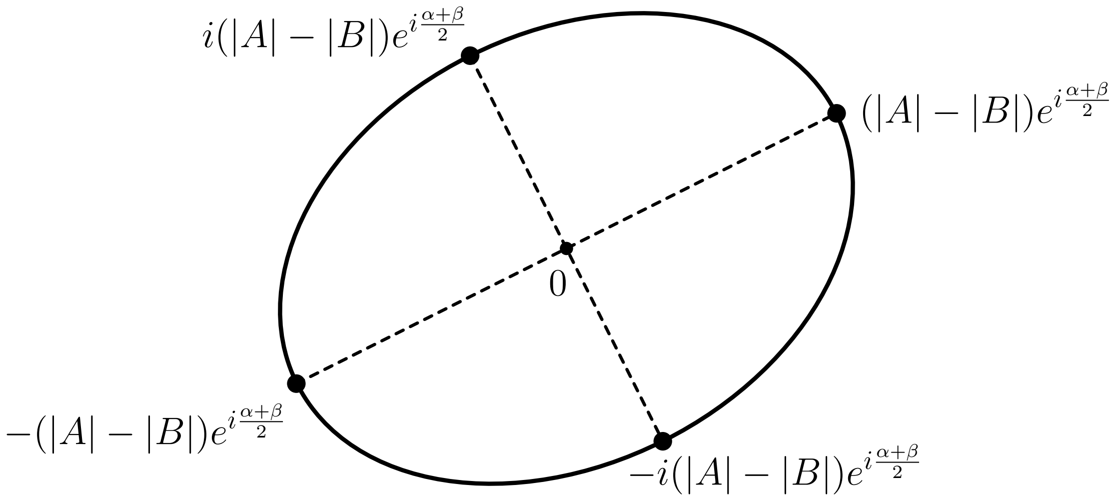
This occurs precisely when

$$\begin{aligned} \theta &= \frac{\beta + \pi - \alpha}{2} \mod \pi \\ &= \frac{\beta - \alpha}{2} + \frac{\pi}{2} \mod \pi. \end{aligned}$$

For such θ we have

$$\begin{aligned} f(e^{i\theta}) &= \pm(|A|-|B|)e^{i\left(\frac{\alpha+\beta}{2}+\frac{\pi}{2}\right)} \\ &= \pm i(|A|-|B|)e^{i\frac{\alpha+\beta}{2}}. \end{aligned}$$

Here's a representative sketch:



Of course, $|A|-|B| \neq 0$. And we have a <u>circle</u> precisely when $A=0$ or $B=0$.

Now assume it's really an ellipse: $AB \neq 0$. Then we have this data:

semimajor axis has length $|A|+|B|$;
semiminor axis has length $\big||A|-|B|\big|$;
center $=0$.

Therefore, the distance from the origin to each focus equals

$$\sqrt{(|A|+|B|)^2-(|A|-|B|)^2} = 2\sqrt{|A||B|}.$$

And the foci are the two points

$$\begin{aligned} \pm 2\sqrt{|A||B|}e^{i\frac{\alpha+\beta}{2}} &= \pm 2\sqrt{|A|e^{i\alpha}|B|e^{i\beta}} \\ &= \pm 2\sqrt{AB}. \end{aligned}$$

Another way of giving this result is that the two foci are the two square roots of the complex number $4AB$:

$$2\sqrt{AB}.$$

Chapter 2

Differentiation

Section A: The Complex Derivative

Now we begin a thrilling introduction to complex analysis. It all starts with a seemingly innocent and reasonable definition of derivative, using complex numbers instead of real numbers. But we shall learn very soon what an enormous step this really is!

> **DEFINITION**: Let f be a complex valued function defined on some neighborhood of a point $z \in \mathbb{C}$. We say that f is *complex-differentiable* at z if
>
> $$\lim_{\substack{h\in\mathbb{C}\\ h\to 0}} \frac{f(z+h)-f(z)}{h}$$
>
> CRUCIAL!
>
> exists.

In case this limit exists, it is called the *complex derivative* of f at z, and is denoted either

$$f'(z) \text{ or } \frac{df}{dz}.$$

This truly seems naive, as it's completely similar to the beginning definition in Calculus. But we shall see that the properties of f which follow from this definition are astonishing!

What makes this all so powerful is that in the difference quotient the denominator $h \in \mathbb{C}$ must be allowed simply to tend to 0, no restrictions on "how" or particular directions: merely $|h| \to 0$.

BASIC PROPERTIES:

- $f'(z)$ exists $\Rightarrow f$ is continuous at z.
  For if $\dfrac{f(z+h)-f(z)}{h}$ has a limit, then since $h \to 0$, the numerator must also have limit 0, so that

$$\lim_{h\to 0} f(z+h) = f(z).$$

- f and g differentiable $\Rightarrow f+g$ is too, and $(f+g)' = f' + g'$.
- PRODUCT RULE: also fg is differentiable, and

$$(fg)' = fg' + f'g.$$

  Proof:

$$\begin{aligned} &\frac{f(z+h)g(z+h)-f(z)g(z)}{h} \\ &\qquad = \underset{\substack{\downarrow \\ f(z)}}{f(z+h)}\,\underset{\substack{\downarrow \\ g'(z)}}{\frac{g(z+h)-g(z)}{h}} \underset{\text{(by continuity of } f)}{+\,g(z)}\,\underset{\substack{\downarrow \\ f'(z)}}{\frac{f(z+h)-f(z)}{h}} \end{aligned}$$

- $\dfrac{dz}{dz} = 1$ and then we prove by induction that for $n = 1, 2, 3, \ldots$

$$\frac{dz^n}{dz} = nz^{n-1}.$$

- QUOTIENT RULE:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2} \text{ provided that } g \neq 0.$$

- CHAIN RULE: If g is differentiable at z and f is differentiable at $g(z)$, then the composite function $f \circ g$ is differentiable at z, and

$$(f \circ g)'(z) = f'(g(z))g'(z).$$

All these properties are proved just as in "real" Calculus, so I have chosen to not write out detailed proofs for them all.

EXAMPLES:

- Möbius transformations – directly from the quotient rule

$$\left(\frac{az+b}{cz+d}\right)' = \frac{ad-bc}{(cz+d)^2} \qquad \text{(notice the determinant!)}$$

- Exponential Function
First for $h \to 0$ we have

$$\frac{e^h - 1}{h} = \sum_{n=1}^{\infty} \frac{h^{n-1}}{n!} = 1 + \frac{h}{2} + \frac{h^2}{6} + \ldots$$

has limit 1 as $h \to 0$. Thus,

$$\frac{e^{z+h} - e^z}{h} = e^z \frac{e^h - 1}{h} \to e^z.$$

Conclusion:

$$\frac{de^z}{dz} = e^z.$$

- Trigonometric and hyperbolic functions – follow immediately from $\exp$:

$$\frac{d\,\sin z}{dz} = \cos z, \qquad \frac{d\,\cos z}{dz} = -\sin z,$$
$$\frac{d\,\sinh z}{dz} = \cosh z, \qquad \frac{d\,\cosh z}{dz} = \sinh z.$$

Section B: The Cauchy-Riemann Equation

$$\boxed{\frac{d}{dz} \text{ and } \frac{\partial}{\partial x} \text{ and } \frac{\partial}{\partial y}}$$

By an audacious – but useful – abuse of notation we write

$$f(z) = f(x+iy) \stackrel{!}{=} f(x,y).$$

This sets up a correspondence between a function defined on $\mathbb{C}$ and a function defined on $\mathbb{R}^2$, but we use the same name for these functions!

Now suppose that $f'(z)$ exists. In the definition, we then restrict h to be real …the limit still exists, of course, and we compute

$$f'(z) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(z+h) - f(z)}{h} = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{f(x+h,y) - f(x,y)}{h}$$
$$= \frac{\partial f}{\partial x}(x,y).$$

Likewise, let $h = it$ be pure imaginary:

$$f'(z) = \lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{f(z+it) - f(z)}{it} = \lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{f(x, y+t) - f(x,y)}{it}$$
$$= \frac{1}{i} \frac{\partial f}{\partial y}(x,y).$$

We thus conclude that

$$f'(z) = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

This second equality is a famous relationship, called

$$\boxed{\begin{array}{c} \textbf{THE CAUCHY-RIEMANN EQUATION:} \\ \boldsymbol{\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}} \end{array}}$$

WARNING – everyone else calls this the Cauchy-Riemann equations, after expressing f in terms of its real and imaginary parts as $f = u + iv$. Then we indeed get 2 real equations:

$$\boxed{\begin{array}{c} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{array}}$$

(I prefer "equation," as that form shows a natural switch from x to iy.)

In a very precise sense, the converse is also valid, as we now discuss.

We suppose f is differentiable at (x, y) <u>in a multivariable calculus sense</u>. This means that not only do the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at (x, y), but also they provide the coefficients for a <u>good linear approximation</u> to $f(z+h)-f(z)$ for small $|h|$:

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}^2}} \frac{f(z+h) - f(z) - \dfrac{\partial f}{\partial x}(z)h_1 - \dfrac{\partial f}{\partial y}(z)h_2}{|h|} = 0.$$

Now assume that the Cauchy-Riemann equation is satisfied. Then we may replace $\dfrac{\partial f}{\partial y}$ by $i\dfrac{\partial f}{\partial x}$ and conclude that

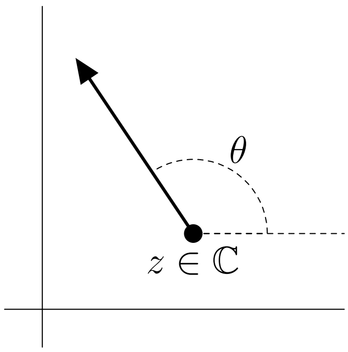
$$\lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{f(z+h)-f(z)-\frac{\partial f}{\partial x}(z)(h_1+ih_2)}{h} = 0.$$

I.e.,

$$\lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{f(z+h)-f(z)}{h} = \frac{\partial f}{\partial x}(z).$$

Therefore, we conclude that $f'(z)$ exists, so f is differentiable in the complex sense! Moreover, $f'(z) = \dfrac{\partial f}{\partial x}(z)$.

Connection to directional derivatives of Calculus III:



Suppose that we wish to find the directional derivative of a function f at a point z in the direction of the vector $e^{i\theta}$. This is described by the expression

$$\frac{d}{dt} f(z+te^{i\theta})\Big|_{t=0},$$

where $t \in \mathbb{R}$. If f is differentiable at z, then the chain rule holds, and we easily calculate

$$\frac{d}{dt} f(z+te^{i\theta})\Big|_{t=0} = f'(z+te^{i\theta})e^{i\theta}\Big|_{t=0} = f'(z)e^{i\theta}.$$

Thus if a function is differentiable at a point and we wish to calculate a directional derivative there, then we simply multiply the derivative of the function by the unit vector in the desired direction!

Cauchy-Riemann equation in polar coordinates:

We employ the usual polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \left(z = re^{i\theta}\right) \quad (r>0,\ \text{of course})$$

and then again abuse notation by writing $f = f(x,y)$ as

$$f = f(r\cos\theta, r\sin\theta),$$

and then computing the r and θ partial derivatives of this composite function and designating them as $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial \theta}$ (terrible!). Then the chain rule gives

$$\begin{cases} \dfrac{\partial f}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r} = \dfrac{\partial f}{\partial x}\cos\theta + \dfrac{\partial f}{\partial y}\sin\theta, \\ \dfrac{\partial f}{\partial \theta} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial \theta} = \dfrac{\partial f}{\partial x}(-r\sin\theta) + \dfrac{\partial f}{\partial y}(r\cos\theta). \end{cases}$$

Now suppose f satisfies the Cauchy-Riemann equation and substitute $\dfrac{\partial f}{\partial y} = i\dfrac{\partial f}{\partial x}$:

$$\begin{cases} \dfrac{\partial f}{\partial r} = \dfrac{\partial f}{\partial x}(\cos\theta + i\sin\theta), \\ \dfrac{\partial f}{\partial \theta} = \dfrac{\partial f}{\partial x}(-r\sin\theta + ir\cos\theta). \end{cases}$$

Thus,

$$\begin{cases} \dfrac{\partial f}{\partial r} = \dfrac{\partial f}{\partial x}e^{i\theta}, \\ \dfrac{1}{ir}\dfrac{\partial f}{\partial \theta} = \dfrac{\partial f}{\partial x}e^{i\theta}. \end{cases}$$

We conclude that

$$\boxed{\dfrac{\partial f}{\partial r} = \dfrac{1}{ir}\dfrac{\partial f}{\partial \theta}}$$

polar coordinate form of the Cauchy-Riemann equation

Our calculations show that since $\dfrac{\partial f}{\partial x} = f'$,

$$f'(z) = e^{-i\theta}\frac{\partial f}{\partial r} = \frac{1}{ir}e^{-i\theta}\frac{\partial f}{\partial \theta}.$$

EXERCISE: Prove that $\dfrac{\partial f}{\partial r} = \dfrac{1}{ir}\dfrac{\partial f}{\partial \theta}$ implies the original Cauchy-Riemann equation.

Complex Logarithm:

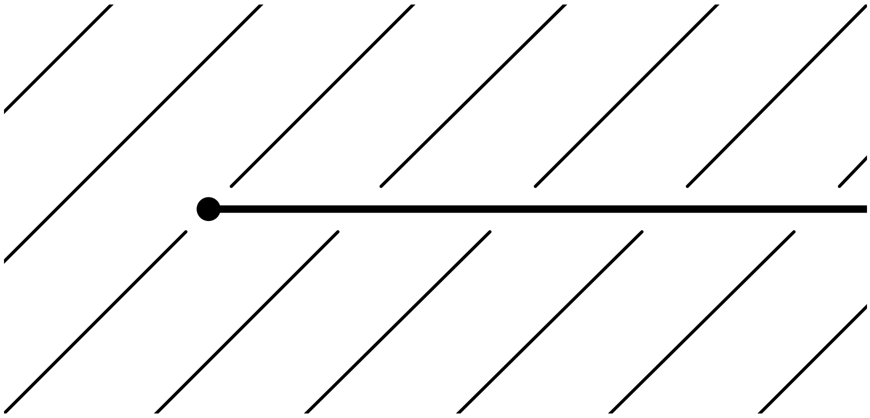
We have derived the defining equation

$$\log z = \ln|z| + i\arg z.$$

In terms of polar coordinates,

$$\log z = \ln r + i\theta.$$

We pause to discuss an easy but crucial idea. When we are faced with the necessity of using $\log$ or $\arg$, we almost always work in a certain region of $\mathbb{C} \setminus \{0\}$ in which it is possible to define $\arg z$ <u>in a continuous manner</u>. A typical situation might be the following: <u>exclude the nonnegative real axis</u> and define $\arg z$ so that $0 < \arg z < 2\pi$:



Then we would have e.g.

$$\log(-1) = \pi i, \quad \log(ei) = 1 + i\frac{\pi}{2}, \quad \log(-i) = \frac{3\pi i}{2}, \quad \text{etc.}$$

In such a situation $\log z$ is also a well-defined function of z, and the polar form of the Cauchy-Riemann equation applies immediately:

$$\begin{aligned} \frac{\partial}{\partial r} \log z &= \frac{\partial}{\partial r}(\ln r + i\theta) = \frac{1}{r}, \\ \frac{\partial}{\partial \theta} \log z &= \frac{\partial}{\partial \theta}(\ln r + i\theta) = i; \\ \text{thus} \qquad \frac{\partial}{\partial r} \log z &= \frac{1}{ir}\frac{\partial}{\partial \theta} \log z. \end{aligned}$$

Thus, $\log z$ has a complex derivative, which equals $e^{-i\theta}\frac{1}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}$.

We have thus obtained the expected formula

$$\frac{d \log z}{dz} = \frac{1}{z}.$$

(Be sure to notice that although $\log z$ is ambiguous, the ambiguity is the form of an <u>additive constant</u> $2\pi i n$, so $\frac{d}{dz}$ annihilates that constant.)

Section C: Holomorphic Functions

Now an extremely important definition will be given and discussed:

Definition: Let $D \subset \mathbb{C}$ be an open set, and assume that $D \xrightarrow{f} \mathbb{C}$ is a function which is of class C^1. That is, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are defined at each point in D and are themselves continuous functions on D.

Suppose also that the complex derivative $f'(z)$ exists at every point $z \in D$.

Then we say that f is a *holomorphic* function on D.

So of course, we have at our disposal quite an array of holomorphic functions:

- $\exp$, as well as $\sinh, \cosh, \sin$ and $\cos$;
- all Möbius transformations;
- $\log$;
- all <u>polynomials</u> in z: $f(z) = a_0 + a_1 z + \ldots + a_n z^n$;
- all <u>rational</u> functions in z: $\dfrac{\text{polynomial}}{\text{polynomial}}$.

REMARKS:

1. We do not acutally need to <u>say</u> that D is an open set! The very existence of $f'(z)$ is that
$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}$$
and this requires $f(z+h)$ to be defined for all sufficiently small $|h|$, and thus that f be defined in some neighborhood of z.

2. The assumption that $f \in C^1$ can be dispensed with, as a fairly profound theorem implies that it follows from just the assumption that $f'(z)$ <u>exists</u> for every $z \in D$. (We won't need this refinement in this book.) It's called Goursat's theorem.

3. "**Holomorphic**" is not a word you will see in most basic books on complex analysis. Usually those books use the word "**analytic**."

However, I want us to use "**analytic**" function to refer to a function which in a neighborhood of each z_0 in its domain can be represented as a power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

with a positive radius of convergence.

- It is pretty easy to prove (and we shall do so) that every analytic function is holomorphic.
- A much more profound theorem will also be proved – that every holomorhic function is analytic.

DEFINITION (from Wikipedia):
`https://en.wikipedia.org/wiki/Holomorphic_function`

In mathematics, a **holomorphic function** is a complex-valued function of one or more complex variables that is complex differentiable in a neighborhood of every point in its domain. The existence of a complex derivative in a neighborhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal to its own Taylor series (*analytic*). Holomorphic functions are the central objects of study in complex analysis.

Though the term *analytic function* is often used interchangeably with "holomorphic function," the word "analytic" is defined in a broader sense to denote any function (real, complex, or of more general type) that can be written as a convergenet power series in a neighborhood of each point in its domain. The fact that all holomorphic functions are complex analytic functions, and vice versa, is a major theorem in complex analysis.

PROBLEM 2-1. Let D be the open half plane

$$D = \{z \in \mathbb{C} | \operatorname{Re}(z) > 0\}.$$

Let f be the function defined on D by $f(z) = z^2$. Of course, f is holomorphic.

1. Prove that f is a bijection of D onto a set $D' \subset \mathbb{C}$.
2. What is D'?

3. The inverse function f^{-1} maps D' onto D. We'll actually prove a general theorem asserting that inverses of holomorphic functions are always holomorphic. But in this problem, I want you to prove directly that f^{-1} is holomorphic.

4. For every real numer $0 < a < \infty$ let L_a be the straight line
$$L_a = \{z \in \mathbb{C} | \operatorname{Re}(z) = a\}.$$
Prove that the images $f(L_a)$ are parabolas.

5. Prove that the *focus* of each parabola $f(L_a)$ is the origin.

6. For each real number b let M_b be the ray
$$M_b = \{z \in D | \operatorname{Im}(z) = b\}.$$
Since f is conformal, the sets $f(M_b)$ and the parabolas $f(L_a)$ are orthogonal to one another.

   Describe the sets $f(M_b)$.

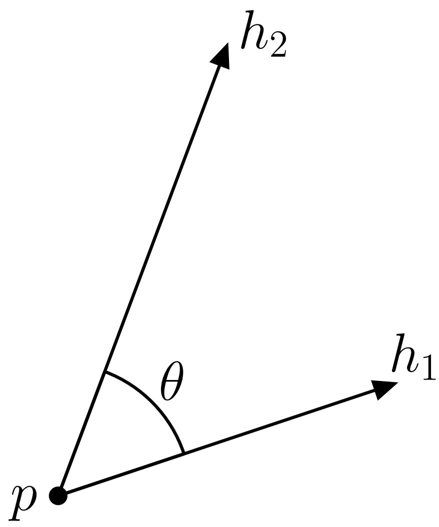
Section D: Conformal Transformations

Roughly speaking, the adjective conformal refers to the preservation of angles. More specifically, consider a situation in which a function F from one type of region to another is differentiable in the vector calculus sense. And consider a point p and its image $F(p)$. Calculus then enables us to move tangent vectors at p to tangent vectors at $F(p)$... some sort of notation like this is frequently used:

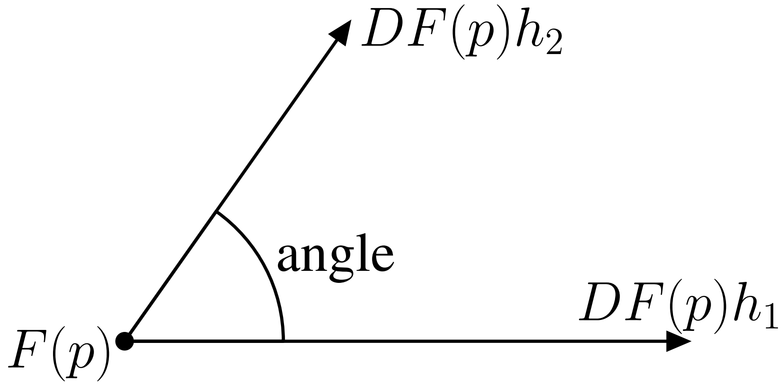
$$h = (\text{a tangent vector at } p) \longrightarrow DF(p)h.$$

Here $DF(p)$ is often called the *Jacobian matrix* of F at p, and the symbol $DF(p)h$ refers to multiplication of a matrix and a vector.

Then if h_1 and h_2 are nonzero tangent vectors at p, they have a certain angle θ between them:



We are interested in the angle between the images under F of these tangent vectors:



If this angle is also θ and this happens at every p and for all tangent vectors, we say that F is a *conformal* transformation. Tersely:

conformal means *angle preserving*

Examples from multivariable calculus:

Mercator projections of the earth;
stereographic projections.

Now we particularize this for holomorphic functions. So assume that f is holomorphic and that for a fixed point z we know that $f'(z) \neq 0$. Let the polar form of this number be

$$f'(z) = Ae^{i\alpha} \qquad \text{(where } A > 0, \alpha \in \mathbb{R}\text{)}.$$

By definition

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}.$$

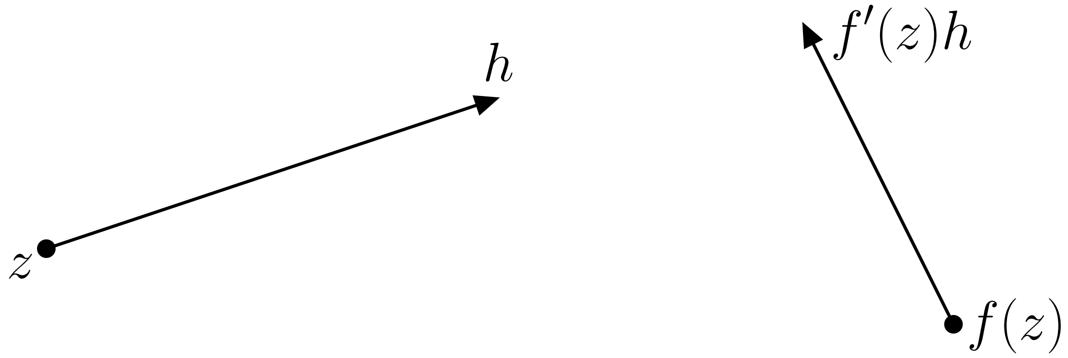
Rewrite this relationship as

$$f(z+h) = f(z) + f'(z)h \quad \text{approximately.}$$

This means that f transforms a <u>tangent vector</u> h at z to the vector at $f(z)$ given by

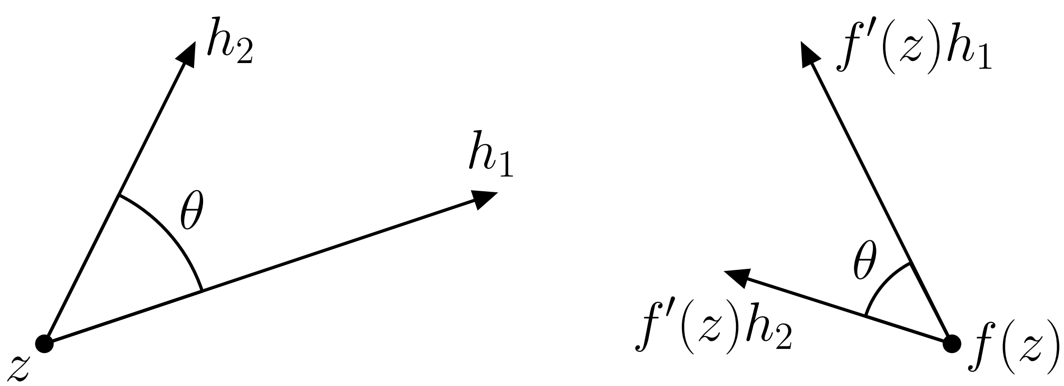
$$f'(z)h.$$

In other words, <u>directions</u> h at z are transformed to <u>directions</u> $f'(z)h$ at $f(z)$:



This action does two things to h: (1) multiplies its modulus by A, and (2) rotates it by the angle α.

We conclude immediately that f preserves angles:

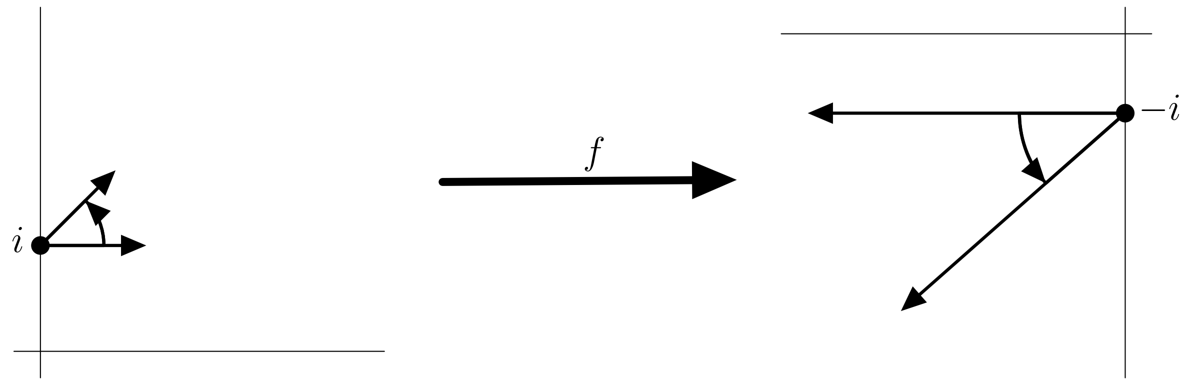


The moduli of all the infinitesimal vectors at z are multiplied by the same positive number A.

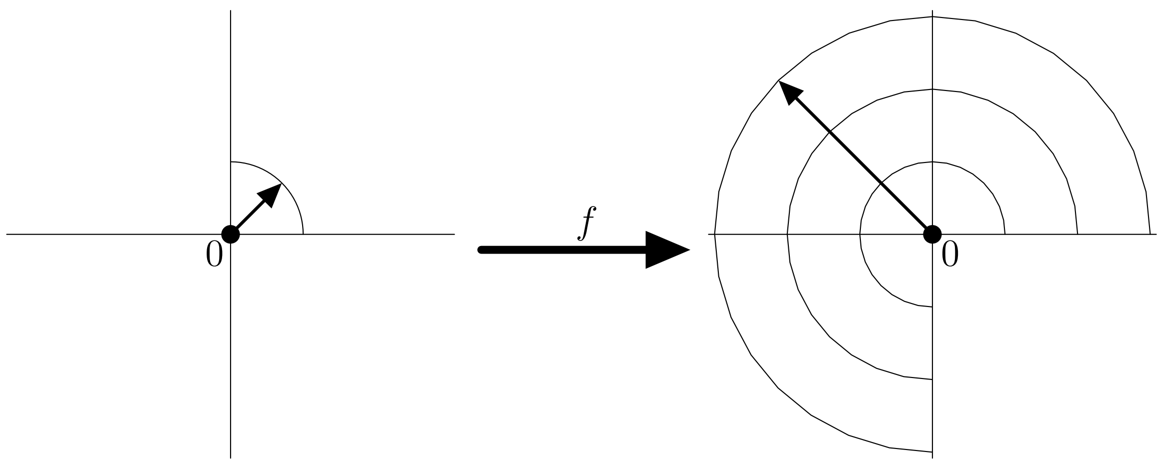
SUMMARY: Every holomorphic function f is conformal at every z with $f'(z) \neq 0$. Infinitesimal vectors at z are magnified by the positive number $|f'(z)|$.

Example:

$$f(z) = z^3;$$
$$z = i \to f'(i) = -3;$$
$$f(i) = -i.$$



But notice that $f'(0) = 0$ and f does not preserve angles at 0 — instead, it multiplies them by 3.



Section E: (Complex) Power Series

(1) Infinite series of complex numbers

We shall need to discuss $\sum_{n=0}^{\infty} a_n$, where $a_n \in \mathbb{C}$. Convergence of such series is no mystery at all. We form the sequence $\{s_N\}_{N=1}^{\infty}$ of partial sums

$$s_N = a_0 + \ldots + a_N,$$

and just demand that

$$\lim_{N\to\infty} s_N = L \text{ exists.}$$

Then we say

$$\sum_{n=0}^{\infty} a_n = L \text{ is convergent.}$$

Equivalently, we could reduce everything to two real series, require that they converge, and then

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \mathrm{Re}(a_n) + i \sum_{n=0}^{\infty} \mathrm{Im}(a_n).$$

Necessarily, if a series converges, then $\lim_{n\to\infty} a_n = 0$ (for $a_N = s_N - s_{N-1} \to L - L = 0$).

The converse is, of course, false: the "harmonic series" $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges.

Absolute convergence is what we will usually see. We say that $\sum_{n=0}^{\infty} a_n$ *converges absolutely* if $\sum_{n=0}^{\infty} |a_n|$ converges. Then there is an important

THEOREM: *If a series converges absolutely, then it converges.*

(Note: The basic calculus proof relies on the completeness of $\mathbb{R}$.)

(2) Most important example of a power series — the **GEOMETRIC SERIES**

$$\sum_{n=0}^{\infty} z^n, \text{ where } z \in \mathbb{C}.$$

By our necessity condition, if this series converges, then $z^n \to 0$. That is, $|z|^n = |z^n| \to 0$. That is, $|z| < 1$.

Conversely, suppose $|z| < 1$. Then

$$s_n = 1 + z + \ldots + z^N = \frac{1 - z^{N+1}}{1 - z} \qquad (z \neq 1 \text{ of course})$$
$$= \frac{1}{1-z} - \frac{z^{N+1}}{1-z}.$$

Now simply note that $\left| -\dfrac{z^{N+1}}{1-z} \right| = \dfrac{|z|^{N+1}}{|1-z|} \to 0$, because $|z| < 1$.

SUMMARY: $\sum\limits_{n=0}^{\infty} z^n$ converges $\Leftrightarrow$ $|z| < 1$. And then it converges absolutely, and

$$\boxed{\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.}$$

(3) DEFINITION: A *power series centered at* z_0 is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the coefficients a_n are complex numbers.

(★) Usually in developing the properties of such series, we will work with the center $z_0 = 0$.

Simple warning: the first term in this series is not really $a_0(z - z_0)^0$, but it is actually a lazy way of writing the constant a_0. A more legitimate expression would be

$$a_0 + \sum_{n=1}^{\infty} a_n (z - z_0)^n \quad \ldots \text{no one ever bothers.}$$

THEOREM: (easy but crucial!): *If a power series*

$$\sum_{n=0}^{\infty} a_n z^n$$

converges when $z = z_1$, *and if* $|z_2| < |z_1|$, *then it converges absolutely when* $z = z_2$.

(easy) Proof:

$$\sum_{n=0}^{\infty} a_n z_1^n \text{ converges} \Rightarrow \lim_{n \to \infty} a_n z_1^n = 0 \Rightarrow |a_n z_1^n| \leq \text{a constant } C \text{ for all } n \geq 0.$$

Therefore,

$$|a_n z_2^n| \leq C|z_1|^{-n}|z_2|^n = C\left(\frac{|z_2|}{|z_1|}\right)^n.$$

Since $\frac{|z_2|}{|z_1|} < 1$, the geometric series $\sum_{n=0}^{\infty}\left(\frac{|z_2|}{|z_1|}\right)^n$ converges. Therefore,

$$\sum_{n=0}^{\infty} |a_n z_2^n| \text{ converges.}$$

That is,

$$\sum_{n=0}^{\infty} a_n z_2^n \text{ converges absolutely.}$$

QED

RADIUS OF CONVERGENCE

It is an easy but extremely important fact that every power series has associated with it a unique $0 \leq R \leq \infty$ such that

$$\begin{cases} |z| < R \Rightarrow \text{the power series converges absolutely at } z, \\ |z| > R \Rightarrow \text{the power series diverges at } z. \end{cases}$$

This is a quick result from what we have just proved.

There is actually a formula for R in general, but it will not be needed by us. Just to be complete, here is that formula:

$$\boxed{R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}} \qquad \text{(Cauchy-Hadamard)}$$

Useful observation: suppose $|z| < R$, where R is the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$. Choose any z_1 such that $|z| < |z_1| < R$. Then from the preceding proof we have the estimate

$$|a_n| \leq C|z_1^{-n}|.$$

Now consider the quantity $na_n z^n$:

$$|na_n z^n| \leq Cn\left(\frac{|z|}{|z_1|}\right)^n.$$

Since $\frac{|z|}{|z_1|} < 1$, the real series

$$\sum_{n=0}^{\infty} n\left(\frac{|z|}{|z_1|}\right)^n$$

converges. (We can actually appeal to the basic calculus ratio test to check this.) Therefore,

$$\sum_{n=0}^{\infty} |na_n z^n| < \infty.$$

Thus, not only does $\sum_{n=0}^{\infty} a_n z^n$ converge absolutely, but the series with larger coefficients na_n also converges absolutely…remember, $|z| < R$.

CONCLUSION: multiplying the coefficients a_n of a power series by n does not change the radius of convergence.

RATIO TEST

We just mentioned this result of basic calculus, namely, suppose that a series of positive numbers $\sum_{n=0}^{\infty} c_n$ has the property that

$$\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \ell \text{ exists.}$$

Then,

$$\begin{cases} \ell < 1 \Rightarrow \text{the series converges,} \\ \ell > 1 \Rightarrow \text{the series diverges.} \end{cases} \qquad \left(\begin{array}{c} \ell = 1 : \text{no conclusion} \\ \text{in general} \end{array} \right)$$

And now we apply this to power series $\sum_{n=0}^{\infty} a_n z^n$ with the property that

$$\boxed{\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \ell \text{ exists}}$$

Then we can apply the ratio test to the series $\sum_{n=0}^{\infty} |a_n z^n|$, since

$$\lim_{n\to\infty} \frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} = \ell |z|.$$

Thus,

$$\begin{cases} \ell|z| < 1 \Rightarrow \text{convergence,} \\ \ell|z| > 1 \Rightarrow \text{divergence.} \end{cases}$$

That is, the radius of convergence of the power series equals

$$\boxed{R = \frac{1}{\ell}}$$

EXAMPLES:

- $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad R = \infty;$
- $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad R = 1;$
- $\sum_{n=0}^{\infty} n! z^n \qquad R = 0.$

Also, convergence for $|z| = R$ can happen variously:

$$\begin{cases} \sum_{n=0}^{\infty} z^n & \text{diverges for all } |z| = 1; \\ \sum_{n=1}^{\infty} \frac{z^n}{n^2} & \text{converges for all } |z| = 1; \\ \sum_{n=1}^{\infty} \frac{z^n}{n} & \text{diverges for } z = 1, \text{ converges for all other } |z| = 1. \end{cases}$$

(Note: in this last example we do not know convergence at the present time in this book, but we'll see it soon enough.)

<u>SIMPLE PROPERTIES OF POWER SERIES</u>

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R_1, and let $g(z) = \sum_{n=0}^{\infty} b_n z^n$ have radius of convergence R_2.

$\boxed{\text{SUM}}$ $f(z)+g(z) = \sum_{n=0}^{\infty} (a_n+b_n) z^n$ has radius of convergence $\geq \min(R_1, R_2)$.

$\boxed{\text{PRODUCT}}$ $f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$ has radius of convergence $\geq \min(R_1, R_2)$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. (Cauchy Product)

$\boxed{\text{DERIVATIVE}}$ For $|z| < R_1$, the function f has a complex derivative, and $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ has the <u>same</u> radius of convergence.

(★) We will soon be able to prove the fact about products and this fact about $f'(z)$ with very little effort, almost no calculation involved. But I want to show you a direct proof for $f'(z)$. So let $|z| < R_1$ be fixed and $h \in \mathbb{C}$ with small modulus, so that in particular $|z| + |h| < R_1$. Then we compute

$$
\begin{aligned}
f(z+h) - f(z) - h\sum_{n=1}^{\infty} na_n z^{n-1} &= \sum_{n=1}^{\infty} \left[a_n(z+h)^n - a_n z^n - na_n z^{n-1}h\right] \\
&= \sum_{n=2}^{\infty} a_n \left[(z+h)^n - z^n - nz^{n-1}h\right] \\
\text{binomial theorem} \qquad &= \sum_{n=2}^{\infty} a_n \left[\sum_{k=0}^{n} \binom{n}{k} z^{n-k}h^k - z^n - nz^{n-1}h\right] \\
&= \sum_{n=2}^{\infty} a_n \left[\sum_{k=2}^{n} \binom{n}{k} z^{n-k}h^k\right] \\
&= h^2 \sum_{n=2}^{\infty} a_n \left[\sum_{k=2}^{n} \binom{n}{k} z^{n-k}h^{k-2}\right].
\end{aligned}
$$

Divide by h:

$$\frac{f(z+h) - f(z)}{h} - \sum_{n=1}^{\infty} na_n z^{n-1} = h\sum_{n=2}^{\infty} a_n \left[\sum_{k=2}^{n} \binom{n}{k} z^{n-k}h^{k-2}\right].$$

It follows easily that $f'(z)$ exists and equals $\sum_{n=1}^{\infty} na_n z^{n-1}$.

Therefore,

Every power series is holomorphic on its open disc of convergence.

PROBLEM 2-2. A power series centered at 0 is often called a *Maclaurin* series.

In the following exercises simplify your answers as much as possible.

1. Find the Maclaurin series for $\dfrac{1}{(1-z)^3}$.
2. Find the Maclaurin series for $\left(\dfrac{z}{3-z}\right)^2$.
3. Find the Maclaurin series for $e^z \sin z$.
4. Let $\omega = e^{2\pi i/3}$. $(1 + \omega + \omega^2 = 0)$.
   Find the Maclaurin series for $\dfrac{e^z + e^{\omega z} + e^{\omega^2 z}}{3}$.

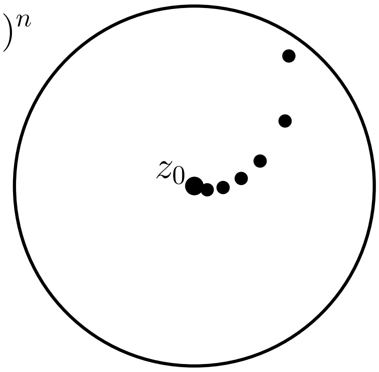
5. Find explicitly $\sum_{n=0}^{\infty}(-1)^n\frac{(z-\pi i)^n}{n!}$.

6. Find explicitly $\sum_{n=0}^{\infty}\frac{z^{5n}}{5^n}$.

<u>MORE BASIC RESULTS ABOUT POWER SERIES</u>:

First, a very simple theorem which will have profound consequences!

THEOREM: *Suppose that* $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ *is a power series with a positive radius of convergence. And suppose that* $f(z)=0$ *for an infinite sequence of points* z *converging to* z_0. *Then* $f=0$. *In other words,* $a_n=0$ *for all* n.



Proof: We assume $z_0=0$ with no loss of generality. Out proof is by contradiction, so we suppose that not all $a_n=0$. Then we have $a_N\neq 0$ for a smallest N, so that

$$
\begin{aligned}
f(z) &= \sum_{n=N}^{\infty}a_nz^n\\
&= z^N\sum_{n=N}^{\infty}a_nz^{n-N}\\
&=: z^Ng(z),
\end{aligned}
$$

where $g(z)$ is the power series

$$
\begin{aligned}
g(z) &= \sum_{k=0}^{\infty}a_{N+k}z^k\\
&= a_N+a_{N+1}z+\ldots.
\end{aligned}
$$

Then $f(z)=0$ and $z\neq 0\Rightarrow g(z)=0$. Therefore, our hypothesis implies that $g(z)=0$ for an infinite sequence of points z converging to 0. But $\lim_{z\to 0}g(z)=g(0)=a_N$. Thus, $a_N=0$. Contradiction.

QED

TAYLOR SERIES:

Again, we suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ is a power series with positive radius of convergence. Then we observe

$$\begin{aligned} f(z_0) &= a_0; \\ f'(z) &= \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}, \text{ so } f'(z_0) = a_1; \\ f''(z) &= \sum_{n=2}^{\infty} n(n-1) a_n (z-z_0)^{n-2}, \text{ so } f''(z_0) = 2a_2. \end{aligned}$$

In this manner, we find

$$f^{(k)}(z_0) = k! a_k.$$

Therefore,

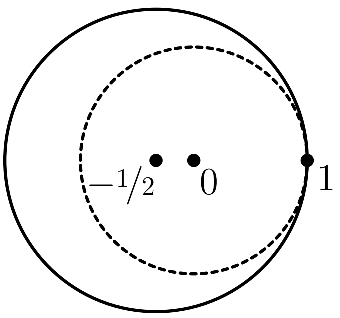
$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n}$$

The right side of this equation is called *the Taylor series of f centered at z_0*.

(If $z_0 = 0$, it is called the Maclaurin series of f.)

Changing the center of power series:
First, a couple of examples:

Example 1: $f(z) = \sum_{n=0}^{\infty} z^n$ for $|z| < R = 1$, the geometric series.



Let's investigate an expansion of $f(z)$ centered instead at $-\frac{1}{2}$. thus, we write

$$\begin{aligned} f(z) &= \frac{1}{1-z} && \text{(sum of geometric series)} \\ &= \frac{1}{3/2 - (z + 1/2)} \\ &= \frac{2}{3} \frac{1}{1 - \frac{z+1/2}{3/2}} \\ &= \frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{z + 1/2}{3/2} \right)^n && \text{(a different geometric series),} \end{aligned}$$

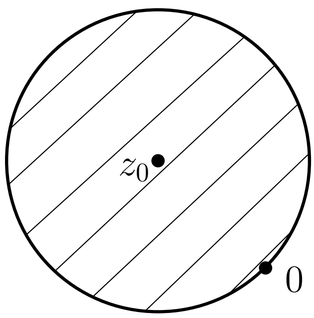
and this series converges in the disk $\left|z+\frac{1}{2}\right| < \frac{3}{2}\cdots$. Therefore,

$$f(z) = \sum_{n=0}^{\infty} \frac{(z+{}^1\!/_2)^n}{({}^3\!/_2)^{n+1}}.$$

<u>Example 2</u>: $f(z) = \frac{1}{z}$, and we want to express this in a power series centered at $z_0 \neq 0$. Then as in the preceding example, we write

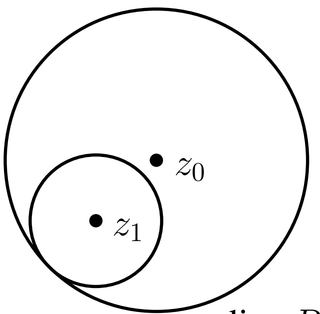
$$\begin{aligned} f(z) &= \frac{1}{z_0 + (z - z_0)} \\ &= \frac{1}{z_0}\frac{1}{1+\frac{z-z_0}{z_0}} \\ &= \frac{1}{z_0}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-z_0}{z_0}\right)^n && \text{(geometric series)} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{z_0^{n+1}}(z-z_0)^n, \end{aligned}$$

a Taylor series with radius of convergence $|z_0|$:



A very general <u>theorem</u>:

> *Let $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series with radius of convergence R, and assume $|z_1 - z_0| < R$. Then $f(z) = \sum_{n=0}^{\infty} b_n(z-z_1)^n$ and the radius of convergence of this new series is $\geq R - |z_1 - z_0|$.*



Although it is easy enough to prove this theorem with basic manipulations we already know, such a proof is tedious and boring. We will soon be able to prove this theorem and many others elegantly with almost no effort at all!

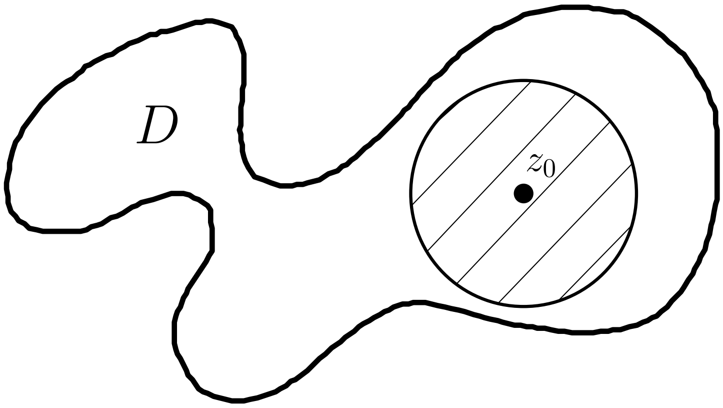
By the way, notice that in Example 1, $R = 1$. amd the new power series has radius of convergence $\frac{3}{2}$.

These ideas lead us to an important:

DEFINITION: Suppose f is a $\mathbb{C}$-valued function defined on an open subset $D \subset \mathbb{C}$, and suppose that for every $z_0 \in D$ we are able to write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ for all } |z - z_0| < R(z_0),$$

where $R(z_0)$ is some positive number. Then we say that f is *(complex) analytic* on D.



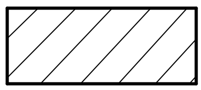
It is then quite clear that ***every analytic function is holomorphic***.

After we obtain Cauchy's integral formula, we will see that the exact converse is valid:

★ **every holomorphic function is analytic!**

We conclude this chapter with the important Taylor series for logarithm. We'll treat $\log(1-z)$. The principle involved here is based on simple single-variable calculus.

LEMMA: *Suppose f has partial derivatives of first order which satisfy $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$ on a rectangle $(x_0, x_1) \times (y_0, y_1)$. Then f is constant on that rectangle.*



Theorem: *Suppose $D \subset \mathbb{C}$ is an open connected set and $D \xrightarrow{f} \mathbb{C}$ has partial derivatives of first order which satisfy*

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \text{ on } D.$$

Then f is constant on D.

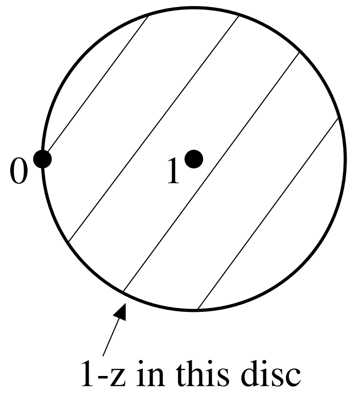
Proof: By the lemma, f is constant on all closed rectangles contained in D. Since D is *connected*, f is constant on D.

QED

COROLLARY: *Suppose $D \subset \mathbb{C}$ is an open connected set and $D \xrightarrow{f} \mathbb{C}$ is holomorphic on D with $f'(z) = 0$ for all $z \in D$. Then f is constant.*

Illustration: For $|z| < 1$ the number $1 - z$ can be chosen to have $-\frac{\pi}{2} < \arg(1-z) < \frac{\pi}{2}$. Then

$$\frac{d}{dz}\log(1-z) = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n = -\frac{d}{dz}\sum_{n=1}^{\infty}\frac{z^n}{n}.$$



Thus $\log(1-z) + \sum_{n=1}^{\infty}\frac{z^n}{n}$ satisfies the hypothesis of the corollary for $|z| < 1$, and is thus constant. At $z = 0$ it equals 0. Therefore

$$\boldsymbol{\log(1-z) = -\sum_{n=1}^{\infty}\frac{z^n}{n} \quad \text{for } |z| < 1.}$$

Chapter 3

Integration

In this chapter we begin with a review of multivariable calculus for $\mathbb{R}^2$, stressing the concept of **line integrals** and especially as they arise in Green's theorem. We can easily derive what Green's theorem looks like using complex notation. A huge result will then be easily obtained: the **Cauchy Integral Theorem**.

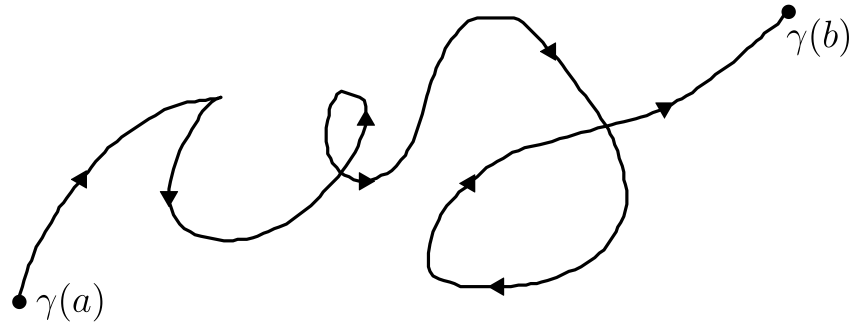
Section A: Line Integrals

<u>REVIEW OF VECTOR CALCULUS</u>:

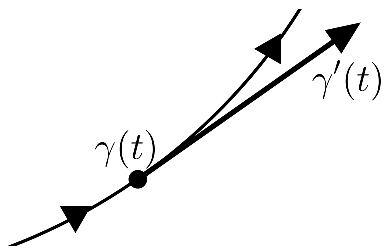
The particular thing we need is called *line integration* or *path integration* or *contour integration*. It is based on curves in $\mathbb{R}^n$, which we'll typically denote by γ. These will need to be given a *parametrization* (at least in theory, if not explicitly) so that γ can be thought of as a function defined on an interval $[a, b] \subset \mathbb{R}$ with values in $\mathbb{R}^n$:

$$[a, b] \xrightarrow{\gamma} \mathbb{R}^n.$$

We'll need γ to be piecewise C^1. Its shape in $\mathbb{R}^n$ may look something like this:



Notice that as t varies from a to b, $\gamma(t)$ moves in a definite direction. And $\gamma'(t) = \dfrac{d\gamma}{dt}$ represents a vector in $\mathbb{R}^n$ which is tangent to the curve. Thinking of t as time, this vector is called the *velocity* of the curve at time t.



For any $1 \leq j \leq n$ we then define the *line integral* of a function f *along* γ, in the x_j direction as

$$\int_\gamma f dx_j := \int_a^b f(\gamma(t))\gamma_j'(t)dt.$$

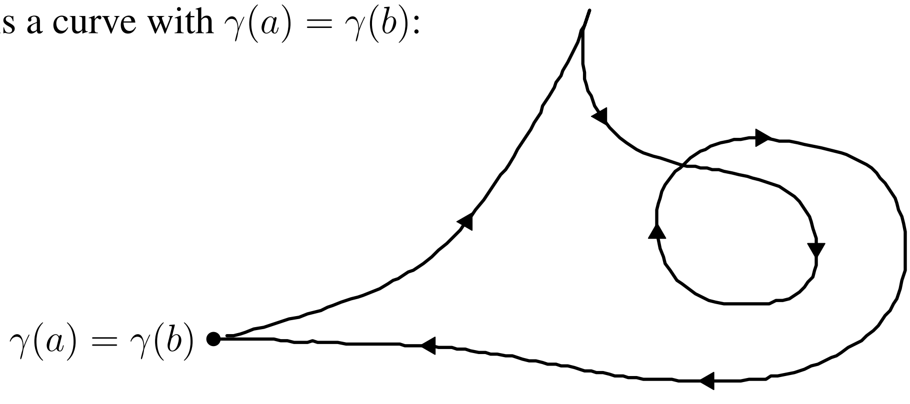
Here we are using the standard coordinate representation

$$\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t)).$$

The chain rule shows that this result is independent of "reasonable" changes of parametrization. But if we replace t by $-t$, the curve is traced in the opposite direction, so that

$$\int_{\substack{\text{REVERSED}\\ \gamma}} f dx_j = -\int_\gamma f dx_j.$$

A *loop* is a curve with $\gamma(a) = \gamma(b)$:

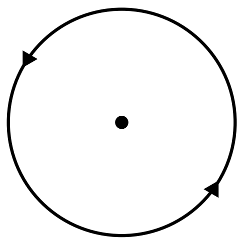


<u>Complex-valued f</u>: No difficulty with this at all, as the integral of a complex-valued function is given as

$$\int_a^b (g(t) + ih(t))dt = \int_a^b g(t)dt + i\int_a^b h(t)dt.$$

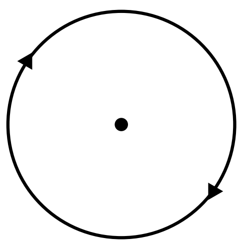
Special notation for $\mathbb{R}^2$: Usually we use x and y instead of x_1 and x_2.

Example:



$$\begin{aligned}\int_{\substack{\text{CCW unit}\\ \text{circle}}} \frac{1}{z}dx &= \int_0^{2\pi} \frac{1}{e^{i\theta}} d(\cos\theta)\\ &= \int_0^{2\pi} e^{-i\theta}(-\sin\theta)d\theta\\ &= \int_0^{2\pi}(\cos\theta - i\sin\theta)(-\sin\theta)d\theta\\ &= 0 + i\int_0^{2\pi}\sin^2\theta d\theta = \pi i.\end{aligned}$$

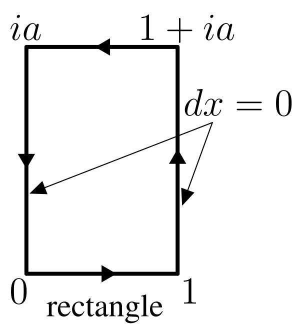
Example: Let the path γ be the clockwise circle with center 0 and radius r. Then



$$\begin{aligned}\int_\gamma \frac{1}{z^2}dy &= -\int_0^{2\pi}\frac{1}{(re^{i\theta})^2}d(r\sin\theta)\\ &= -\frac{1}{r}\int_0^{2\pi} e^{-2i\theta}\cos\theta d\theta\\ &= -\frac{1}{r}\int_0^{2\pi} e^{-2i\theta}\frac{e^{i\theta}+e^{-i\theta}}{2}d\theta\\ &= -\frac{1}{2r}\int_0^{2\pi}(e^{-i\theta}+e^{-3i\theta})d\theta\\ &= 0.\end{aligned}$$

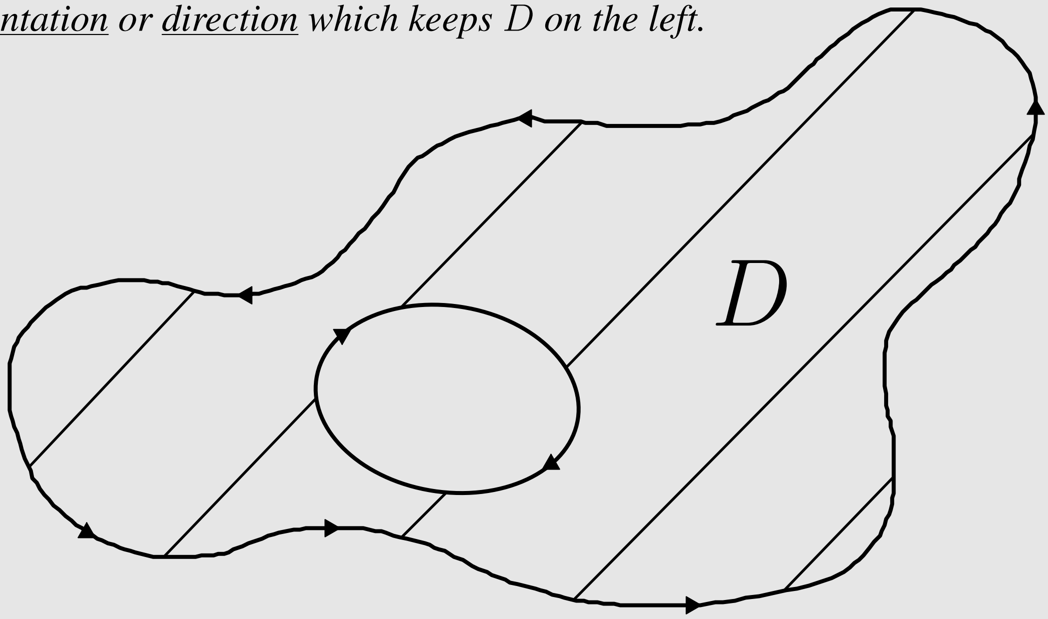
Example:

$$\begin{aligned}\int_\gamma e^z dx &= \int_0^1 e^x dx + 0 + \int_1^0 e^{x+ia}dx + 0\\ &= (1-e^{ia})\int_0^1 e^x dx\\ &= (1-e^{ia})(e-1).\end{aligned}$$

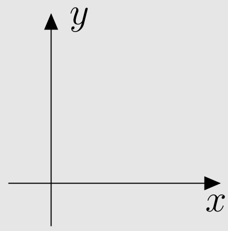


Of special importance to us is **Green's Theorem**:

If D is a "reasonably nice" bounded region, then we can consider ∂D, the boundary of D, as a curve or a union of curves, and we always give it the orientation or direction which keeps D on the left.



Notice the orientation of the coordinate axes! :



Then for a C^1 function f we have

$$\iint_D \frac{\partial f}{\partial x} dxdy = \int_{\partial D} f dy,$$
$$\iint_D \frac{\partial f}{\partial y} dxdy = -\int_{\partial D} f dx.$$

Usually these are presented as a single formula:

GREEN: $$\iint_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dxdy = \int_{\partial D} (f dx + g dy)$$

Remember: f and g are allowed to be complex-valued functions.

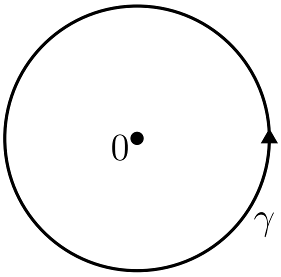
Complex line integrals: Not only can the functions we are integrating be complex valued, but also we can integrate with respect to dz: just think $dz = d(x+iy) = dx + idy$. Then we write

$$\int_\gamma f dz = \int_\gamma f dx + i \int_\gamma f dy.$$

Most important example:

$$\int_\gamma \frac{1}{z} dz \ldots \text{parametrize with } z = re^{i\theta},\ 0 \le \theta \le 2\pi :$$

$$\begin{aligned}
\int_\gamma \frac{1}{z}dz &= \int_0^{2\pi} \frac{1}{re^{i\theta}} d(re^{i\theta}) \\
&= \int_0^{2\pi} \frac{1}{re^{i\theta}} (rie^{i\theta}d\theta) \\
&= \int_0^{2\pi} id\theta = 2\pi i.
\end{aligned}$$



CCW circle of radius r
centered at 0

$$\boxed{\int_\gamma \frac{\mathbf{1}}{\boldsymbol{z}}\boldsymbol{dz} = \mathbf{2}\boldsymbol{\pi i}}$$

<u>Another derivation of this formula</u>: Using the facts that $0 \le \arg z \le 2\pi$ and $\log z = \ln r + i \arg z$, we get

$$\begin{aligned}
\int_\gamma \frac{dz}{z} &= \int_\gamma d(\log z) \\
&\overset{\text{FTC}}{=} \log z \Big|_{z=re^{0i}}^{z=re^{2\pi i}} \\
&= (\ln r + 2\pi i) - (\ln r + 0i) \\
&= 2\pi i.
\end{aligned}$$

<u>Special application of Green</u>: use a function f and $g = if$:

$$\iint_D \left(i\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy = \int_{\partial D} f dx + i f dy.$$

Rewrite:

$$\boxed{\int_{\partial D} f dz = i \iint_D \left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y} \right) dxdy}$$

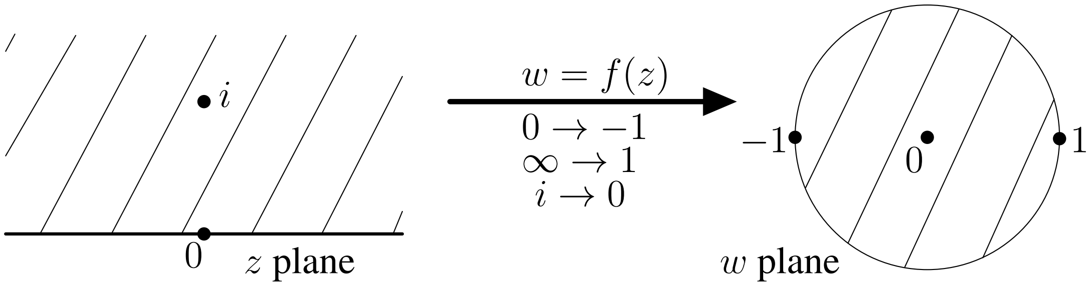
Hmmm: notice the interesting combination in the integrand on the right side! (Think about Cauchy-Riemann!)

PROBLEM 3-1. We know that there is a unique Möbius transformation f of $\hat{\mathbb{C}}$ which satisfies

$$\begin{cases} f(0) = -1, \\ f(\infty) = 1, \\ f(i) = 0. \end{cases}$$

This Möbuis function is called the ***Cayley transformation***.

1. Write explicitly $f(z) = \dfrac{az+b}{cz+d}$ (i.e. find a, b, c, d).
2. Prove that $f(\mathbb{R} \cup \{\infty\}) =$ the unit circle.
3. Prove that f(open upper half plane) $=$ open unit disc.



4. For several values of $y > 0$ sketch the image of the horizontal straight lines in the upper half plane.

$$\{f(x+iy) | x \in \mathbb{R} \cup \{\infty\}\}.$$

Next, we mention this simple concept:

<u>The fundamental theorem of calculus and line integrals</u>

There's a simple theorem in $\mathbb{R}^n$ vector calculus concerning the line integral of a conservative vector field. Its proof relies on the FTC and looks like this:

$$\boxed{\int_a^b \frac{d}{dt} f(\gamma(t))dt = f(\gamma(b)) - f(\gamma(a))}$$

THE FUNDAMENTAL THEOREM OF CALCULUS AND LINE INTEGRALS

Let γ be a curve in $\mathbb{C}$ and f a holomorphic function. Then

(FTC) $\boxed{\int_\gamma f'(z)dz = f(\text{final point of } \gamma) - f(\text{initial point of } \gamma)}$

Proof: Let $\gamma = \gamma(t)$ for $a \le t \le b$. Then by definition

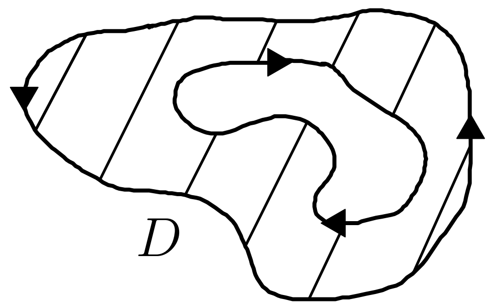
$$\begin{aligned}\int_\gamma f'(z)dz &= \int_a^b f'(\gamma(t))\gamma'(t)dt \\ &\overset{\text{chain rule}}{=} \int_a^b \frac{d}{dt}\left(f(\gamma(t))\right)dt \\ &\overset{\text{FTC}}{=} f(\gamma(t))\Big|_a^b \\ &= f(\gamma(b)) - f(\gamma(a)).\end{aligned}$$

QED

Section B: The Cauchy Integral Theorem

At the end of Section A we used Green's theorem to prove that

$$\int_{\partial D} f dz = i \iint_D \left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) dxdy.$$



Notice that if f is holomorphic, then the Cauchy-Riemann equation, $\dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}$, gives a zero integrand on the right side of the Green equation, so that $\int_{\partial D} f dz = 0$. We now state this as a separate theorem:

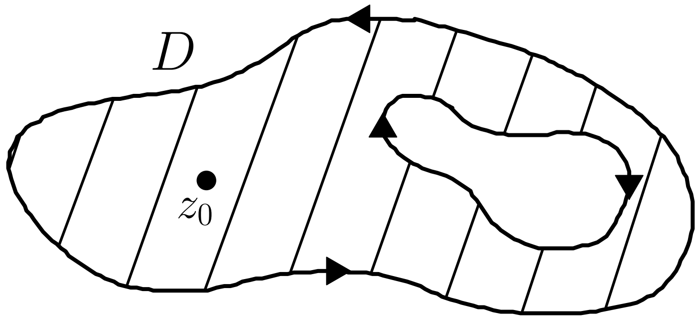
THE CAUCHY INTEGRAL THEOREM

Suppose $D \subset \mathbb{C}$ is a "reasonably nice" bounded open set with boundary ∂D consisting of finitely many curves oriented with D on the left. Suppose f is a holomorphic function defined on an open set containing $D \cup \partial D$. Then

$$\int_{\partial D} f dz = 0.$$

We are now going to use this theorem to prove a truly amazing theorem, *Cauchy's integral formula*, which will be the basis for much of our subsequent study.

We assume the hypothesis exactly as above, but in addition we assume that a point $z_0 \in D$ is fixed... remember that D is open, so $z_0 \notin \partial D$:



We want to apply the Cauchy integral theorem to the function

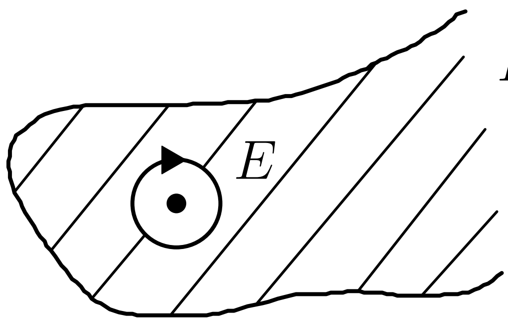
$$\frac{f(z)}{z - z_0},$$

but this function is not even defined at z_0.

The way around this difficulty is extremely clever, and also a strategy that is often used in similar situations not just in complex analysis, but also in partial differential equations and other places. It is the following

ruse: extract a small disc centered at z_0**!** Namely, let E the be the closed disc of radius ε centered at z_0:

$$E = \{z \in \mathbb{C} \big| |z - z_0| \leq \varepsilon\}.$$



(E is called a *safety disc*.)

Then for sufficiently small ε we see that $E \subset D$, since D is open, and we may apply the Cauchy integral theorem to the difference

$$D \setminus E.$$

We obtain

$$0 = \int\limits_{\partial(D \setminus E)} \frac{f(z)}{z - z_0} dz.$$

Now $\partial(D \backslash E)$ is the disjoint union of ∂D and ∂E, so we have, using the correct orientation,

$$0 = \int_{\partial D} \frac{f(z)}{z - z_0} dz + \int_{\substack{\partial E \\ \text{CW circle}}} \frac{f(z)}{z - z_0} dz.$$

Move the second integral to the left side and reverse the direction of the circle ∂E:

$$\int_{\substack{\partial E \\ \text{CCW circle}}} \frac{f(z)}{z - z_0} dz = \int_{\partial D} \frac{f(z)}{z - z_0} dz.$$

Fascinating equation! The right side is independent of ε, and thus so is the left side!

Parametrize ∂E: $z = z_0 + \varepsilon e^{i\theta}, 0 \leq \theta \leq 2\pi$, so the left side equals:

$$\int_0^{2\pi} \frac{f(z_0 + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}} \varepsilon i e^{i\theta} d\theta = i \int_0^{2\pi} f(z_0 + \varepsilon e^{i\theta}) d\theta.$$

This can be rewritten as:

$$2\pi i \text{ times } \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \varepsilon e^{i\theta}) d\theta = 2\pi i \text{ times } \underline{\text{the average of } f \text{ on } \partial E}.$$

This does not depend on ε! Yet, it has a clear limit as $\varepsilon \to 0$, since f is continuous at z_0: namely, $2\pi i f(z_0)$. Therefore,

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz.$$

BEWARE: notation change coming up — z_0 is replaced by z,
z is replaced by ZETA: ζ.

Final result

THE CAUCHY INTEGRAL FORMULA

Same hypothesis as the Cauchy integral theorem. Then for every $z \in D$

$$\boxed{f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta}$$

PROBLEM 3-2. Give examples of two power series centered at 0 with the following properties:

$$\begin{aligned} &f(z) \text{ has radius of convergence } 1, \\ &g(z) \text{ has radius of convergence } 2, \\ &f(z)g(z) \text{ has radius of convergence } 10. \end{aligned}$$

Section C: Consequences of the Cauchy Integral Formula

We now derive very quickly many astonishing consequences of the Cauchy integral formula.

(1) Holomorphic functions are C^∞

This is rather stunning given that the definition of holomorphic required f to be of class C^1 and satisfy the Cauchy-Riemann equation. The key to this observation is that the dependence of $f(z)$ on z has now been relegated to the simple function $\dfrac{1}{\zeta - z}$:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

For $z \in D$ (open set) and $\zeta \in \partial D$, the function $\dfrac{1}{\zeta - z}$ is quite well behaved and we have for fixed ζ:

$$\frac{d}{dz}\frac{1}{\zeta - z} = \frac{1}{(\zeta - z)^2}.$$

Therefore, by performing $\dfrac{d}{dz}$ through the integral sign we obtain

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

We already knew $f'(z)$ existed, but now our same observation shows that $f'(z)$ has a complex derivative (we didn't know that before!), and that

$$f''(z) = \frac{2}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^3} d\zeta.$$

Continuing in this manner, we see that

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

QED

In particular,

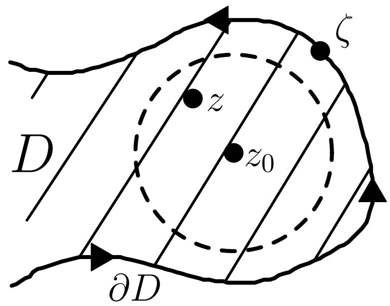
(2) f holomorphic $\Rightarrow$ f' is holomorphic

Now we can also fulfill the promise made near the end of Chapter 2 (page 39):

(3) Every holomorphic function is analytic

Once again, the key to this is the nature of $\dfrac{1}{\zeta - z}$. We establish a power series expansion in a disc centered at an arbitrary point $z_0 \in D$. As D is open, there exists $a > 0$ such that $|\zeta - z| \geq a$ for all $\zeta \in \partial D$. We then suppose that

$$|z - z_0| < a.$$



Looking for geometric series, we have

$$\begin{aligned}
\frac{1}{\zeta - z} &= \frac{1}{\underset{\text{BIG}}{(\zeta - z_0)} - \underset{\text{SMALL}}{(z - z_0)}} \\
&= \frac{1}{\zeta - z_0}\frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} \\
&= \frac{1}{\zeta - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n \\
&= \sum_{n=0}^{\infty}\frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}.
\end{aligned}$$

Since $\left|\dfrac{z-z_0}{\zeta-z_0}\right| \le \dfrac{|z-z_0|}{a} < 1$ for all $\zeta \in \partial D$, we have *uniform convergence* of the geometric series (rate of convergence same for all $\zeta \in \partial D$) and we conclude that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\partial D} f(\zeta)\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\zeta-z_0)^{n+1}}\,d\zeta \\ &= \sum_{n=0}^{\infty} c_n(z-z_0)^n, \qquad \text{(interchanged order of summation and integration)} \end{aligned}$$

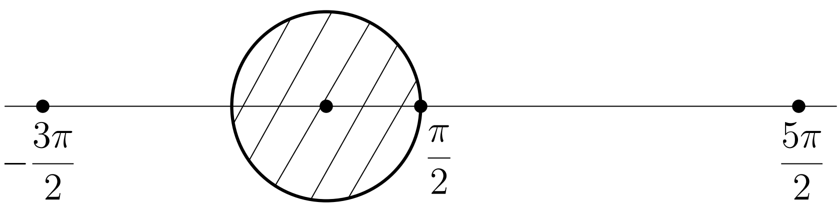
where the coefficients are given by

$$c_n = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta.$$

(By the way, notice from part 1 above that $c_n = \dfrac{1}{n!}f^{(n)}(z_0)$. Therefore, we have actually derived the Taylor series for f.)

Clearly, the radius of convergence of this power series is at least a... though of course, it might be larger.

EXAMPLE: $\dfrac{1}{1-\sin z}$ is holomorphic wherever $\sin z \neq 1$. And $\sin\dfrac{\pi}{2} = 1$, so Problem 1-8 yields $\sin z = 1 \Leftrightarrow z = \left(2k+\dfrac{1}{2}\right)\pi$. We conclude that



$$\frac{1}{1-\sin z} = \sum_{n=0}^{\infty} c_n z^n \text{ with radius of convergence } \frac{\pi}{2}.$$

(Did not need to calculate any of the coefficients.)

Next, a converse to Cauchy's integral theorem

(4) MORERA'S THEOREM

Suppose f is a continuous function defined on an open set $D \subset \mathbb{C}$, with the property that for all loops γ contained in D,

$$\int_\gamma f(z)dz = 0.$$

Then f is holomorphic.

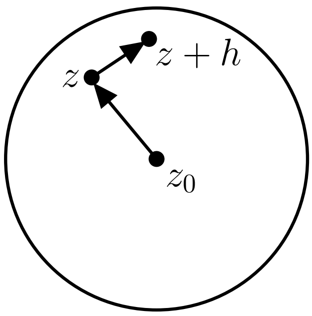
(This theorem and its proof are similar to the result in vector calculus relating zero line integrals of a vector field to the vector field's being conservative, i.e., being a gradient field.)

Proof: This theorem is local in nature, so it suffices to prove it for the case in which D is a disc. Let z_0 be the center of D, and defined the function $g(z)$ on D by

$$g(z) = \int_\gamma f(\zeta)d\zeta, \text{ where } \gamma = \text{any path in } D \text{ from } z_0 \text{ to } z.$$

Our hypothesis guarantees that $g(z)$ depends only on z, not on the choice of γ. Now assume $z \in D$ is fixed and $h \in \mathbb{C}$ is so small that $z + h \in D$: then $g(z + h)$ can be calculated using the straight line from z_0 to z and then from z to $z + h$:

$$g(z+h) = g(z) + \int_z^{z+h} f(\zeta)d\zeta.$$



Parametrize the line segment from z to $z + h$ as $z + th$, $0 \leq t \leq 1$. Then

$$\begin{aligned} g(z+h) - g(z) &= \int_0^1 f(z+th)hdt \\ &= h\int_0^1 f(z+th)dt. \end{aligned}$$

Therefore,

$$\frac{g(z+h) - g(z)}{h} = \int_0^1 f(z+th)dt.$$

Since f is continuous at z, the right side of this equation has limit $f(z)$ when $h \to 0$. Thus, the left side has the same limit. We conclude that

$$g'(z) \text{ exists, and } g'(z) = f(z).$$

Since f is continuous, so is g'. Thus g *is holomorphic*. By (2), f is holomorphic.

QED

★ REMARK: the proof of Morera's theorem shows that the only hypothesis actually needed is that f be continuous and that in small discs contained in D,

$$\int_\gamma f dz = 0$$

for all triangles γ contained in the disc!

PROBLEM 3-3.

$$\boldsymbol{\sec z} \qquad \left(:= \frac{1}{\cos z}\right)$$

This function is holomorphic in some disc centered at 0. Therefore, it has a Maclaurin representation near 0.

1. Prove that only even terms z^{2n} are in this representation.
2. Find its radius of convergence.
3. ★ This expansion is customarily expressed in this form:

$$\sec z = \sum_{n=0}^{\infty} \frac{s_n z^{2n}}{(2n)!}.$$

Prove that all $s_n > 0$. The s_n's are called *secant numbers*.

Here are given $s_0, s_1, \ldots, s_{16}$:

$$\begin{gathered}1, 1, 5, 61, 1385, 50521, 2702765, 199360981, 1939151214\\ 2404879675441, 370371188237525, 69348874393137901\\ 15514534163557086905, 4087072509293123892361,\\ 1252259641403629865468285, 441543893249023104553682821\\ 177519391579539289436664789665\end{gathered}$$

(https://oeis.org/search?q=secant+numbers&language=english&go=Search)

$\underline{\boldsymbol{e^{z+w} = e^z e^w}}$ bis

We gave Proof #1 on page 3. Now two more proofs.

Proof #2:

For fixed $w \in \mathbb{C}$ consider the function

$$f(z) := e^{z+w}e^{-z}.$$

This holomorphic function has $f'(z) = e^{z+w}e^{-z} - e^{z+w}e^{-z} = 0$ by the product rule, so $f(z) = \text{constant}$. This constant $= f(0) = e^w$. Thus,

$$e^{z+w}e^{-z} = e^w \text{ for all } w \text{ and all } z.$$

When $w = 0$, we obtain $e^z e^{-z} = 1$, so that $e^{z+w} = e^w e^z$.

QED

Proof #3:

- Let $w \in \mathbb{R}$ be fixed. Then the analytic function of z,

$$e^{z+w} - e^z e^w,$$

equals 0 for all real z from basic calculus. This occurrence of an infinity of zeroes near $0 \Rightarrow$ the analytic function is 0: (see Section E of Chapter 2, pg.40)

$$e^{z+w} - e^z e^w = 0 \text{ for all } z \in \mathbb{C}, \text{ all } w \in \mathbb{R}.$$

- Now let $z \in \mathbb{C}$ be fixed. Then the analytic function of w,

$$e^{z+w} = e^z e^w$$

equals 0 for all real w, as we've just proved. Therefore, as above, it's 0 for all $w \in \mathbb{C}$.

QED

Basic estimates for complex integrals:

a. Consider a complex-valued function $f = f(t)$ for all $a \le t \le b$, and its integral

$$I := \int_a^b f(t)dt.$$

Write I in polar form

$$I = |I|e^{i\theta} \text{ for some } \theta \in \mathbb{R}.$$

Then

$$\begin{aligned}
|I| &= e^{-i\theta} I \\
&= e^{-i\theta} \int_a^b f(t)dt && \text{(def. of } I\text{)} \\
&= \int_a^b e^{-i\theta} f(t)dt && (e^{-i\theta} \text{ is a constant}) \\
&= \operatorname{Re} \int_a^b e^{-i\theta} f(t)dt && \text{(it's already real)} \\
&= \int_a^b \operatorname{Re}(e^{-i\theta} f(t))dt && \text{(def. of complex integration)} \\
&\le \int_a^b \left|e^{-i\theta} f(t)\right| dt \\
&= \int_a^b |f(t)|\, dt.
\end{aligned}$$

Thus, we have

$$\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|\, dt.$$

b. Line integrals: let the curve γ be parametrized as $\gamma = \gamma(t)$ for $a \le t \le b$. Assume $|f(t)| \le C$ for all $z = \gamma(t)$. Then

$$\begin{aligned}
\left|\int_\gamma f(z)dz\right| &= \left|\int_a^b f(\gamma(t))\gamma'(t)dt\right| \\
&\le \int_a^b |f(\gamma(t))|\, |\gamma'(t)|\, dt \quad \text{(by } \mathbf{a}\text{)} \\
&= C \int_a^b |\gamma'(t)|\, dt \\
&= CL, \text{ where } L = \text{length of } \gamma.
\end{aligned}$$

Thus, we have

$$\left|\int_\gamma f dz\right| \le \max_\gamma |f| \cdot \text{length of } \gamma.$$

P.S. More generally, we see that $\left|\int_\gamma f dz\right| \le \int_\gamma |f||dz|$, where

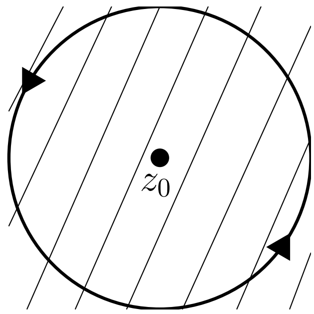
$$|dz| = |dx + idy| = \sqrt{(dx)^2 + (dy)^2} = d(\text{arclength}).$$

Now we continue with consequences of the Cauchy integral formula. So far we have listed 4 of them, so now we come to

5. Mean value property of holomorphic functions:

Let f be holomorphic on an open set $D \subset \mathbb{C}$ and suppose a closed disc $|z - z_0| \leq r$ is contained in D. Then the Cauchy formula gives in particular

$$f(z_0) = \frac{1}{2\pi i} \int_{\substack{|\zeta - z_0| = r \\ \text{(CCW)}}} \frac{f(\zeta)}{\zeta - z_0} d\zeta.$$



The usual parametrization $\zeta = z_0 + re^{i\theta}$ yields

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} ire^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta, \text{ the average of } f \text{ on the circle.} \end{aligned}$$

Before the next result, here's an important bit of terminology.

> an *entire* function (or *entire holomorphic function*) is a function which is defined and holomorphic on all of $\mathbb{C}$.

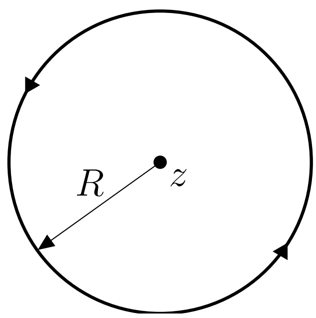
6. LIOUVILLE'S THEOREM

An entire function which is bounded must be constant.

Proof: Let $f = f(z)$ be entire and suppose $|f(z)| \leq C$ for all $z \in \mathbb{C}$, where C is constant.

Let $z \in \mathbb{C}$ be arbitrary, and apply Cauchy's formula using the disc with center z and radius R. Then from page 56 we have

$$f'(z_0) = \frac{1}{2\pi i} \int_{\substack{|\zeta - z_0| = R \\ \text{CCW}}} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$



Therefore, we estimate

$$\begin{aligned}
|f'(z)| &\le \frac{1}{2\pi} \int\limits_{|\zeta - z| = R} \frac{C}{|\zeta - z|^2} |d\zeta| \\
&= \frac{1}{2\pi} \int\limits_{|\zeta - z| = R} \frac{C}{R^2} |d\zeta| \\
&= \frac{1}{2\pi} \frac{C}{R^2} \cdot \text{length of circle} \\
&= \frac{1}{2\pi} \frac{C}{R^2} \cdot 2\pi R \\
&= \frac{C}{R}.
\end{aligned}$$

Simply let $R \to \infty$ to conclude that $f'(z) = 0$. Thus $f' = 0$ on all of $\mathbb{C}$, so f is constant.

QED

Here is a natural place to talk about <u>harmonic</u> functions. These in general are functions u defined on $\mathbb{R}^n$ which satisfy Laplace's equation

$$\nabla^2 u = 0.$$

In a standard orthonormal coordinate system, this equation is

$$\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0.$$

Holomorphic functions are harmonic. For the Cauchy-Riemann equation

$$\begin{aligned}
\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y} \Rightarrow \frac{\partial^2 f}{\partial x^2} &= \frac{1}{i} \frac{\partial^2 f}{\partial x \partial y} \\
&= \frac{1}{i} \frac{\partial^2 f}{\partial y \partial x} \\
&= \frac{1}{i} \frac{\partial}{\partial y} \left(\frac{1}{i} \frac{\partial f}{\partial y} \right) \\
&= -\frac{\partial^2 f}{\partial y^2},
\end{aligned}$$

so that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Here we insert an elegant proof of the

Fundamental Theorem of Algebra

Let P be a polynomial with complex coefficients and positive degree. Then there exists $z \in \mathbb{C}$ such that $P(z) = 0$.

Proof: We suppose to the contrary that for all $z \in \mathbb{C}$, $P(z) \neq 0$. Normalize P to be "monic"—that is,

$$P(z) = z^N + c_1 z^{N-1} + \cdots + c_n,$$

where $N \geq 1$. Then

$$\lim_{z \to \infty} \frac{P(z)}{z^N} = 1.$$

Therefore, the function $\frac{1}{P}$ is a <u>bounded</u> entire function. Aha! **Liouville's theorem** implies that it is <u>constant</u>! Therefore, $P(z)$ is constant. That's a contradiction.

QED

REMARK: Since $P(z_1) = 0$ for some z_1, it's simple polynomial algebra which shows that the polynomial $P(z)$ is <u>divisible</u> by the polynomial $z - z_1$: $P(z) = (z - z_1)Q(z)$, where Q is a polynomial of one less degree than P. If Q has positive degree, then again we conclude that for some z_2, $Q(z) = (z - z_2)R(z)$, where R is again a polynomial. Continuing in this way we have a factorization of P into linear factors

$$P(z) = c \prod_{k=1}^{N} (z - z_k).$$

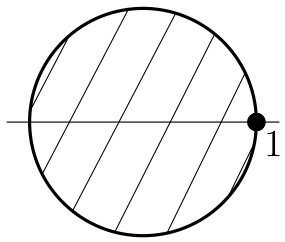
(Some z_k's may be repeated, of course.)

Later we'll give a much different proof of the FTA in which the complete factorization will appear instantaneously!

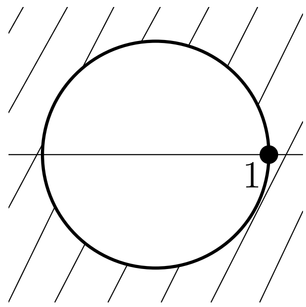
Before we continue with consequences of the Cauchy integral formula, we pause to rethink the holomorphic function $\frac{1}{1-z}$. For $|z| < 1$ we can simply write

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \text{ the geometric series.}$$

This equation is valid $\Leftrightarrow |z| < 1$.



Now suppose $|z| > 1$. Then 1 is dominated by z, so we write

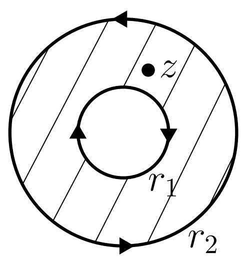


$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{-z}\frac{1}{1-\frac{1}{z}} \\ &= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \\ &= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} \\ &= -\sum_{n=1}^{\infty}\frac{1}{z^n},\end{aligned}$$

valid $\Leftrightarrow |z| > 1$.

The procedure we have just employed is useful in the following more general situation: Suppose f is holomorphic in an open set D which contains a closed annulus $r_1 \le |z| \le r_2$. For $r_1 < |z| < r_2$ we then employ the Cauchy integral formula to write $f(z)$ in terms of line integrals along $|z| = r_2$ counterclockwise and along $|z| = r_1$ clockwise:

$$f(z) = \frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=r_2 \\ \text{CCW}}}\frac{f(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=r_1 \\ \text{CW}}}\frac{f(\zeta)}{\zeta - z}d\zeta.$$



- For $|\zeta| = r_2$ we write

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\frac{1}{1-\frac{z}{\zeta}} = \sum_{n=0}^{\infty}\frac{z^n}{\zeta^{n+1}},$$

so that the corresponding integral becomes

$$\frac{1}{2\pi i}\sum_{n=0}^{\infty} z^n \int\limits_{\substack{|\zeta|=r_2 \\ \text{CCW}}}\frac{f(\zeta)}{\zeta^{n+1}}d\zeta.$$

- For $|\zeta| = r_1$ we write

$$\frac{1}{\zeta - z} = \frac{-1}{z}\frac{1}{1-\frac{\zeta}{z}} = -\sum_{n=0}^{\infty}\frac{\zeta^n}{z^{n+1}},$$

so that the corresponding integral becomes

$$-\frac{1}{2\pi i}\sum_{n=0}^{\infty} z^{-n-1} \int\limits_{\substack{|\zeta|=r_1\\ \text{CW}}} f(\zeta)\zeta^n d\zeta.$$

We can of course change the sign by performing the line integral the opposite direction.

We also change the dummy index n in the latter series by $-n-1=k$, so that k ranges from $-\infty$ to -1, with the result being

$$\frac{1}{2\pi i}\sum_{k=-\infty}^{-1} z^k \int\limits_{\substack{|\zeta|=r_1\\ \text{CCW}}} \frac{f(\zeta)}{\zeta^{k+1}}d\zeta.$$

One more adjustment: the function $\dfrac{f(\zeta)}{\zeta^{n+1}}$ is holomorphic in the complete annulus $r_1 \le |\zeta| \le r_2$, so its path integral over a circle of radius r is <u>independent</u> of r, thanks to Cauchy's integral theorem. We therefore obtain our final result,

$$\bigstar \qquad f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \text{ for } r_1 < |z| < r_2,$$

where

$$c_n = \frac{1}{2\pi i}\int\limits_{\substack{|\zeta|=r\\ \text{CCW}}} \frac{f(\zeta)}{\zeta^{n+1}}d\zeta \quad (r_1 < r < r_2).$$

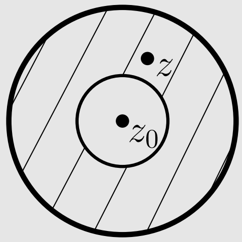
<u>TERMINOLOGY</u>: a series of the form $\bigstar$, containing z^n for both positive and negative indices n, is called a *Laurent series*.

We now formulate what we have accomplished. As usual, we may immediately generalize to an arbitrary center z_0 instead of 0.

7. LAURENT EXPANSION THEOREM

Let $0 \le R_1 < R_2 \le \infty$, and assume that f is a holomorphic function in the open annulus centered at z_0:

$$R_1 < |z - z_0| < R_2.$$



Then for all z in this annulus

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

where c_n is given by

$$c_n = \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta,$$

and r is any radius satisfying $R_1 < r < R_2$.

Here's an important quick corollary:

8. RIEMANN'S REMOVABLE SINGULARITY THEOREM

Let f be a holomorphic function defined in a "punctured" disc $0 < |z - z_0| < R$, and assume f is bounded. Then there is a limit $f(z_0) := \lim_{z \to z_0} f(z)$ and the resulting function is holomorphic in the full disc $|z - z_0| < R$.

Proof: Suppose $|f(z)| \leq C$ for $0 < |z - z_0| < R$. Apply the Laurent expansion theorem with $R_1 = 0$ and $R_2 = R$. Then for any index $n \leq -1$, we can estimate c_n this way: for any $0 < r < R$,

$$\begin{aligned} |c_n| &= \left| \frac{1}{2\pi i} \int_{|\zeta - z_0| = r} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \right| \\ &\leq \frac{1}{2\pi} \frac{C}{r^{n+1}} \cdot \text{length of circle} \\ &= \frac{C}{r^n}. \end{aligned}$$

But when $r \to 0$, $\dfrac{C}{r^n} \to 0$ since $n < 0$. Thus $c_n = 0$ for all $n < 0$. Therefore, we have the result that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad \text{for } 0 < |z - z_0| < R.$$

Clearly then, $\lim\limits_{z \to z_0} f(z) = c_0$ and if we define $f(z_0) = c_0$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad \text{for } |z - z_0| < R.$$

QED

PROBLEM 3-4.

The Bernoulli numbers

1. Show that the function of z given as $\dfrac{z}{e^z - 1}$ has a removable singularity at the origin.

2. Therefore, this function has a Maclaurin expansion, which we write in this form:
$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n,$$
where the B_n's are called the **Bernoulli numbers**. Find the radius of convergence of this series.

3. Use the equation $z = (e^z - 1) \sum_{n=0}^{\infty} \dfrac{B_n}{n!} z^n$ to derive a recursion formula for the B_n's:
$$B_0 = 1$$
$$B_1 = -\frac{1}{2}$$
$$\sum_{n=0}^{k-1} \binom{k}{n} B_n = 0 \quad \text{for } k \geq 2.$$

4. Prove that $B_n = 0$ for all odd $n \geq 3$.

   HINT: examine $\dfrac{z}{e^z - 1} + \dfrac{z}{2}$.

Isolated singularities

Let $z_0 \in \mathbb{C}$ be fixed, and suppose f is a function which is holomorphic for $0 < |z - z_0| < R$. Then f is said to have a singularity at z_0, simply because

$f(z_0)$ is undefined. We actually say that f has an isolated singularity at z_0, since f is holomorphic in the disc $|z - z_0| < R$ except at z_0 (where it is undefined).

We then know that f has a Laurent expansion of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad \text{valid for } 0 < |z - z_0| < R.$$

There is a convenient classification of isolated singular points according to the appearance of c_n with $n < 0$ in the Laurent expansion. They are divided into 3 distinct categories as follows:

R = *removable singularities*, meaning that for all $n < 0$, $c_n = 0$.

P = *poles*, meaning that some $c_n \neq 0$ with $n < 0$, but there are only finitely many such $c_n \ldots$ all the remaining c_n with $n < 0$ are 0.

E = *essential* singularities, meaning that $c_n \neq 0$ for infinitely many $n < 0$.

It is of utmost importance to have a complete understanding of these categories, so we devote the next few considerations to this.

REMOVABLE SINGULARITIES

In this case, the Laurent expansion is

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \quad \text{for } |z - z_0| < R.$$

The right side of this equation defines a function analytic for the entire disc $|z - z_0| < R$. Therefore, we remove the singularity of f defining $f(z_0) = c_0$.

Of course, we have the great **removable singularity theorem** of Riemann, which asserts that if we assume only that f is bounded near z_0, then its singularity at z_0 is removable. Thus, we have these equivalent situations:

- the singularity at z_0 is removable
- f is bounded near z_0
- $\lim_{z \to z_0} f(z)$ exists

Except in the trivial case that $f = 0$, not all $c_n \neq 0$. Say that $c_N \neq 0$ with $N \geq 0$ minimal. Then we may write

$$\begin{aligned} f(z) &= c_N(z-z_0)^N + \text{higher order terms} \\ &= (z-z_0)^N g(z), \end{aligned}$$

where g is holomorphic and $g(z_0) \neq 0$. We then say that *f has a zero at z_0 of order N*. (Of course, $N = 0$ is allowed.)

POLES

In this case there exists $N < 0$ such that $c_N \neq 0$ but all c_n before that are 0. Therefore, we may write

$$\begin{aligned} f(z) &= \sum_{n=N}^{\infty} c_n(z-z_0)^n \\ &= (z-z_0)^N \sum_{k=0}^{\infty} c_{N+k}(z-z_0)^k \\ &= (z-z_0)^N g(z), \end{aligned}$$

where g is holomorphic for $|z - z_0| < R$ and $g(z_0) \neq 0$. We then say that *f has a pole at z_0 of order $-N$* ... in this case $-N \geq 1$.

We then have these equivalent situations:

- the singularity at z_0 is a pole
- $\lim_{z \to z_0} f(z) = \infty$

(The former of these implies that latter, but we'll soon prove the reverse implication.)

EXAMPLES:

- $\csc z$ has a pole at 0 of order 1
- $\sec z$ has a pole at $\frac{\pi}{2}$ of order 1
- $\frac{1}{e^z - 1}$ has a pole at 0 of order 1
- $\frac{z}{e^z - 1}$ has a removable singularity at 0

- $\dfrac{1}{z(e^{z^2}-1)}$ has a pole at 0 of order 3

ESSENTIAL SINGULARITIES

The Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

is neither of the first two kind: z_0 is neither a removable singularity nor a pole.

In this situation, the behavior of f as $z \to z_0$ is quite interesting:

CASORATI-WEIERSTRASS THEOREM

Suppose f has an essential singularity at z_0. Then for any $w \in \hat{\mathbb{C}}$ there exists a sequence $z_1, z_2, \ldots$ such that

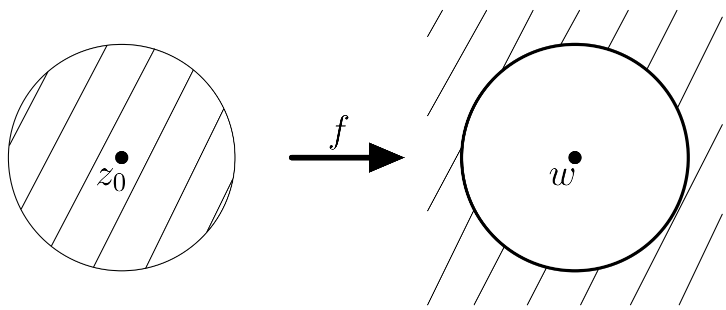
$$\lim_{k\to\infty} z_k = z_0 \quad \textit{and} \quad \lim_{k\to\infty} f(z_k) = w.$$

Proof: We proceed by contradiction. Thus, we suppose there exists $w \in \hat{\mathbb{C}}$ for which no sequence $\{z_k\}$ exists with

$$z_k \to z_0 \quad \text{and} \quad f(z_k) \to w.$$

This means that for z near z_0, $f(z)$ cannot be arbitrarily close to $w \ldots$ in other words, $f(z)$ must stay a positive distance away from w.

- If $w = \infty$, this means that $f(z)$ must be bounded for z near z_0. The removable singularity situation holds, so z_0 is a removable singularity for f. <u>Contradiction</u>.
- If $w \in \mathbb{C}$, then there exists $r > 0$ such that for z sufficiently near z_0, $|f(z) - w| > r$.



Then consider the function $\dfrac{1}{f(z)-w}$ in this neighborhood of z_0. It is <u>bounded</u> (by $\dfrac{1}{r}$) and thus its singularity at z_0 is removable. That is, it agrees with a holomorphic function near z_0 and it may be written as

$$(z-z_0)^N g(z),$$

where g is holomorphic and nonzero. Therefore,

$$f(z)-w=(z-z_0)^{-N}\frac{1}{g(z)}.$$

As $\dfrac{1}{g(z)}$ is holomorphic, this equation shows that the Laurent series f has no terms $(z-z_0)^n$ for $n<-N$. <u>Contradiction</u>.

QED

DISCUSSION: This classification into the 3 types of isolated singularities is quite definitive and complete. However, as wonderful as the Casorati-Weierstrass theorem is, it doesn't come close to the much more profound result known as

PICARD'S GREAT THEOREM: *if f has an isolated essential singularity at z_0, then for every $w\in\mathbb{C}$ with at most one exception, there exists a sequence $z_k\to z_0$ such that $f(z_k)=w$ for all $k=1,2,3,\ldots$.*

(The example $e^{\frac{1}{z}}$ has the exception $w=0$.)
This theorem is "beyond the scope of this book."

PROBLEM 3-5.

1. Show that $\sinh z=w$ has a solution z for every w. Do this by deriving a "formula" for z in terms of w. This formula will involve a choice of <u>square root</u> and choice of <u>log</u> but don't worry about these details at the present time.

2. Do the same for the equation $\tanh z=w$, but notice that there's one exception (actually, two) for w.

3. The function $\sin\dfrac{1}{z}$ has an essential singularity at 0. Verify directly for this function the truth of the great Picard theorem.

Complex Powers:

The goal is to devise a reasonable definition of z^α where α is allowed to be complex. Though it makes no sense to "raise z to the power α," we still use that terminology.

> WARNING: when dealing with this subject it's very important not to use the notation e^z in the usual way, but instead to use the terminology from the beginning of the book,
>
> $$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

A good way to figure out what our definition should be is the use of desired properties of logarithm, namely

$$\log(z^\alpha) = \alpha \log z.$$

Then we use the "inverse" of $\log$ to come to our definition:

$$\boxed{z^n := \exp(\alpha \log z)}$$

This makes sense as long as $z \neq 0$, so we'll always make that assumption. Of course, z^α is usually not a unique complex number, due to the ambiguity in $\log z$. For other values could be

$$\exp(\alpha(\log z + 2n\pi i)) = \exp(\alpha \log z) \exp(2n\pi i\alpha).$$

This will be independent of the integer $n \Leftrightarrow \alpha$ is an integer, and then z^α has its usual meaning.

Now we list some properties of this definition.

1. If α is an integer, z^α has its usual meaning. For all other $\alpha \in \mathbb{C}$, z^α is ambiguous, no matter what z is. In particular,

$$\boxed{z^0 = 1}$$

2. $1^\alpha = \exp(\alpha \log 1) = \exp(\alpha(2n\pi i))$, so

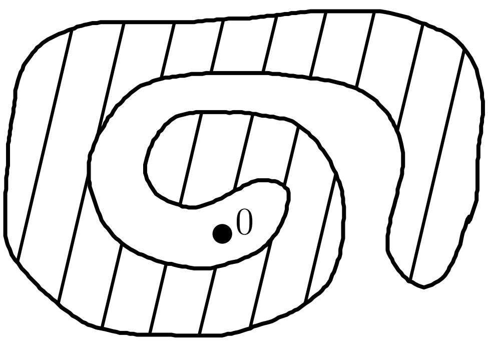
$$\boxed{1^\alpha \text{ has all the values } \exp(2n\pi i\alpha)}$$

3. $i^i = \exp(i \log i) = \exp\left(i\left(0 + \frac{i\pi}{2} + 2n\pi i\right)\right) = \exp\left(-\frac{\pi}{2} - 2n\pi\right)$, so i^i has all the values $\exp\left(\left(\text{even integer } - \frac{1}{2}\right)\pi\right)$. (and all are <u>real</u> numbers.)

4. $\boxed{z^\alpha z^\beta = z^{\alpha+\beta}}$ provided the same $\log z$ is used in all 3 places it appears.

5. $\boxed{z^\alpha w^\alpha = (zw)^\alpha \text{ — sort of true: be careful!}}$

6. In open sets of $\mathbb{C}$ which do not contain 0 and which do not "wind around 0," $\log z$ can be defined in terms of a continuous value for $\arg z$. Then $\log z$ becomes a holomorphic function, as we know, so also the composite function <u>z^α is holomorphic</u>.



And we compute its derivative by the chain rule:

$$(z^\alpha)' = \exp(\alpha \log z)(\alpha \log z)' = z^\alpha \frac{\alpha}{z},$$

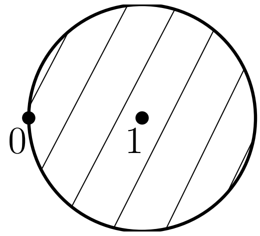
and we write

$$\boxed{\frac{dz^\alpha}{dz} = \alpha z^{\alpha-1}} \qquad \text{(same } \log z \text{ on each side).}$$

7. **Taylor series**

As in the above discussion, we take $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$ in the disc $|z-1| < 1$. Then we have inductively for $n = 0, 1, 2, \ldots$

$$\left(\frac{d}{dz}\right)^n z^\alpha = \alpha(\alpha-1)\cdots(\alpha-n+1)z^{\alpha-n}.$$



In particular, at $z = 1$ we find $\alpha(\alpha-1)\cdots(\alpha-n+1)$, since $1^\alpha = 1$. So we obtain the Taylor series

$$z^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}(z-1)^n \quad \text{for } |z-1| < 1.$$

That coefficient is given this notation

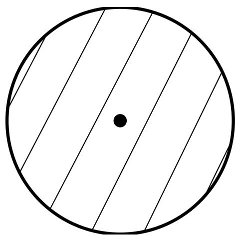
$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!},$$

and is still called a *binomial coefficient*:

$$\binom{\alpha}{0} = 1, \quad \binom{\alpha}{1} = \alpha, \quad \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2}, \quad \text{etc.}$$

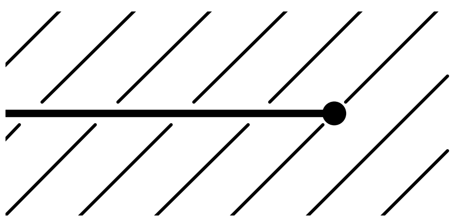
Replacing z by $1 + z$ yields a "binomial" formula,

$$\boxed{(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n} \quad \text{for } |z| < 1.$$



(If $\alpha = 0, 1, 2, \ldots$ this series is *finite*, going only from $0 \le n \le \alpha$. It's a polynomial, and this result is the classical *binomial formula*. Otherwise, the radius of convergence equals 1.)

PRINCIPLE DETERMINATION OF ARG & LOG: this is what we say when we are in the open set $\mathbb{C} \setminus (-\infty, 0]$:



$$-\pi < \arg z < \pi;$$
$$1^\alpha = 1, \log x \text{ is real for } 0 < x < \infty$$

PROBLEM 3-6. Prove this polynomial identity:

$$\binom{\alpha+\beta}{n} = \sum_{n=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k}.$$

HINT: very easy problem!

BRANCH POINTS

All the holomorphic functions which somehow involve $\log z$ have a definite type of "singular" behavior near 0. But these are clearly not isolated singularities. For these functions are not actual functions (single-valued) in any region which includes all z satisfying $0 < |z| < \varepsilon$.

Instead, we say that these functions have a *branch point* at 0. That's a well-chosen descriptive word, for as we follow their behavior on a loop surrounding 0 they can exhibit a change because of the change in a continuous determination of $\arg z$.

For instance, $z^{\frac{1}{2}}$ has 2 values, $z^{\frac{1}{3}}$ has 3 values; $z^{\frac{m}{n}}$ has n values, assuming the integers m and n have no common prime factor. However, if α is irrational, z^{α} has infinitely many values.

We say that the above functions have branching of order $2, 3, n$ and ∞, respectively

You can imagine that things can become more and more complicated. For instance, think about $z^{\sqrt{z}}$ near the origin.

Of course, we can also see branch points at other points, such as in the function $(z-1)^{\frac{1}{2}} + (z+1)^{\frac{1}{2}}$, which has branch points at 1 and -1. Or $(z^2+1)^{\frac{1}{2}}$ with branch points at i and $-i$.

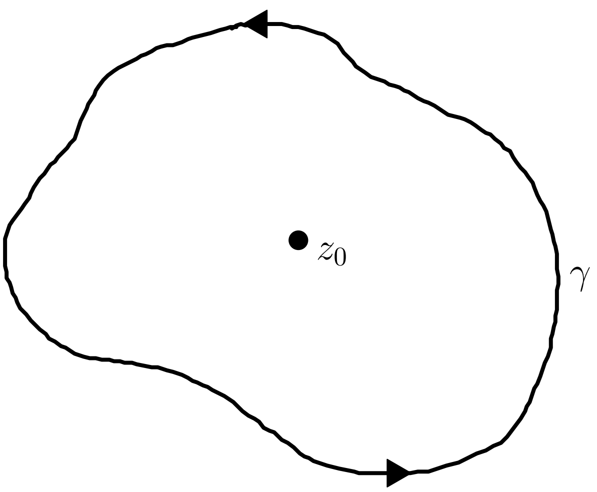
Chapter 4

Residues (Part I)

We are preparing to do some truly amazing things with our theory, but first we need an important definition. This is all in the context of a *holomorphic function with an isolated singularity at* z_0. Let f be such a function. We are then going to define a complex number based on this situation, but we do it in 3 separate ways, and we'll observe that these 3 ways yield the same number.

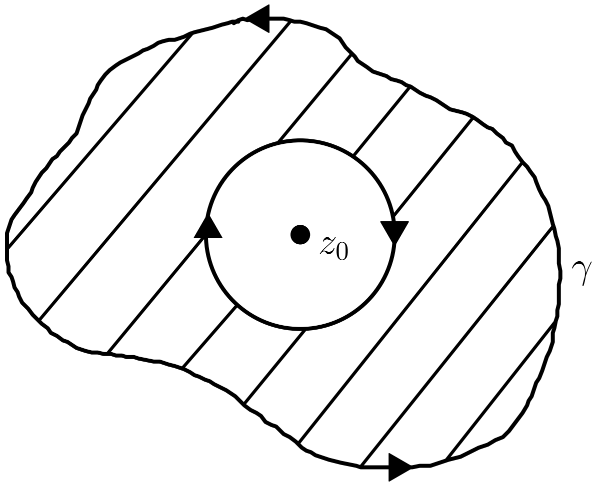
Section A: Definition of Residues

Definition **1**: Let γ be any small loop surrounding z_0 one time in the CCW sense: then our number equals



$$\boxed{\frac{1}{2\pi i}\int_\gamma f(z)dz}$$

This number does not depend on γ, thanks to Cauchy's integral theorem. To see this, surround z_0 with a very small circle: in the region between γ and this circle we have from Cauchy's theorem



$$\int_\gamma f(z)dz + \int_{\substack{\text{CW}\\\text{circle}}} f(z)dz = 0.$$

Thus

$$\int_\gamma f(z)dz = \int_{\substack{\text{CCW}\\\text{circle}}} f(z)dz,$$

so the left side does not depend on γ.

<u>Definition 2</u>: Consider the Laurent expansion of f near z_0:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n.$$

Then our number equals

$$\boxed{c_{-1}}$$

That is, we focus our attention on the Laurent expansion

$$f(z) = \cdots + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + \cdots$$

and it's c_{-1} we use.

This agrees with the first definition since

$$\begin{aligned}\int_\gamma f(z)dz &= \sum_{n=-\infty}^{\infty} c_n \int_\gamma (z-z_0)^n dz = c_{-1}\int_\gamma \frac{dz}{z-z_0}\\ &= 2\pi i c_{-1}.\end{aligned}$$

<u>Definition 3</u>: This definition relies on trying to integrate $f(z)$ as an "indefinite integral" near z_0. The trouble is precisely with the term $\dfrac{c_{-1}}{z-z_0}$. For this creates an undefined expression in

$$\int f(z)dz = \sum_{n\neq -1} \frac{c_n(z-z_0)^{n+1}}{n+1} + \int \frac{c_{-1}}{z-z_0}dz.$$

undefined

Thus, the number we want is the unique $a \in \mathbb{C}$ such that

$\boxed{f(z) - \frac{a}{z-z_0} = \text{the derivative of a holomorphic function in the region } 0 < |z-z_0| < r, \text{ for some } r > 0.}$

Definition: The *residue of f at z_0* is the number defined in all 3 of the above definitions. We denote it as

$$\boxed{\operatorname{Res}(f, z_0)}$$

We now list a number of properties and examples.

1. If f is holomorphic at z_0 (i.e., z_0 is a removable singularity of f), then
$$\operatorname{Res}(f, z_0) = 0.$$
2. $\operatorname{Res}\left(\frac{1}{z-z_0}, z_0\right) = 1$ (most basic case).
3. $\operatorname{Res}\left((z-z_0)^n, z_0\right) = 0$ if $n \in \mathbb{Z}, n \neq -1$.
4. $\operatorname{Res}\left(e^{\frac{1}{z}}, 0\right) = 1.$
5. $\operatorname{Res}\left(\sin\frac{1}{z}, 0\right) = 1.$
6. $\operatorname{Res}\left(\cos\frac{1}{z}, 0\right) = 0.$
7. If f is an even function
$$\operatorname{Res}(f, 0) = 0.$$
8. Suppose f has a simple pole (i.e., pole of order 1) at z_0. Then we have
$$f(z) = \frac{c_{-1}}{z-z_0} + \sum_{n=0}^{\infty} c_n(z-z_0)^n,$$

so that

$$(z - z_0)f(z) = c_{-1} + (z - z_0)\sum_{n=0}^{\infty} c_n(z - z_0)^n,$$

and the left side has limit c_{-1} as $z \to z_0$:

SIMPLE POLE $$\boxed{\mathrm{Res}(f, z_0) = \lim_{z \to z_0}(z - z_0)f(z)}$$

9. A corollary is now the following, which is <u>the handiest special case</u>! Suppose we know that

$$f(z) = \frac{a(z)}{b(z)},$$

where both numerator and denominator are holomorphic at z_0, and $b(z_0) = 0$, $b'(z_0) \neq 0$. Then z_0 is a simple pole for f, and our previous result yields

$$\begin{aligned}\mathrm{Res}(f, z_0) &= \lim_{z \to z_0} \frac{(z - z_0)a(z)}{b(z)} \\ &= \lim_{z \to z_0} \frac{a(z)}{\frac{b(z)-b(z_0)}{z-z_0}} \\ &= \frac{a(z_0)}{b'(z_0)}.\end{aligned}$$

For the record,

$$\boxed{\mathrm{Res}\left(\frac{a(z)}{b(z)}, z_0\right) = \frac{a(z_0)}{b'(z_0)}} \qquad \text{if } \begin{array}{l} b(z_0) = 0, \\ b'(z_0) \neq 0. \end{array}$$

10. $\mathrm{Res}(\cot z, 0) = 1$ since $\cot z = \dfrac{\cos z}{\sin z}$ and $\begin{array}{l}\cos 0 = 1, \\ \sin 0 = 0, \\ \sin'(0) = 1.\end{array}$

$$\boxed{\mathrm{Res}(\cot z, n\pi) = 1} \qquad \left(\frac{\cos n\pi}{\cos n\pi}\right)$$

11. $\boxed{\mathrm{Res}(\csc z, n\pi) = (-1)^n}$

12. $\mathrm{Res}\left(\dfrac{1}{e^z - 1}, 0\right) = 1.$

13. $\mathrm{Res}\left(\dfrac{z}{e^z - 1}, 0\right) = 0.$

14. Let's compute the residue at 0 of $\dfrac{1}{a-z}e^{\frac{1}{z}}$, where $a \neq 0$. None of our easy examples apply, so we resort to series:

$$\frac{1}{a-z} = \frac{1}{a}\frac{1}{1-\frac{z}{a}} = \sum_{n=0}^{\infty}\frac{z^n}{a^{n+1}},$$
$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{1}{n!z^n}.$$

Multiply these series and look for the $\dfrac{1}{z}$ terms:

$$\frac{1}{a}\frac{1}{1!} + \frac{1}{a^2}\frac{1}{2!} + \frac{1}{a^3}\frac{1}{3!} + \cdots .$$

This equals $-1 + \sum_{k=0}^{\infty}\dfrac{1}{a^k k!}$, so

$$\operatorname{Res}\left(\frac{1}{a-z}e^{\frac{1}{z}}, 0\right) = e^{\frac{1}{a}} - 1.$$

Incidentally,

$$\operatorname{Res}\left(\frac{1}{a-z}e^{\frac{1}{z}}, a\right) = -e^{\frac{1}{a}}.$$

Now we come to a major theorem. Before stating it, let's be sure we completely understand the context.

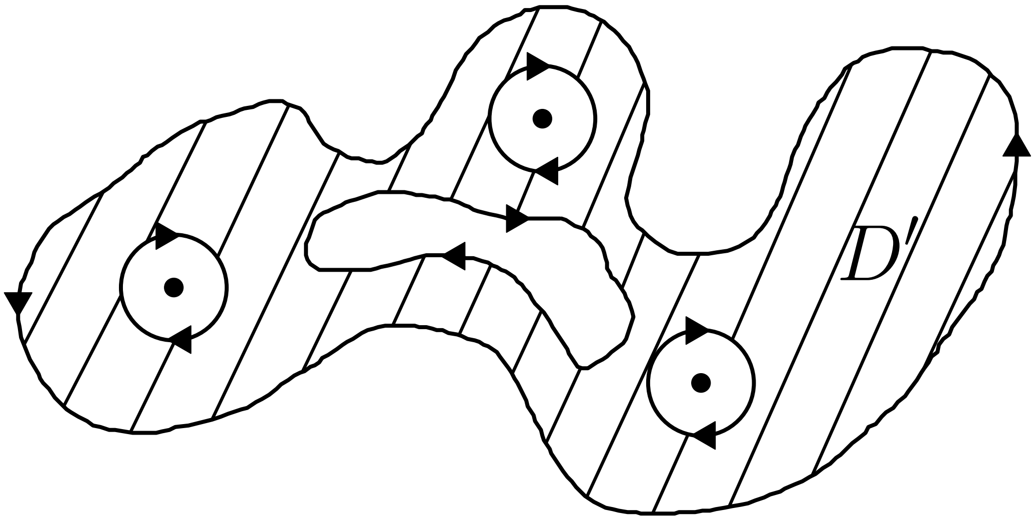
As in the Cauchy integral theorem, we deal with a "nice" bounded open set $D \subset \mathbb{C}$, whose boundary ∂D consists of finitely many curves. We also assume that f is holomorphic on an open set containing $D \cup \partial D$ except for finitely many isolated singularities $z_1, \ldots, z_n$, all contained in the open set D.

Then we have the

RESIDUE THEOREM

$$\frac{1}{2\pi i}\int_{\partial D} f(z)dz = \sum_{k=1}^{n} \operatorname{Res}(f, z_k)$$

The proof is an easy application of the Cauchy integral theorem if we first remove from D small, closed discs E_k centered at the z_k's. Let the resulting open set be denoted



$$D' = D \setminus \bigcup_{k=1}^{n} E_k.$$

Since f is holomorphic on D', Cauchy's theorem yields

$$0 = \frac{1}{2\pi i} \int_{\partial D'} f(z)dz.$$

But $\partial D' = \partial D \cup \bigcup\limits_{k=1}^{n} \partial E_k$, so we obtain

$$0 = \frac{1}{2\pi i} \int_{\partial D} f(z)dz + \sum_{k=1}^{n} \frac{1}{2\pi i} \int\limits_{\substack{\partial E_k \\ \text{CW}}} f(z)dz.$$

But

$$\begin{aligned} \frac{1}{2\pi i} \int\limits_{\substack{\partial E_k \\ \text{CW}}} f(z)dz &= -\frac{1}{2\pi i} \int\limits_{\substack{\partial E_k \\ \text{CCW}}} f(z)dz \\ &= -\operatorname{Res}(f, z_k). \qquad \text{(def. of residue)} \end{aligned}$$

This proves the theorem.

QED

PROBLEM 4-1.

1. For any positive integer n, find all the singularities of the function $\dfrac{1}{z^n+1}$ and calculate all the corresponding residues. Also, compute the sum of all the residues.

2. Suppose that f has a pole at z_0 of order $\leq N$. The function

$(z-z_0)^N f(z)$ has a removable singularity at z_0. Prove that

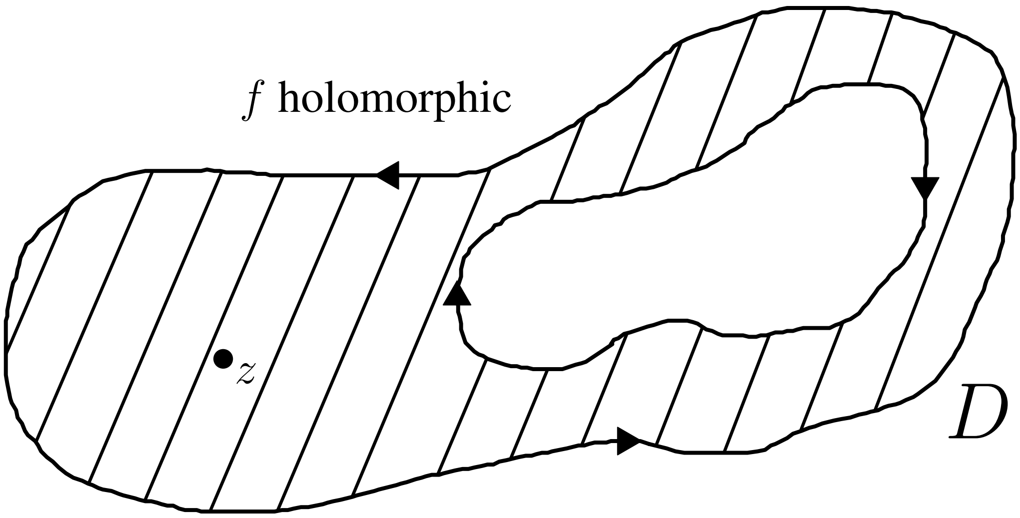
$$\mathrm{Res}(f,z_0) = \left.\frac{\left(\frac{d}{dz}\right)^{N-1}\left[(z-z_0)^N f(z)\right]}{(N-1)!}\right|_{z=z_0}.$$

3. Using the principal determination of $\log$, calculate the residues of $\dfrac{\log z}{(z^2+1)^2}$ at each of its singularities.

4. For any nonnegative integer n, calculate the residues of $(z^2+1)^{-n-1}$ at each of its poles. Present your answer with the binomial coefficient $\dbinom{2n}{n}$ displayed prominently.

5. Find the residues of $\csc^n z$ at $z=0$ for $n = 1, 2, 3, 4, 5$.
(That is, $\left(\dfrac{1}{\sin z}\right)^n$.)

REMARK ABOUT THE RESIDUE THEOREM: <u>It contains the Cauchy integral formula</u>. (Of course, we actually used the Cauchy integral theorem in its proof.) For consider the usual scene for the Cauchy formula:



We write the expected integral $\dfrac{1}{2\pi i}\displaystyle\int_{\partial D}\frac{f(\zeta)}{\zeta - z}d\zeta$ and notice that for fixed $z \in D$ the function $\dfrac{f(\zeta)}{\zeta - z}$ is a holomorphic function of ζ with one isolated singularity, at z. And it's the easy case!

$$\operatorname{Res}\left(\frac{f(\zeta)}{\zeta - z}, z\right) = \frac{f(z)}{\frac{d}{d\zeta}(\zeta - z)} = \frac{f(z)}{1} = f(z).$$

So indeed, the residue theorem $\Rightarrow$

$$\frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta = f(z).$$

WHAT'S AHEAD FOR US: The residue theorem is an amazing tool for accomplishing all sorts of things in complex analysis. It can produce wonderful <u>theoretical</u> results and also astonishing <u>computations</u>. We could present these in either order. However, I prefer the computational aspects first, because these techniques will give us lots of practice in dealing with our new concept of residues, and I think will also give us a nice change of pace in the middle of the book.

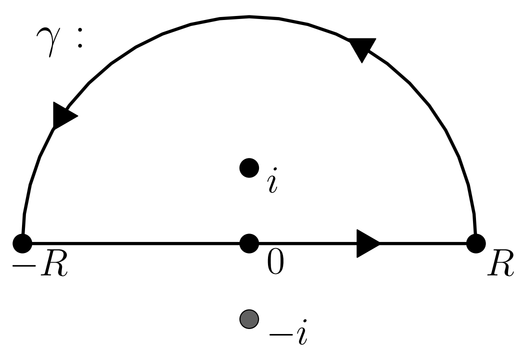
So here we go!

Section B: Evaluation of Some Definite Integrals

1. <u>An example for babies</u>:

A quite elementary integral in basic calculus is $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx = \pi$. Now we approach it in an entirely different way, much more complicated than actually necessary for such a problem... but the technique will lead the way for more interesting situations.

- Define the holomorphic function $f(z) = \dfrac{1}{z^2+1}$. This function has isolated singularities at i and $-i$.
- Devise a clever path. Here it is:



for large R we hope to approximate the desired integral, and we have a pole of f inside.

The residue theorem gives immediately

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{1}{z^2+1}dz &= \operatorname{Res}\left(\frac{1}{z^2+1}, i\right)\\ &= \frac{1}{2z}\bigg|_{z=i}\\ &= \frac{1}{2i}.\end{aligned}$$

Thus,

$$\int_\gamma \frac{dz}{z^2+1} = \pi.$$

- Let $R \to \infty$. We have

$$\int_{-R}^{R} \frac{dx}{x^2+1} + \int_{\text{semicircle}} \frac{dz}{z^2+1} = \pi.$$

The real integral is just what we want. We do not want to evaluate the semicircular integral, but instead to show that it has limit 0 as $R \to \infty$. So, we employ the basic estimate for line integrals:

(see Ch 3, pg. 60)

$$\left|\int_{\text{semicircle}} \frac{1}{z^2+1}\right| \le \max\left|\frac{1}{z^2+1}\right| \cdot \text{length of curve}.$$

The length of the curve is πR. And for $|z| = R$ we have from the triangle inequality

$$|z^2+1| \ge |z^2| - 1 = |z|^2 - 1 = R^2 - 1.$$

Thus, we achieve the estimate

$$\frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty.$$

- Final result:

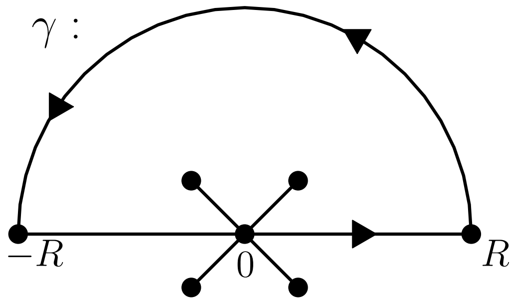
$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} = \pi.$$

2. A more challenging example:

Use the above template for $\dfrac{1}{z^4+1}$.

The denominator has (simple) zeros at the fourth roots of $-1 = e^{\pi i}$:

$$e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi i}{4}},$$



and the residues of f at each one equal

$$\frac{1}{4z^3} = \frac{z}{4z^4} = -\frac{z}{4}.$$

The residue theorem $\Rightarrow$

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{dz}{z^4+1} &= \text{sum of residues at } e^{\frac{\pi i}{4}} \text{ and } e^{\frac{3\pi i}{4}} \\ &= -\frac{e^{\frac{\pi i}{4}}}{4} - \frac{e^{\frac{3\pi i}{4}}}{4} \\ &= -\frac{\frac{1+i}{\sqrt{2}} + \frac{-1+i}{\sqrt{2}}}{4} \\ &= \frac{-i\sqrt{2}}{4}.\end{aligned}$$

Thus,

$$\int_\gamma \frac{dz}{z^4+1} = 2\pi i\left(\frac{-i\sqrt{2}}{4}\right) = \frac{\pi}{\sqrt{2}}.$$

Again, the integral on the semicircle has modulus bounded by

$$\frac{\pi R}{R^4-1} \to 0.$$

Conclusion:

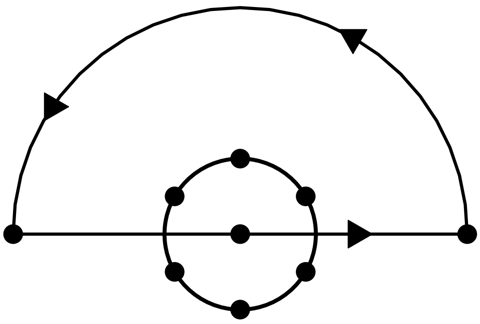
$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}}$$

3. Another:

Use $f(z) = \dfrac{1}{z^6+1}$.

Six poles this time:

$$e^{\frac{\pi i}{6}}, e^{\frac{\pi i}{2}}, e^{\frac{5\pi i}{6}}, e^{\frac{7\pi i}{6}}, e^{\frac{3\pi i}{2}}, e^{\frac{11\pi i}{6}}.$$



Everything works the same way. The sum of the three residues inside of D is

$$\begin{aligned}\sum \frac{1}{6z^5} = \sum \frac{z}{6z^6} &= -\frac{1}{6}\sum z \\ &= -\frac{1}{6}\left[e^{\frac{\pi i}{6}} + i + e^{\frac{5\pi i}{6}}\right] \\ &= -\frac{1}{6}\left[\frac{\sqrt{3}}{2} + \frac{i}{2} + i + \frac{-\sqrt{3}}{2} + \frac{i}{2}\right] \\ &= -\frac{1}{6}\cdot 2i = -\frac{i}{3}.\end{aligned}$$

So the integral we obtain is $2\pi i\left(-\frac{i}{3}\right) = \frac{2\pi}{3}$:

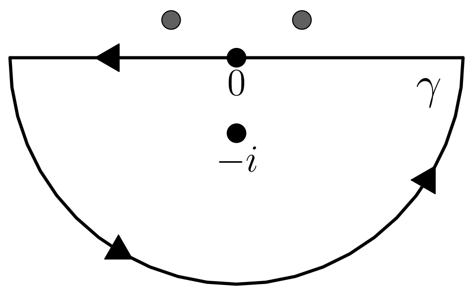
$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}}$$

4. Another:

Let's try $f(x) = \dfrac{1}{z^3 - i}$.

The three poles are roots of $z^3 = i = e^{\frac{\pi i}{2}} = e^{\frac{5\pi i}{2}} = e^{\frac{9\pi i}{2}}$. So we obtain

$$z = e^{\frac{\pi i}{6}}, e^{\frac{5\pi i}{6}}, e^{\frac{3\pi i}{2}}.$$



This time we save a small amount of work by using the lower semicircle, so that we deal with one residue only. The residue at $-i$ equals

$$\frac{1}{3z^2} = \frac{1}{3(-i)^2} = -\frac{1}{3}.$$

So the residue theorem gives $\int_\gamma \frac{1}{z^3-i}dz = -\frac{2\pi i}{3}$.

Again, the integral on the semicircle tends to 0 as $R \to \infty$, so our limiting equation is

$$\int_\infty^{-\infty} \frac{dx}{x^3-i} = -\frac{2\pi i}{3}.$$

Reverse the direction:

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^3-i} = \frac{2\pi i}{3}}$$

(P.S. we could have solved it this way:

$$\int_{-\infty}^{\infty} \frac{dx}{x^3-i} = \int_{-\infty}^{\infty} \frac{x^3+i}{x^6+1}dx = 0 + i\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi i}{3}$$

from the preceding example.)

5. <u>Long detailed discussion of another example</u>: $\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx,\ \alpha \in \mathbb{R}$.

   a. Convergence issues:

      Near ∞ the integrand is approximately $x^{\alpha-2}$, so we require $\underline{\underline{\alpha < 1}}$.
      Near 0 the integrand is approximately $x^{\alpha-1}$, so we require $\underline{\underline{\alpha > 0}}$.

      Thus, $\underline{\underline{0 < \alpha < 1}}$.

   b. Choose a holomorphic function:

      Let $f(z) = \frac{z^{\alpha-1}}{z+1}$, but we realize we'll have to cope with the ambiguity in

$$\begin{aligned} z^\alpha &= \exp(\alpha \log z) = \exp(\alpha(\ln|z| + i\arg z)) \\ &= |z|^\alpha \exp(i\alpha \arg z). \end{aligned}$$

   c. Residues:

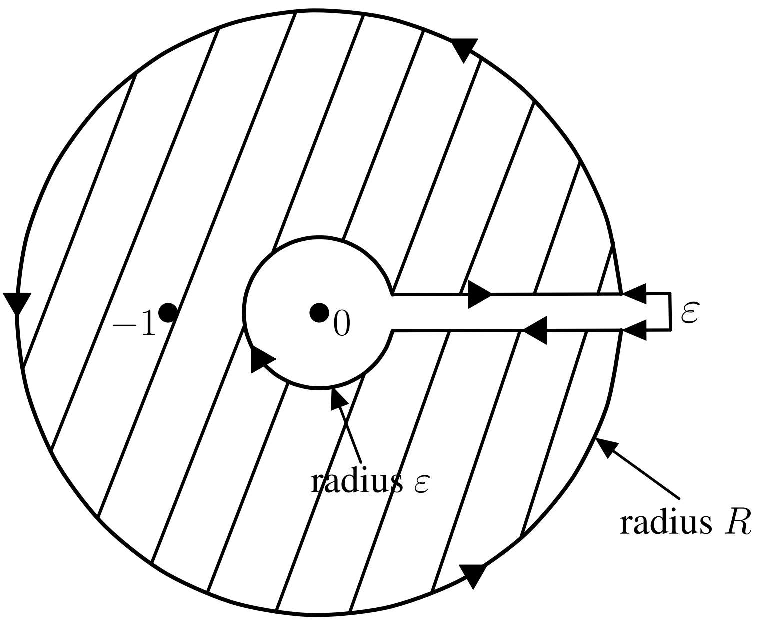
$z = 0$ is a branch point, not an isolated singularity. There is an isolated singularity at $z = -1$, and the residue is easy:

$$\begin{aligned}\operatorname{Res}(f, -1) &= \frac{(-1)^{\alpha-1}}{1} = -(-1)^{\alpha} \\ &= -\exp(i\alpha \arg(-1)).\end{aligned}$$

d. Path of integration:

Rather tricky! We need to have the positive real axis as part of the path, we need it to surround -1, and we need 0 to be outside!

Here is what we do:



γ is the boundary of the shaded region.

In this region we should use the choice of $\arg z$ so that $0 < \arg z < 2\pi$. Then we have

$$\operatorname{Res}(f, -1) = -\exp(i\alpha\pi) = -e^{i\alpha\pi}.$$

e. The residue theorem yields immediately that

$$\int_{\gamma} \frac{z^{\alpha-1}}{z+1} dz = -2\pi i e^{i\alpha\pi}.$$

f. $R \to \infty$

The integral on the large circle is bounded by

$$\begin{aligned}\max_{|z|=R} |f(z)| \cdot 2\pi R &= \max_{|z|=R} \frac{R^{\alpha-1}}{|z+1|} \cdot 2\pi R \\ &\leq \frac{R^{\alpha-1}}{R-1} \cdot 2\pi R \overset{\text{approx}}{=} 2\pi R^{\alpha-1}.\end{aligned}$$

Since $\underline{\alpha < 1}$, this tends to 0 as $R \to \infty$.

g. $\varepsilon \to 0$

The integral on the small circle is bounded by

$$\max_{|z|=\varepsilon} \frac{\varepsilon^{\alpha-1}}{|z+1|} \cdot 2\pi\varepsilon \leq \frac{\varepsilon^{\alpha-1}}{1-\varepsilon} \cdot 2\pi\varepsilon \overset{\text{approx}}{=} 2\pi\varepsilon^{\alpha}.$$

Since $\underline{\alpha > 0}$, this tends to 0 as $\varepsilon \to 0$.

REMARK: our criteria for convergence of the desired integral match perfectly with what is needed in the line integral as $R \to \infty$, $\varepsilon \to 0$.

h. We have remaining two integrals along the positive real axis. In the "upper" one we have $z = x$ with argument 0, so it becomes

$$\int_0^\infty \frac{z^{\alpha-1}}{z+1} dz = \int_0^\infty \frac{x^{\alpha-1}}{x+1} dx =: I.$$

But in the "lower one" $z = x$ with argument 2π, so it becomes

$$-\int_0^\infty \frac{z^{\alpha-1}}{z+1} dz = -\int_0^\infty \frac{x^{\alpha-1} e^{i(\alpha-1)2\pi}}{x+1} dx = -e^{i\alpha 2\pi} I.$$

i. Summary: the equation in e. becomes in the limit

$$I - e^{i\alpha 2\pi} I = -2\pi i e^{i\alpha\pi}.$$

Solve for I:

$$\begin{aligned}I &= \frac{-2\pi i e^{i\alpha\pi}}{1 - e^{i\alpha 2\pi}} = \frac{2\pi i}{e^{i\alpha\pi} - e^{-i\alpha\pi}} \\ &= \frac{\pi}{\sin \alpha\pi}.\end{aligned}$$

CONCLUSION: $$\boxed{\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx = \frac{\pi}{\sin \alpha\pi} \quad \text{for } 0<\alpha<1}$$

This is just about the easiest example of this type of analysis, but I have included great detail to justify all the reasoning. After some practice this should become almost routine for you.

REMARK: Problem 4-1, #5... to find the residues

$$\operatorname{Res}(\csc^n z, 0) \quad \text{for } n = 1, 2, \dots.$$

The easy cases are

$$n = 1: \text{ simple pole, residue } = 1.$$
$$n = 2, 4, 6, \dots \text{ even function, residue } = 0.$$

$n=3$: Here's a beautiful and elegant technique, which I choose to call *integration by parts*. It's based on the fact that if f is a holomorphic function on a closed path γ, then

$$\int_\gamma f'(z)dz = 0.$$

This is a FTC fact, see Chapter 3 Section B, page 50. Now apply this to a product fg of holomorphic functions:

$$0 = \int_\gamma (fg)'dz = \int_\gamma f'gdz + \int_\gamma fg'dz,$$

or

$$\int_\gamma f'gdz = -\int_\gamma fg'dz. \qquad \text{"INTEGRATION BY PARTS"}$$

In the case of functions with isolated singularities at z_0, when we use a small circle γ surrounding z_0 we obtain

$$\boxed{\operatorname{Res}(f'g, z_0) = -\operatorname{Res}(fg', z_0)}$$

Now we try this on $\csc^3 z$ near $z_0 = 0$. Then

$$\sin^{-3} z = \left(\frac{\sin^{-2} z}{-2}\right)' \sec z,$$

so that

$$\begin{aligned}
\operatorname{Res}(\sin^{-3} z, 0) &= \operatorname{Res}\left(\frac{\sin^{-2} z}{2} \sec' z, 0\right) \\
&= \operatorname{Res}\left(\frac{\sin^{-2} z}{2} \sec z \tan z, 0\right) \\
&= \operatorname{Res}\left(\frac{\sin^{-1} z}{2} \sec^2 z, 0\right).
\end{aligned}$$

Look what just happened! We started with a pole of order 3, and now we have a pole of order 1! So, we're in the easy case, and

$$\operatorname{Res}\left(\frac{\sec^2 z}{2\sin z}, 0\right) = \frac{1}{2\cos 0} = \frac{1}{2}.$$

REMARK: "Integration by parts" is somewhat a misnomer. For Definition 3 gives the result immediately that a derivative of a holomorphic function has zero residue, since on page 78 we simply take $a = 0$. Nonetheless, I like the IBP name for this principle, as the result is so reminiscent of such a procedure.

6. Integrals of a certain form:

$$\int_0^\infty \frac{a(x)}{b(x)}dx$$

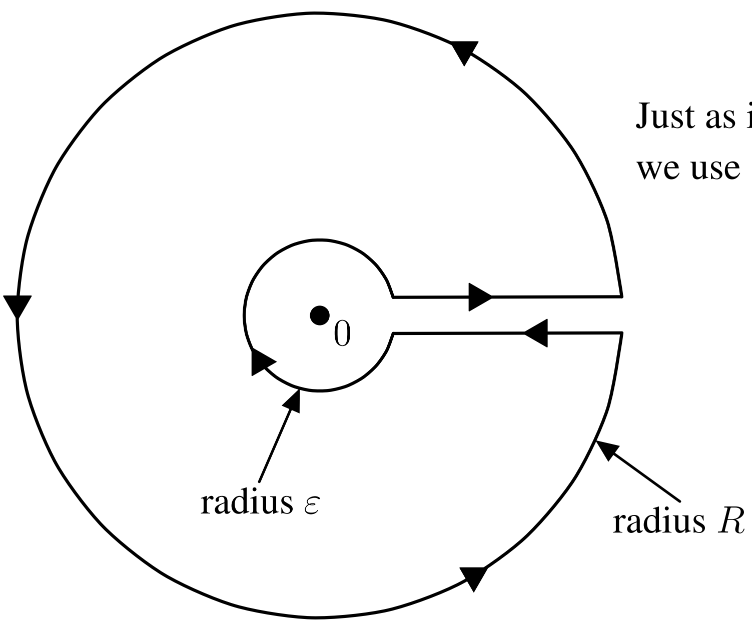
where:
- a, b are polynomials
- degree $b \geq$ degree $a + 2$
- $b \neq 0$ on $[0, \infty) \subset \mathbb{R}$

(Notice that the integration extends only over the positive real axis!)
Such situations can always be handled with residue theory, so we'll derive a general result and then apply it to a particular example.

The approach is quite clever! We define

$$\boxed{f(z) = \frac{a(z)}{b(z)} \log z}$$

and use this type of path:



Just as in example 5 on page 88, we use $0 < \arg z < 2\pi$.

The integrals on the two circles tend to 0 in the limit as $\varepsilon \to 0$ and $R \to \infty$. The extra factor of $\log z$ is of very little concern, since

$$\big|\log z\big| = \big|\ln|z| + i\arg z\big| \leq \big|\ln|z|\big| + 2\pi.$$

So, when $|z| = R$ we have

$$|\log z| \leq \ln R + 2\pi \leq 2\ln R,$$

and for $|z| = \varepsilon$

$$|\log z| \leq |\ln \varepsilon| + 2\pi \leq 2\ln\frac{1}{\varepsilon}$$

for $R \to \infty$ and $\varepsilon \to 0$. Thus, on the circle $|z| = R$ we have

$$\left|\int_{|z|=r} f(z)dz\right| \leq \frac{\text{constant}}{R^2} \cdot \ln R \cdot 2\pi R \leq \frac{\text{constant} \cdot \ln R}{R},$$

which tends to 0 as $R \to \infty$... since $\ln R \to \infty$ much slower than R. And on the circle $|z| = \varepsilon$ we have a similar estimate

$$\left|\int_{|z|=\varepsilon} f(z)dz\right| \leq \text{constant} \cdot \ln\frac{1}{\varepsilon} \cdot 2\pi\varepsilon \leq \text{constant} \cdot \varepsilon\ln\frac{1}{\varepsilon}.$$

Again, this tends to 0 as $\varepsilon \to 0$.

Thus, we apply the residue theorem and then let $R \to \infty$, $\varepsilon \to 0$, and obtain in the limit

$$\int_0^\infty \frac{a(x)}{b(x)}\ln x dx - \int_0^\infty \frac{a(x)}{b(x)}\left[\ln x + 2\pi i\right]dx$$
$$= 2\pi i \cdot \text{sum of the residues of } f(z).$$

Notice that on the path $(0, \infty)$ <u>above</u> the axis, $\log z = \ln x$ $(\arg z = 0)$, but on the path $(\infty, 0)$ <u>below</u> the axis, $\log z = \ln|x| + 2\pi i$, because $\arg z = 2\pi i$.

So when we subtract the integrals, $\ln x$ disappears, and we're left with

$$-2\pi i\int_0^\infty \frac{a(x)}{b(x)}dx.$$

Divide by $-2\pi i$ to achieve the formula

$$\boxed{\int_{-\infty}^\infty \frac{a(x)}{b(x)}dx = -\text{ the sum of all residues of } \frac{a(z)}{b(z)}\log z}$$

(Here $0 < \arg z < 2\pi$.)
(Though it doesn't matter!)

EXAMPLE: Let $a > 0$ and consider

$$I = \int_0^\infty \frac{x}{(x+a)(x^2+1)} dx.$$

All our requirements are met. The poles are located at $-a, i$, and $-i$, and they are all simple! When the residues are computed we obtain

$$\begin{aligned}\operatorname{Res}\left(\frac{z}{(z+a)(z^2+1)}\log z, -a\right) &= \frac{-a\log(-a)}{1\cdot(a^2+1)} \\ &= \frac{-a\left(\ln a + \boxed{\pi i}\right)}{a^2+1};\end{aligned}$$

at i we have the residue

$$\frac{i\log i}{(i+a)2i} = \frac{i\boxed{\dfrac{i\pi}{2}}}{(a+i)2i} = \frac{i\pi}{4(a+i)} = \frac{i\pi(a-i)}{4(a^2+1)}.$$

And at $-i$

$$\frac{-i\log(-i)}{(-i+a)2(-i)} = \frac{-i\boxed{\dfrac{3i\pi}{2}}}{(a-i)(-2i)} = \frac{3i\pi}{4(a-i)} = \frac{3i\pi(a+i)}{4(a^2+1)}.$$

When we add these three residues, we obtain

$$\begin{aligned}&\frac{1}{a^2+1}\left\{(-a\ln a - \underline{\underline{a\pi i}}) + \frac{\underline{\underline{i\pi a}} + \pi}{4} + \frac{\underline{\underline{3i\pi a}} - 3\pi}{4}\right\} \\ &\qquad = \frac{1}{a^2+1}\left\{-a\ln a - \frac{\pi}{2}\right\}.\end{aligned}$$

Therefore, we obtain from the formula the result that for $a > 0$

$$\boxed{\int_0^\infty \frac{x}{(x+a)(x^2+1)} dx = \frac{a\ln a + \frac{\pi}{2}}{a^2+1}}$$

REMARK: *Many examples we demonstrate can actually be done with single variable calculus. this is a good example, as are 1,2,3,4. We could even find the indefinite integral first. Regardless, these techniques are exceedingly beautiful even in such cases!*

PROBLEM 4-2. For any integer $n \geq 2$, compute the integral

$$\int_0^\infty \frac{dx}{x^n + x^{n-1} + \ldots + x + 1}.$$

DISCUSSION: This fits what we have just done, with $a(z) = 1$ and $b(z) = z^n + z^{n-1} + \ldots + z + 1$. Here's an approach using lots of calculations:

- Poles:

  $b(z) = \dfrac{z^{n+1} - 1}{z - 1}$ so poles occur when $z^{n+1} = 1$ (and $z \neq 1$). I suggest defining

  $$\omega = e^{\frac{2\pi i}{n+1}},$$

  so that poles are ω^k for $1 \leq k \leq n$.

- Residues of $\dfrac{\log z}{b(z)}$ are happily

  $$\frac{\log \omega^k}{b'(\omega^k)} = \frac{k \log \omega}{b'(\omega^k)} = \frac{2\pi i}{n+1} \frac{k}{b'(\omega^k)}.$$

  For $z = \omega^k$ we have

  $$b'(z) = -\frac{z^{n+1} - 1}{(z-1)^2} + \frac{(n+1)z^n}{z-1} = \frac{n+1}{z^2 - z} \qquad \text{(WHY?)}$$

- Formula of page 93 $\Rightarrow$ our integral equals

  $$-\sum_{k=1}^{n} \frac{2\pi i k}{n+1} \frac{\omega^{2k} - \omega^k}{n+1} = \frac{2\pi i}{(n+1)^2} \left(\sum_{k=1}^{n} k\omega^k - \sum_{k=1}^{n} k\omega^{2k} \right).$$

- Algebra: SHOW THAT the term '( )' on the preceding line equals

  $$zb'(z)\Big|_{\omega^2}^{\omega} = (n+1)\left(\frac{\omega}{\omega^2 - \omega} - \frac{\omega^2}{\omega^4 - \omega^2} \right).$$

- Finish: Greatly simplify to get the final answer in the form

$$\frac{\pi}{(n+1)\sin(??)}.$$

7. Principal value integrals refer to reasonable attempts to define a sort of integral even when the integrand is not actually integrable. There are two different situations where this may occur. Here are illustrations of these types:

Type 1: $\int_{-1}^{e} \frac{1}{x} dx$ does not actually exist, since the two "sub-integrals" do not exists:

$$\int_{-1}^{0} \frac{1}{x} dx = -\infty \text{ and } \int_{0}^{e} \frac{1}{x} dx = +\infty.$$

So what we may do is first delete a symmetric interval about 0 and then perform a limit:

$$\begin{aligned} \int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{e} \frac{dx}{x} &= \ln|x|\Big|_{-1}^{-\varepsilon} + \ln|x|\Big|_{\varepsilon}^{e} \\ &= (\ln\varepsilon - 0) + (1 - \ln\varepsilon) \\ &= 1. \end{aligned}$$

So, the limit as $\varepsilon \to 0$ does exist, and is called the *principal value integral*. Notation:

$$\text{PV} \int_{-1}^{e} \frac{dx}{x} = 1.$$

Type 2: $\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx$ again does not exist, for

$$\int_{-\infty}^{0} \frac{x}{x^2+1} dx = -\infty \text{ and } \int_{0}^{\infty} \frac{x}{x^2+1} dx = +\infty.$$

But we can define a principal value by integrating from $-R$ to R and then letting $R \to \infty$. Notation:

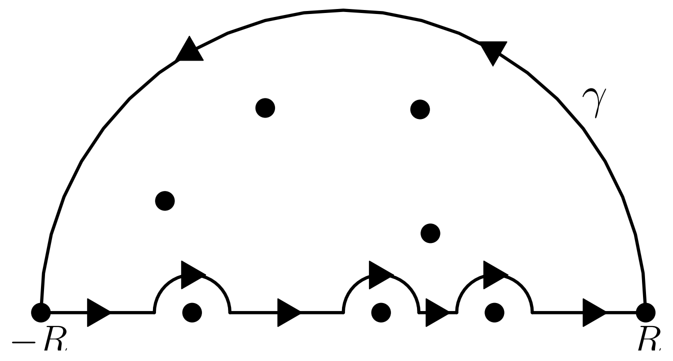
$$\text{PV} \int_{-\infty}^{\infty} \frac{x}{x^2+1} dx = 0.$$

The residue theorem can often be of use in calculating such integrals. I'll give as an example a typical situation, and we'll see others.

So, assume $f(z)$ is a rational funciton of z for which degree of denominator ≥ 2 + degree of numerator, just as on page 92.

Furthermore, if f has any real poles, assume they are simple.

We shall then apply the residue theorem when our line integral has this form:



We are of course familiar with the large semicircle. The new twist is that we have semicircles of small radius centered at the real (simple) poles of f.

The radius R is so large that all the real poles of f are between $-R$ and R, and all the poles with imaginary part greater than 0 satisfy $|z| < R$. The residue theorem then gives immediately the equation

$$\int_\gamma f(z)dz = 2\pi i\mathcal{R},$$

where we have denoted

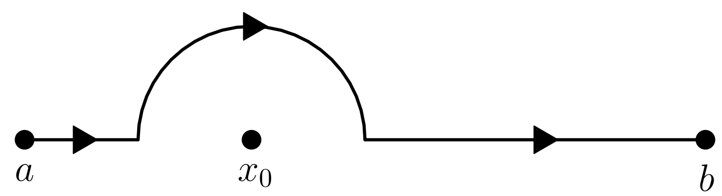
$$\mathcal{R} = \sum_{\mathrm{Im}(z)>0} \mathrm{Res}(f, z).$$

Because of the restriction on degrees of denominator and numerator of f, we can let $R \to \infty$ and obtain

$$\int_{\gamma_\varepsilon} f(z)dz = 2\pi i\mathcal{R},$$

where γ_ε represents the real axis with semicircular arcs of radius ε situated about the real poles of f.

It is fascinating to see what happens when we let $\varepsilon \to 0$. We can deal with each real pole individually. So, look at a pole at $x_0 \in \mathbb{R}$ and the portion of γ_ε from a to b, when $a < x_0 < b$:



Parametrize the semicircle as $z = x_0 + \varepsilon e^{i\theta}$, where θ travels from π to 0. We obtain

$$\int_a^{x_0-\varepsilon} f(x)dx + \int_\pi^0 f(x_0+\varepsilon e^{i\theta}) i\varepsilon e^{i\theta} d\theta + \int_{x_0+\varepsilon}^b f(x)dx.$$

The first plus third of these integrals will have the limit as $\varepsilon \to 0$:

$$\text{PV} \int_a^b f(x)dx \ldots \ \underline{\text{provided the limit exists}}!$$

The limit does indeed exist, as we see from analyzing the second integral:

from page 78 we have

$$\text{Res}(f, x_0) = \lim_{z \to x_0} (z - x_0) f(z),$$

since x_0 is a simple pole. Thus

$$\text{Res}(f, x_0) = \lim_{\varepsilon \to 0} \varepsilon e^{i\theta} f(x_0 + \varepsilon e^{i\theta}) \qquad (\underline{\text{uniformly}} \text{ w.r.t. } \theta)$$

and we therefore obtain

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_\pi^0 f(x_0+\varepsilon e^{i\theta}) i\varepsilon e^{i\theta} d\theta &= \int_\pi^0 i \, \text{Res}(f, x_0) d\theta \\ &= -\pi i \, \text{Res}(f, x_0).\end{aligned}$$

Doing this for each pole thus yields the formula

$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)dz = \text{PV} \int_{-\infty}^{\infty} f(x)dx - \pi i \mathcal{R}^*,$$

where

$$\mathcal{R}^* = \sum_{x \in \mathbb{R}} \text{Res}(f, x).$$

Thus, we have derived the result, that under the given restrictions on f,

$$\boxed{\text{PV} \int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{\text{Im}(z)>0} \text{Res}(f, z) + \pi i \sum_{x \in \mathbb{R}} \text{Res}(f, x)}$$

EXAMPLE: $f(z) = \dfrac{1}{z^3+1}$. The real pole is at -1, and the residue there is $\dfrac{1}{3z^2} = \dfrac{1}{3}$. The other pole to consider is $e^{\frac{i\pi}{3}}$, with residue $\dfrac{1}{3z^2} = \dfrac{1}{3} e^{\frac{-2i\pi}{3}} = \dfrac{1}{3}\left(-\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}\right)$. Thus, we have to calculate $2\pi i \left(-\dfrac{1}{6} - \dfrac{i}{2\sqrt{3}}\right) + \dfrac{\pi i}{3} = \dfrac{\pi}{\sqrt{3}}$, so we have

$$\boxed{\text{PV} \int_{-\infty}^{\infty} \frac{dx}{x^3+1} = \frac{\pi}{\sqrt{3}}}$$

PROBLEM 4-3. For $0 < \alpha < 1$ calculate

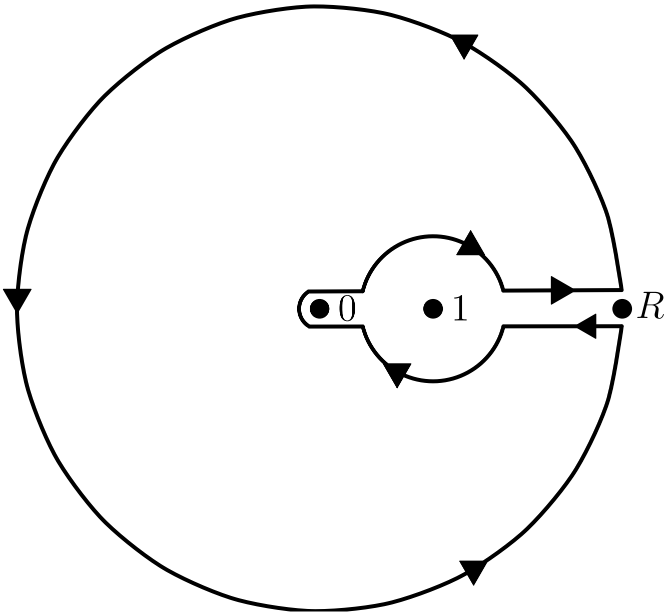
$$\text{PV}\int_0^\infty \frac{x^{\alpha-1}}{1-x}dx.$$

RECOMMENDATION:

$$f(z) = \frac{z^{\alpha-1}}{1-z}$$

with

$$0 < \arg z < 2\pi$$



8. Integrals involving exponentials (and trig functions):

This is an extremely important type of definite integral, and we'll gain extra proficiency in our techniques as well as in our use of exp. First, we'll look at two rather typical examples.

Example 1: $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}dx.$

First, recall the simple equation $|e^w| = e^{\operatorname{Re} w}$. Therefore,

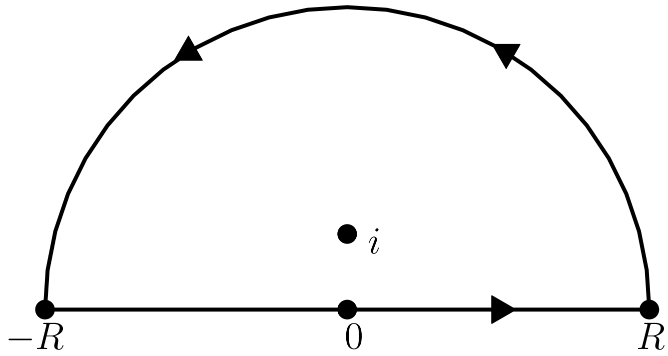
$$\left|e^{iax}\right| = e^{\operatorname{Re}(iax)} = e^{-\operatorname{Im}(ax)} = e^{-x\operatorname{Im}(a)}.$$

So, if $\operatorname{Im}(a) \neq 0$, then the integrand blows up exponentially as $x \to \infty$ or as $x \to -\infty$, and we have no existing integral. Therefore, we definitely must assume that $a \in \mathbb{R}$.

We are led to define $f(z) = \dfrac{e^{iaz}}{z^2+1}$, a holomorphic function with isolated simple poles at $\pm i$.

Let's try our familiar semicircle: knowing we'll need to worry about the path $|z| = R$, we examine

$$\begin{aligned}|f(z)| &= \frac{|e^{iaz}|}{|z^2+1|} = \frac{e^{-\operatorname{Im}(az)}}{|z^2+1|} \\ &\le \frac{e^{-a\operatorname{Im}(z)}}{|z^2|-1} = \frac{e^{-a\operatorname{Im}(z)}}{R^2-1}.\end{aligned}$$



We'll be in bad trouble if $a < 0$! (Since $\operatorname{Im}(z) = R$.) Therefore, we also assume $a \ge 0$. Then we have for $|z| = R$ that

$$|f(z)| \le \frac{1}{R^2-1},$$

and we conclude that since the length of the semicircle is πR, the line integral

$$\left| \int\limits_{\substack{|z|=R \\ \operatorname{Im}(z)>0}} f(z)dz \right| \le \frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty.$$

Thus, the residue theorem implies (after letting $R \to \infty$) that

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= 2\pi i \operatorname{Res}(f(z), i) \\ &= 2\pi i \frac{e^{iai}}{2i} = \pi e^{-a}.\end{aligned}$$

Thus, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx = \pi e^{-a} \text{ for } a \ge 0.$$

Finish: if $a < 0$ we obtain immediately by conjugation the result πe^{a}. Therefore, we have in general

$$\boxed{\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx = \pi e^{-|a|} \text{ for all } a \in \mathbb{R}}$$

What a terrific result! If $a = 0$ this is a very elementary integral since $\arctan$ *has derivative* $\frac{1}{x^2+1}$*. But for $a \ne 0$ there's no convenient indefinite integral.*

<u>Example 2</u>: $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+1}dx$

Again, we must assume that $a \in \mathbb{R}$. We will also first deal with the case $a > 0$. (The case $a = 0$ is quite different, as

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1}dx$$

exists only in the principal value sense — and clearly is 0.)

Thus, we define

$$f(z) = \frac{ze^{iaz}}{z^2+1}.$$

The residue at $z = i$ is $\displaystyle\frac{ie^{iai}}{2i} = \frac{1}{2}e^{-a}$.

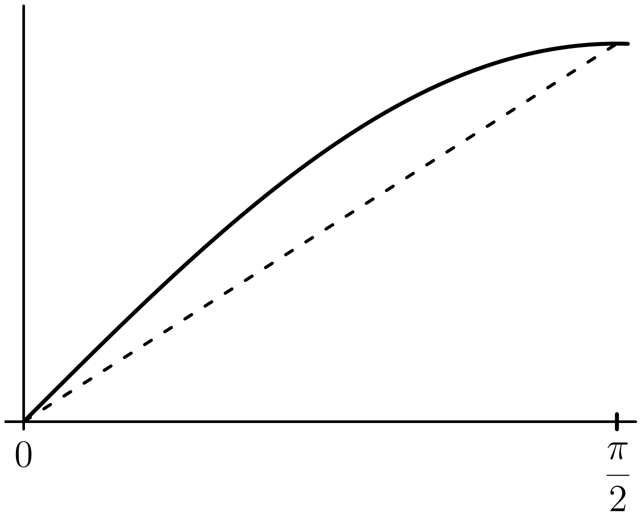
We employ the same semicircular path, and we first try to estimate the line integral along $|z| = R$:

$$\begin{aligned}
\left| \int\limits_{\substack{|z|=R \\ \operatorname{Im} z>0}} f(z)dz \right| &\le \int\limits_{\substack{|z|=R \\ \operatorname{Im} z>0}} \frac{|z|e^{-a\operatorname{Im} z}}{|z|^2-1}|dz| \\
&= \frac{R}{R^2-1}\int_0^{\pi} e^{-aR\sin\theta} R d\theta \\
&= \frac{R^2}{R^2-1}\int_0^{\pi} e^{-aR\sin\theta} d\theta.
\end{aligned}$$

Uh oh! We can no longer simply use the estimate $e^{-aR\sin\theta} \le 1$, so we have to be cleverer. Not knowing how to integrate $e^{-aR\sin\theta}$, we employ a useful estimate. First, we can integrate from 0 to $\dfrac{\pi}{2}$ only and double the answer to get the estimate

$$\frac{2R^2}{R^2-1}\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}d\theta < 3\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}d\theta$$

for large R. Then we estimate $\sin\theta$ <u>from below</u> by observing its graph:



$$\sin\theta \geq \frac{2\theta}{\pi} \text{ for } 0 \leq \theta \leq \frac{\pi}{2}.$$

Therefore, we find

$$\begin{aligned}\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta} d\theta &< \int_0^{\frac{\pi}{2}} e^{\frac{-aR2\theta}{\pi}} d\theta \\ &< \int_0^{\infty} e^{\frac{-aR2\theta}{\pi}} d\theta = \frac{\pi}{2aR}\ \textbf{!}\end{aligned}$$

Conclusion: the line integral of $f(z)$ along the semicircle has modulus no bigger than:

$$\frac{R^2}{R^2-1}\frac{3\pi}{2aR} \to 0 \text{ as } R \to \infty.$$

Thus, we again obtain from the residue theorem:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)dx &= 2\pi i \operatorname{Res}(f(z), i) \\ &= 2\pi i \frac{e^{-1}}{2} \\ &= \pi i e^{-1}.\end{aligned}$$

Here is the result for all $a \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+1} dx = \begin{cases} \pi i e^{-a} & \text{for } a > 0, \\ 0 & \text{for } a = 0 \text{ (PV integral)}, \\ -\pi i e^{a} & \text{for } a < 0. \end{cases}$$

SUMMARY: Using Euler's formula $e^{iax} = \cos ax + i\sin ax$, we see that the symmetry of the integrands gives the two results in the form:

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1} dx = \pi e^{-|a|} \text{ for } a \in \mathbb{R},$$

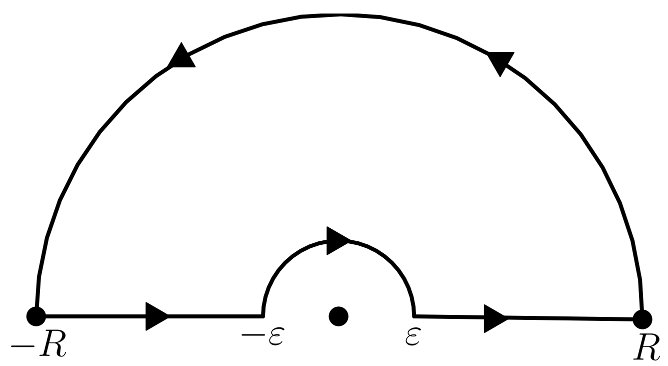
$$\int_{-\infty}^{\infty} \frac{x \sin ax}{x^2+1} dx = \begin{cases} \pi e^{-a} & \text{for } a > 0, \\ 0 & \text{for } a = 0, \\ -\pi e^{a} & \text{for } a < 0. \end{cases}$$

REMARK: The integrals $\int_{-\infty}^{\infty} \frac{x \sin ax}{x^2+1} dx$ are <u>not</u> principal value integrals, as the integrand is an even function of x. However, they are <u>improper</u> integrals as they are not absolutely integrable:

$$\int_{-\infty}^{\infty} \left| \frac{x \sin ax}{x^2+1} \right| dx = \infty \quad (\text{for } a \neq 0).$$

<u>Two more example involving trigonometric functions</u>:

- The first can be found in almost every textbook on complex analysis. It's the integral
$$\int_0^{\infty} \frac{\sin ax}{x} dx, \text{ where as usual } a \in \mathbb{R}.$$
This integral is of course 0 if $a = 0$. Otherwise, it is an <u>improper</u> integral, since
$$\int_0^{\infty} \left| \frac{\sin ax}{x} \right| dx = \infty. \quad \text{(Not hard to show.)}$$
Thus, it has to be interpreted as
$$\lim_{R \to \infty} \int_0^{R} \frac{\sin ax}{x} dx.$$
The choice of holomorphic function is crucial! We must <u>not</u> choose $\frac{\sin az}{z}$, because of its large modulus when $\operatorname{Im} z \neq 0$. Therefore, the logical choice is
$$f(z) = \frac{e^{iaz}}{z}.$$
(This actually introduces a <u>pole</u> at 0.) Prior experience leads us to assume at first that $a > 0$ and to choose a path like this one:



Our earlier estimates on page 101 show that the integral along $|z| = R$ tends to 0, since $a > 0$. Thus, the residue theorem gives

$$\int_{-\infty}^{-\varepsilon} f(x)dx + \int\limits_{\substack{CW\\ semicircle}} f(z)dz + \int_{\varepsilon}^{\infty} f(x)dx = 0.$$

Then our work on page 98 yields in the limit as $\varepsilon \to 0$

$$\text{PV}\int_{-\infty}^{\infty} f(x)dx - \pi i\,\text{Res}(f,0) = 0.$$

This residue is 1, so our result is

$$\text{PV}\int_{-\infty}^{\infty} \frac{e^{iax}}{x}dx = \pi i \text{ for } a > 0.$$

I.e.,

$$\text{PV}\int_{-\infty}^{\infty} \frac{\cos ax + i\sin ax}{x}dx = \pi i.$$

All that survives from this equation is

$$\int_{-\infty}^{\infty} \frac{\sin ax}{x}dx = \pi \text{ for } a > 0.$$

Note: no PV is left, as that goes only with the cosine term. Or we could have taken the imaginary part of each side.

Since the integrand is even as a function of x, we obtain

$$\int_{0}^{\infty} \frac{\sin ax}{x}dx = \frac{\pi}{2} \text{ for } a > 0.$$

Finally, since $\sin ax$ is an odd function of a, our final result is

$$\int_{0}^{\infty} \frac{\sin ax}{x}dx = \begin{cases} \dfrac{\pi}{2} & \text{for } a > 0, \\ 0 & \text{for } a = 0, \\ -\dfrac{\pi}{2} & \text{for } a < 0. \end{cases}$$

- Our second example is the integral

$$\int_{0}^{\infty} \frac{1-\cos ax}{x^2}dx.$$

(This is a proper integral, thanks to the boundedness of the integrand as $z \to 0$ and its x^{-2}-like decay at ∞.)

Our experience leads us to assume $a > 0$ and to choose

$$f(z) = \frac{1 - e^{iaz}}{z^2}.$$

This function has a simple pole at $z = 0$, with

$$\begin{aligned}\operatorname{Res}(f, 0) &= \operatorname{Res}\left(\frac{-iaz + \dots}{z^2}, 0\right) \\ &= -ia.\end{aligned}$$

Using the same path as above and letting $R \to \infty, \varepsilon \to 0$, we obtain

$$\mathrm{PV}\int_{-\infty}^{\infty} \frac{1 - e^{iax}}{x^2} dx = \pi i(-ia) = \pi a.$$

And then Euler's formula gives

$$\int_{-\infty}^{\infty} \frac{1 - \cos ax}{x^2} dx = \pi a \text{ for } a > 0.$$

Final result:

$$\boxed{\int_0^{\infty} \frac{1 - \cos ax}{x^2} dx = \frac{\pi|a|}{2} \text{ for all } a \in \mathbb{R}}$$

(A standard trig identity $\Rightarrow$

$$\int_0^{\infty} \frac{2\sin^2 \frac{ax}{2}}{x^2} dx = \frac{\pi|a|}{2},$$

and thus $(a \to 2a)$

$$2\int_0^{\infty} \frac{\sin^2 ax}{x^2} dx = \pi|a|,$$

giving the result

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2} dx = \pi|a|}\ )$$

PROBLEM 4-14. Calculate

$$\mathrm{PV}\int_0^{\infty} \frac{\cos ax}{x^2 - 1} dx \quad \text{for } a \in \mathbb{R}.$$

PROBLEM 4-15. Calculate

$$\mathrm{PV}\int_0^\infty \frac{x\sin ax}{x^2-1}dx \quad \text{for } a \in \mathbb{R}.$$

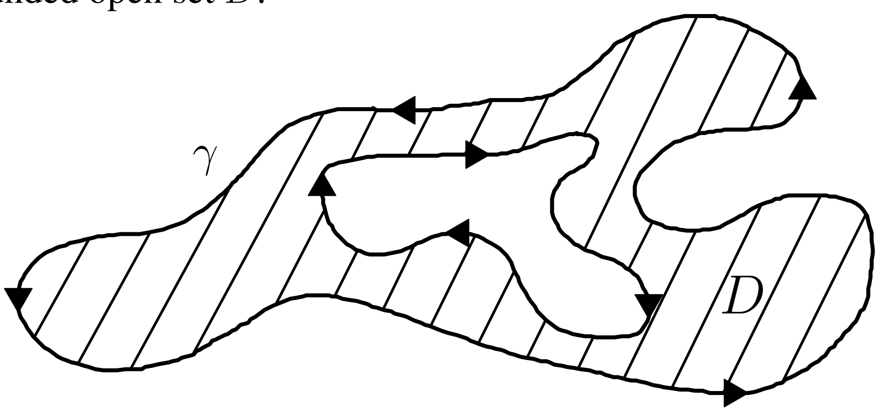
Chapter 5

Residues (Part II)

Section A: The Counting Theorem

We now consider some astonishing theoretical consequences of the residue theorem.

Here is the situation we are going to be discussing. We will have a "nice" simple closed path γ in $\mathbb{C}$. It can be regarded as the oriented boundary of a "nice" bounded open set D:



We also will have a function f defined in an open set containing $D \cup \gamma$, and we assume that f is holomorphic except for finitely many poles (no essential singularities allowed.)

Furthermore, we assume that on γ our function f has no zeros and no poles. It therefore makes sense to form the line integral

$$\int_\gamma \frac{f'(z)}{f(z)} dz.$$

(The denominator is never 0 for $z \in \gamma$ and the numerator is continuous on

γ.)

Then we have

THE COUNTING THEOREM

Assuming the above hypothesis

$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz = \text{NUMBER OF ZEROS OF } f \text{ IN } D$$
$$- \text{ NUMBER OF POLES OF } f \text{ IN } D.$$

Here these numbers are counted according to their multiplicities.

Proof: The residue theorem (page 81) asserts that the LHS of this equation equals

$$\text{the sum of the residues of } \frac{f'(z)}{f(z)} \text{ at all its singularities in } D.$$

So, we must examine these singularities. They occur precisely at points $z_0 \in D$ such that either z_0 is a zero of f or z_0 is a pole of f.

If z_0 is a zero of f of order $m \geq 1$: We then write the Taylor series of f centered at z_0

$$f(z) = \sum_{k=m}^{\infty} c_k(z-z_0)^k \qquad (c_m \neq 0).$$

Then we factor $(z-z_0)^m$ from the RHS to arrive at

$$f(z) = (z-z_0)^m g(z),$$

where g is holomorphic in a neighborhood of z_0 and $g(z_0) \neq 0$. Then

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{(z-z_0)^m g'(z) + m(z-z_0)^{m-1}g(z)}{(z-z_0)m(z)} \\ &= \frac{g'(z)}{g(z)} + \frac{m}{z-z_0}.\end{aligned}$$

(note that g(z) is holomorphic near z_0)

Thus, $\dfrac{f'(z)}{f(z)}$ has a simple pole at z_0 with

$$\operatorname{Res}\left(\frac{f'}{f}, z_0\right) = m.$$

If z_0 is a pole of f of order $n \geq 1$: In the very same manner we can factor f in the form

$$f(z) = (z - z_0)^{-n} h(z),$$

where h is holomorphic in a neighborhood of z_0 and $h(z_0) \neq 0$. Then

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \frac{(z - z_0)h'(z) - n(z - z_0)^{-n-1}h(z)}{(z - z_0)^{-n}h(z)} \\ &= \frac{h'(z)}{h(z)} - \frac{n}{z - z_0}. \qquad \text{(note that h(z) is holomorphic near } z_0\text{)} \end{aligned}$$

Then $\dfrac{f'(z)}{f(z)}$ has a simple pole at z_0 with

$$\operatorname{Res}\left(\frac{f'}{f}, z_0\right) = -n.$$

Conclusion: when we sum the residues of $\dfrac{f'(z)}{f(z)}$ at all its singularities in D, we are obtaining the total order of all zeroes of f in D minus the total order of all the poles of f in D.

QED

An interesting corollary of this result depends on a way of interpreting the line integral of $\dfrac{f'}{f}$. At any point $z_0 \in \gamma$, $f(z_0)$ is not zero and is holomorphic and not zero in a small disc centered at z_0. Thus, there is a continuous determination of $\arg f(z)$ in this disc. (See the discussion on page 27.) And therefore, $\log f(z)$ becomes a holomorphic function, with derivative

$$\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)}.$$

Now if we start at some point $z_0 \in \gamma$ with a choice of $\arg f(z_0)$ and extend that choice continuously as we traverse γ, then $\arg f(z_0)$ at the end of the path will be the original choice $+2\pi N$, for some integer N. Then we may write

$$\int_\gamma \frac{f'(z)}{f(z)} dz = \log f(z)\Big|_{\text{end of } \gamma} - \log f(z)\Big|_{\text{beginning of } \gamma}.$$

But of course, $\log f(z) = \log |f(z)| + i \arg f(z)$, so there is no net change in $\log |f(z)|$ and we have

$$\int_\gamma \frac{f'(z)}{f(z)} dz = i \left(\arg f(z)\right)\Big|_{\text{end of } \gamma} - i \left(\arg f(z)\right)\Big|_{\text{start of } \gamma}.$$

Our counting theorem is thus equivalent to

THE ARGUMENT PRINCIPLE

Under all the same hypothesis,

$$\frac{\text{CHANGE OF ARG } f(z) \text{ AROUND } \gamma}{2\pi}$$
$$= \text{NUMBER OF ZEROS OF } f \text{ IN } D$$
$$- \text{NUMBER OF POLES OF } f \text{ IN } D$$

EXAMPLE: apply this to the holomorphic function $P = P(z)$, where P is a polynomial of degree n, and γ is a large circle $|z| = R$. Then

$$P(z) = cz^n + \text{ lower order terms},$$

so for $|z| = R$

$$\frac{P(z)}{c} = z^n(1 + \text{ small terms}).$$

Let $z = Re^{i\theta}$:

$$\frac{P(z)}{c} = R^n e^{in\theta}(1 + \text{ small terms}),$$

so that $\arg \dfrac{P(z)}{c}$ is approximately $n\theta$. Thus, the change in $\arg \dfrac{P(z)}{c}$ around this circle is approximately $2\pi n$. Thus, the LHS of the argument principle is approximately n. Thus, we have a second proof of the fundamental theorem of algebra:

> a polynomial of degree n has precisely n complex zeros, counted accordining to multiplicity.

PROBLEM 5-1.

Recall the definition of the Bernoulli numbers: from Problem 3-4,

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

In the solution of problem #4 we discover that

$$\frac{z}{2}\coth\left(\frac{z}{2}\right) = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!} z^{2k}.$$

1. Now prove that the Laurent expansion of $\cot z$ centered at 0 is

$$\cot z = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!}(-1)^k 2^{2k} z^{2k-1}.$$

2. Verify the trigonometric identity

$$\tan z = \cot z - 2\cot 2z.$$

3. Now prove that the Maclaurin expansion of $\tan z$ is

$$\tan z = \sum_{k=1}^{\infty} \frac{B_{2k}(-1)^{k-1}}{(2k)!} 2^{2k}\left(2^{2k}-1\right) z^{2k-1}.$$

4. Prove that

$$B_{2k}(-1)^{k-1} > 0 \qquad \text{for all } k \geq 1.$$

Short list:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}$$
$$B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730}.$$

Illustration of the argument principle:
We do two of these…both to locate which quadrants of $\mathbb{C}$ contain zeros of a polynomial.

- $\boldsymbol{f(z) = z^4 - 6z^3 + 18z^2 - 16z + 10}$
  (An example with real coefficients.) Of course, f has 4 zeros.

First, note that f has no real zeros. There are several ways to see this. Here's an ad hoc way: for $z \in \mathbb{R}$

$$\begin{aligned} f(z) &= z^2(z^2 - 6z + 9) + 9z^2 - 16z + 10 \\ &= z^2(z-3)^2 + (9z^2 - 16z + 10) > 0 \end{aligned}$$

always ≥ 0 always positive since discriminant $= 16^2 - 4 \cdot 9 \cdot 10 = 256 - 360 < 0$

We are going to write $f(z) = |f(z)|e^{i\theta(z)}$, where $\theta(z)$ stands for an argument of $f(z)$. We'll do this on the coordinate axes as well as on a large circle $|z| = R$.

$|z| = R$: Quite easy, since $f(Re^{it}) = R^4 e^{4it}(1 + \text{ small quantity})$ for large R. Thus $\theta \sim 4t$, so that as t increases by $\frac{\pi}{2}$, θ increases by 2π, approximately.

Real Axis: Also quite easy, since $f(x) > 0$ for all $x \in \mathbb{R}$. Thus, we actually can choose $\theta \equiv 0$ on the real axis.

Imaginary Axis: This becomes significant (no pun intended):

$$f(iy) = y^4 + 6iy^3 - 18y^2 - 16iy + 10.$$

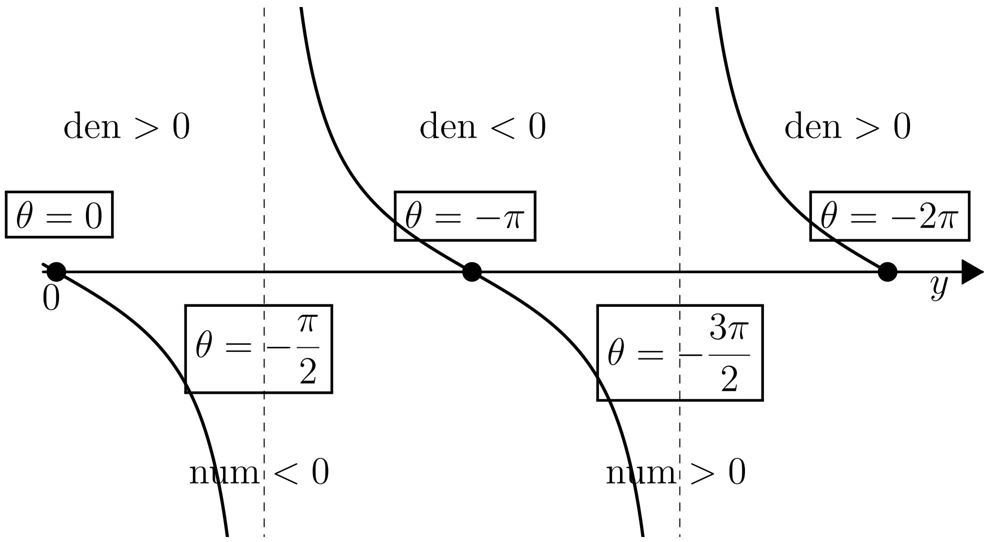
Thus, for $y \in \mathbb{R}$, $f(iy) = (y^4 - 18y^2 + 10) + i(6y^3 - 16y)$. We notice that this is nonzero for $y \in \mathbb{R}$, so θ is well defined and

$$\tan\theta = \frac{6y^3 - 16y}{y^4 - 18y^2 + 10}.$$

We roughly sketch this, noting that it's odd as a function of y, so we can restrict attention to $y > 0$. The denominator changes sign at two places, and we note that when the numerator vanishes for a value of $y > 0$, then $y^2 = \frac{16}{6} = \frac{8}{3}$. There the denominator equals

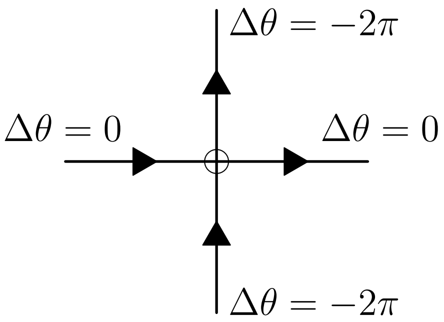
$$\left(\frac{8}{3}\right)^2 - 18 \cdot \frac{8}{3} + 10 = \frac{64}{9} - 38 < 0.$$

Here's a sketch of $\tan\theta$:

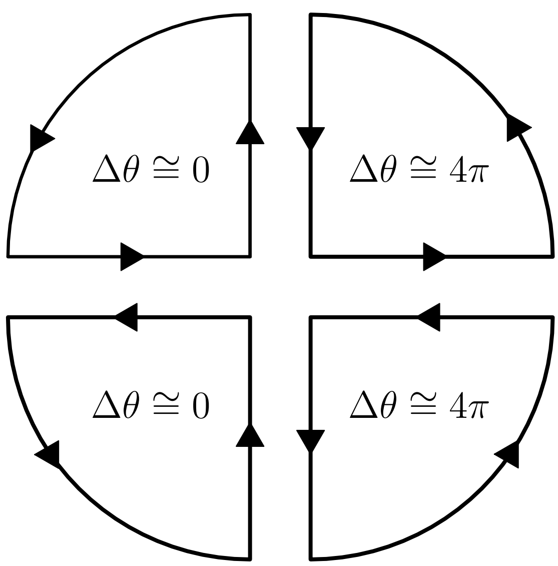


Starting with $\theta = 0$ at $y = 0$, note the values of θ at crucial places.

Summary of changes in θ:



Impose the approximate changes on $|z| = R$, R large:



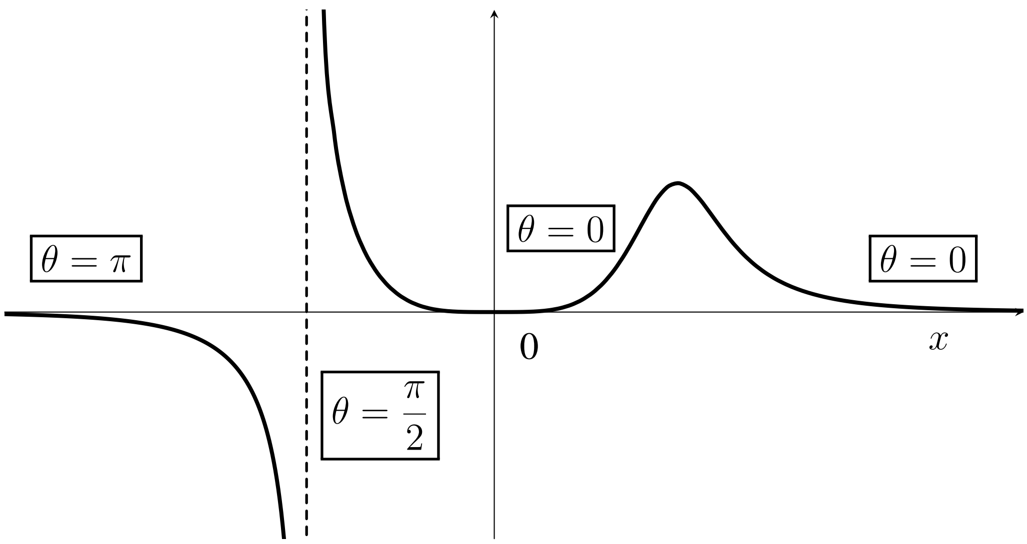
These approximate values are good enough, and the argument principle locates the number of zeros in each quadrant:

0	2
0	2

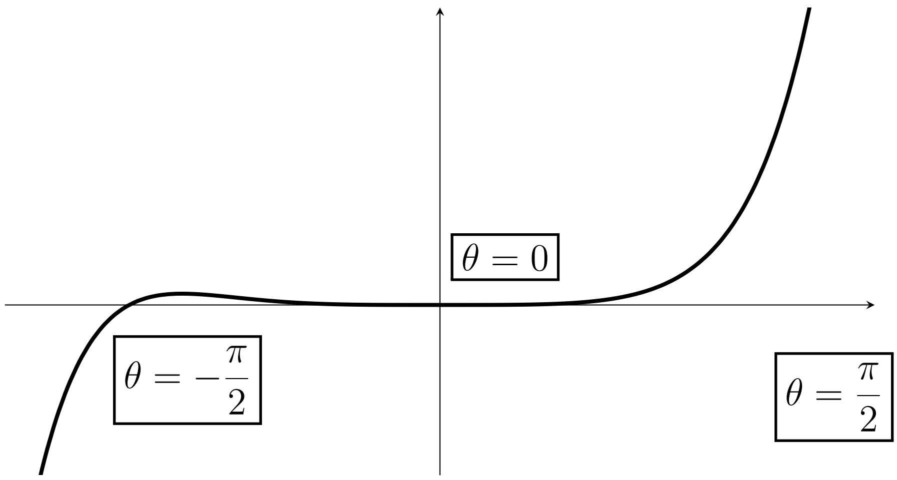
- $\boldsymbol{f(z) = z^9 + 5iz^4 + 7}$
  Same strategy. In this case it's clear there are no zeros on the coordinate axes.

<u>|z|=R</u>: For large R, θ increases by approximately $\dfrac{9\pi}{2}$ on each quadrant.

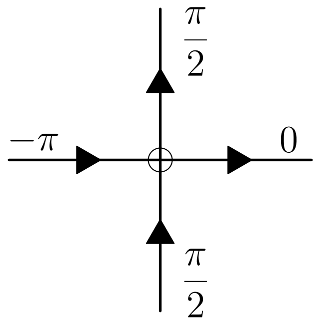
<u>Real axis</u>: $f(x) = x^9 + 7 + 5ix^4, \;\; \tan\theta = \dfrac{5x^4}{x^9+7}$.

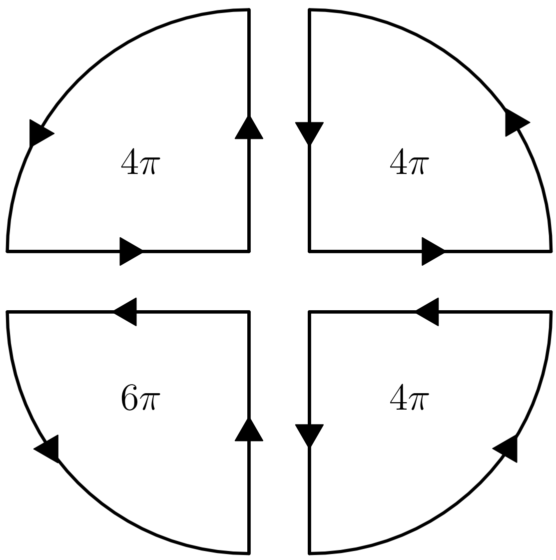


<u>Imaginary axis</u>: $f(iy) = 9iy^7 + 5iy^4 + 7, \;\; \tan\theta = \dfrac{9y^7 + 5y^4}{7}$.



<u>Summary of changes in θ</u>:





Argument principle yields numbers of zeros per quadrant:

2	2
3	2

MATHEMATICA output:

$$\begin{aligned}
&\{\{x \to -1.255504501880301 - 0.271981479350827 84i\},\\
&\{x \to -1.012461828409084 - 0.5403423271892956i\},\\
&\{x \to -0.8920069996513834 + 1.0379290053741679i\},\\
&\{x \to -0.34036461818177277 + 1.0240247108970417i\},\\
&\{x \to 0.08570449919877836 - 1.4126419862113073i\},\\
&\{x \to 0.44338217546397485 - 0.9488109790334857i\},\\
&\{x \to 0.8038223045822586 + 1.1975545074749976i\},\\
&\{x \to 0.9608045914177918 + 0.3811634298839924i\},\\
&\{x \to 1.3780333758572947 - 0.4668948818452832i\}\}
\end{aligned}$$

PROBLEM 5-2

1. Consider the quadratic polynomial $f(z) = z^2 + iz + 2 - i$. It has two zeros. Determine which quadrants they are in. (Use the argument principle, no calculators, no "quadratic formula.")

2. For any positive numbers a, b, c consider the polynomial $f(z) = z^8 + az^3 + bz + c$. How many zeros does it have in the first quadrant?

3. Suppose $\lambda \in \mathbb{C}$ has $\operatorname{Re}(\lambda) > 1$.

   a. Show that the equation
   $$e^{-z} + z = \lambda$$
   has exactly one solution satisfying $\operatorname{Re}(z) \geq 0$.

   b. Show that the solution is real $\Leftrightarrow$ λ is real.

4. Let f be an entire holomorphic function. Use $\log z$ defined by $0 < \arg z < 2\pi$. Prove that for any $R > 0$
   $$\int\limits_{\substack{|z|=R \\ \text{CCW}}} f(z) \log z \, dz = 2\pi i \int_0^R f(x) dx.$$

We continue with the <u>counting theorem</u> and the <u>argument principle</u>. We're now going to derive an especially useful corollary of the results called *Rouché's theorem*. Just for convenience we'll restrict attention to functions holomorphic on an open set containing $D \cup \gamma$ —in other words, our functions will have no isolated singularities—they're simply holomorphic.

(We continue with the picture and notation from the beginning of this chapter.)

Section B: Rouché's Theorem

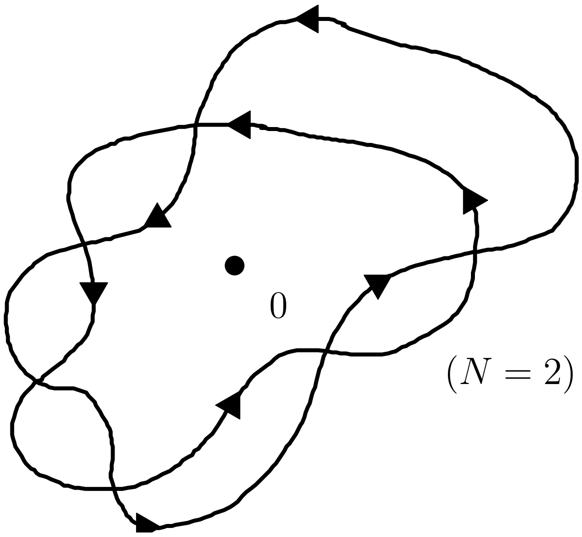
ROUCHÉ'S THEOREM

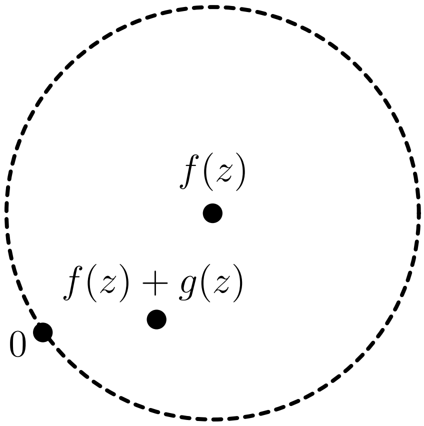
Suppose the function f and g are holomorphic as discussed above, and *assume that*

$$|g(z)| < |f(z)| \quad \textit{for all } z \in \gamma.$$

Then f and $f+g$ have the same number of zeros in D. (As always, counted according to multiplicity.)

Proof using the argument principle: As z traverses γ, $f(z)$ varies in $\mathbb{C} \setminus \{0\}$ and winds around 0 a number N times, where $N =$ the number of zeros of f in D. The assumption $|g(z)| < |f(z)|$ for $z \in \gamma$ always yields a picture like this:





and this prevents $f(z) + g(z)$ from gaining or losing any circuits around 0. Thus, $f(z) + g(z)$ also winds around 0 the same number N times.

QED 1

Proof using the counting theorem: (My favorite proof.) Note again that the hypothesis $|g(z)| < |f(z)|$ on γ implies that both

$$f(z) \neq 0 \text{ on } \gamma$$

and

$$f(z) + g(z) \neq 0 \text{ on } \gamma.$$

The counting theorem thus shows that

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f} dz = \# \text{ of zeros of } f \text{ in } D$$

and

$$\frac{1}{2\pi i} \int_\gamma \frac{(f+g)'}{f+g} dz = \# \text{ of zeros of } f+g \text{ in } D.$$

So we need to show these two integrals are equal. That is, that

$$\int_\gamma \left(\frac{(f+g)'}{f+g} - \frac{f'}{f} \right) dz = 0.$$

The integrand equals

$$\begin{aligned}
\frac{f'+g'}{f+g} - \frac{f'}{f} &= \frac{f(f'+g') - (f+g)f'}{(f+g)f} \\
&= \frac{fg' - gf'}{(f+g)f} \\
&= \frac{fg' - gf'}{f^2} \cdot \frac{f}{f+g} \\
&= \left(\frac{g}{f}\right)' \cdot \frac{f}{f+g}!
\end{aligned}$$

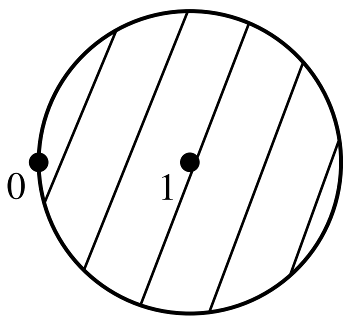
We are thus led to define the function

$$h(z) = \frac{g(z)}{f(z)} \qquad \text{for } z \text{ in an open set containing } \gamma.$$

So, our integrand equals

$$h' \cdot \frac{1}{1+\frac{g}{f}} = \frac{h'}{1+h}.$$

This quotient of course equals $\dfrac{(1+h)'}{1+h}$. Aha! Since $|h(z)| < 1$ by hypothesis, the values $1 + h(z)$ lie in the disc $|w - 1| < 1$.



In particular, we can choose $\arg w$ to be between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$, so that $\log w$ is holomorphic. Thus, with this choice the function $\log(1 + h(z))$ is holomorphic for z near γ. In particular

$$\int_\gamma \frac{h'}{1+h} dz = \int_\gamma \frac{d}{dz}\big(\log(1+h(z))\big) dz = 0$$

because of the fundamental theorem of calculus (Ch 3, page 50).

QED 2

Elegant version: This introduces a parameter $0 \leq t \leq 1$ to gradually move from $f(z)$ to $f(z) + g(z)$. Namely, consider the function $N(t)$ by

$$N(t) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z) + tg'(z)}{f(z) + tg(z)} dz.$$

Thus, $N(t) = \#$ of zeros of $f + tg$ in D. Therefore, $N(t)$ is an integer. But it's also a continuous function of r because of its representation in that particular integral form. Therefore, it must be constant. In particular,

$$N(0) = N(1).$$

QED 3

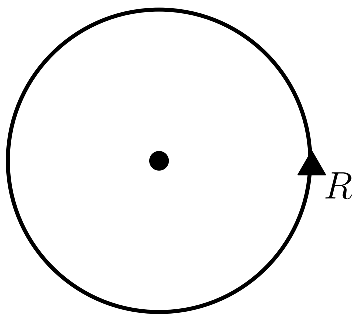
ILLUSTRATIONS:

1. FTA once again! Consider a polynomial of degree n:

   $$P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0.$$

   Let D be the disc with center 0 and radius R. Let

   $$\begin{aligned} f(z) &= z^n, \\ g(z) &= a_{n-1}z^{n-1} + \ldots + a_1 z + a_0. \end{aligned}$$

   For $|z| = R$ with $R > 1$,

   $$|g(z)| \leq (|a_{n-1}| + \ldots + |a_0|)\, R^{n-1} =: CR^{n-1}.$$

   Thus, if also $R > C$, then for $|z| = R$,

   $$|g(z)| < R^n = |f(z)|.$$

   Rouché's theorem applies: $P = f + g$ has the same number of zeros in D as $f = z^n$. The latter has n zeros (at 0). Thus, P has n zeros in D.

2. $\boxed{z^3 + e^z}$

- For $|z| = 2$, $|e^z| = e^{\operatorname{Re} z} \le e^2 < 8 = |z|^3$, so $z^3 + e^z$ has three zeros in $|z| < 2$. (z^3 dominates)
- For $|z| = \frac{3}{4}$, $|e^z| = e^{\operatorname{Re} z} \ge e^{-\frac{3}{4}} = .472\ldots$ and $|z^3| = .421\ldots$ (e^z dominates). Therefore, $z^3 + e^z$ has no zeros in $|z| < \frac{3}{4}$.

Now we begin an interesting investigation of what we might call "mapping properties of holomorphic functions." First, we work an exercise we could have done long ago—Chapter 2, page 24, when we first mentioned the Cauchy-Riemann equation:

THEOREM: *Suppose f is a holomorphic function defined on a connected open set D, and suppose the modulus of f is constant. Then f is constant.*

Proof: We're given that $|f(z)| = C$ (constant) for all $z \in D$. If $C = 0$, there's nothing to prove, since then $f(z) = 0$. So, we assume $C > 0$. Write the hypothesis in the form

$$f(z)\overline{f(z)} = C^2.$$

The product rule gives immediately

$$f_x\bar{f} + f\bar{f}_x = 0$$

(note that $\dfrac{\partial \bar{f}}{\partial x} = \overline{\dfrac{\partial f}{\partial x}}$). That is,

$$\operatorname{Re}\left(f_x\bar{f}\right) = 0. \qquad \bigstar$$

In the same way,

$$\operatorname{Re}\left(f_y\bar{f}\right) = 0.$$

Aha! The Cauchy-Riemann equation is $f_y = if_x$, so the second equation becomes

$$\operatorname{Re}\left(if_x\bar{f}\right) = 0.$$

That is,

$$\operatorname{Im}(f_x\bar{f}) = 0. \qquad \bigstar\bigstar$$

Then $\bigstar$ and $\bigstar\bigstar$ $\Rightarrow$

$$f_x\bar{f} = 0.$$

But $\bar{f} \neq 0$, so we conclude $f_x = 0$. I.e., $f'(z) = 0$ for all $z \in D$. Thus f is constant. (Chapter 2, page 43)

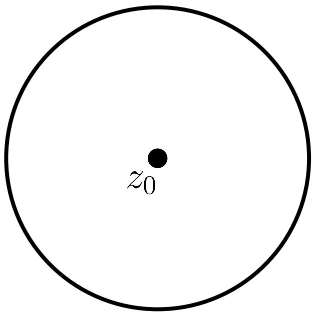
QED

Next, we have the famous

MAXIMUM PRINCIPLE: *Let f be a nonconstant holomorphic function defined on a connected open set D. Then the modulus $|f(z)|$ cannot have a local maximum value at any point of D.*

Proof: Suppose to the contrary that there is a closed disc with center z_0 and radius $a > 0$ such that for all z in the disc

$$|f(z)| \leq |f(z_0)|.$$



The mean value property of holomorphic functions (Chapter 3, page 61) yields the equation

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta \qquad \text{for } 0 \leq r \leq a.$$

Therefore,

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(z_0 + re^{i\theta}) \right| d\theta.$$

Thus,

$$0 \leq \frac{1}{2\pi} \int_0^{2\pi} \left(\left| f(z_0 + re^{i\theta}) \right| - |f(z_0)| \right) d\theta,$$

but the integrand here is continuous and ≤ 0 — since its integral is ≥ 0, it must be the case that the integrand is exactly 0: thus,

$$\left| f(z_0 + re^{i\theta}) \right| = |f(z_0)| \qquad \text{for all } 0 \leq \theta \leq 2\pi.$$

And this is true for all $0 < r \leq a$. Thus, $|f(z_0 + w)| = |f(z_0)|$ for all complex w such that $|w| \leq a$. Thus, $|f|$ is <u>constant</u> on this disc. The preceding theorem $\Rightarrow f$ is constant on this disc. Since D is connected, f is constant on D.

QED

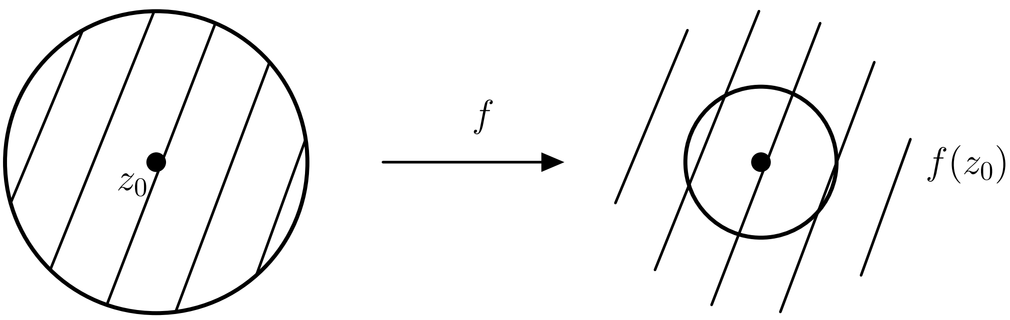
MINIMUM PRINCIPLE: *Let f be a nonconstant holomorphic function defined on a connected open set D. Suppose that the modulus $|f(z_0)|$ has a local minimum value at a point $z_0 \in D$. Then $f(z_0) = 0$.*

Proof: We reason by contradiction, so we assume $f(z_0) \neq 0$. Then consider the function $\frac{1}{f}$. It's defined in a neighborhood of z_0 (since $f(z) \neq 0$ close enough to z_0 and is nonconstant and holomorphic). And has a local maximum at z_0, contradicting the maximum principle. Therefore, $f(z_0) = 0$.

QED

Now we are going to discuss function which are **open mappings**. This is a description requiring f to have the following equivalent properties:

1. For every open set E, the image $f(E)$ is open.
2. For every point z_0 in the domain of definition of f, there exist $\delta > 0$ and $\varepsilon > 0$ such that for all w for which $|w - f(z_0)| < \varepsilon$, there exists z for which $|z - z_0| < \delta$ and $f(z) = w$.



Think this way: "f preserves openness."

EXAMPLE: $\mathbb{R} \xrightarrow{f} \mathbb{R}$ strictly increasing and continuous

NONEXAMPLE: $\mathbb{R} \xrightarrow{\sin} \mathbb{R}$

NONEXAMPLE: constant function

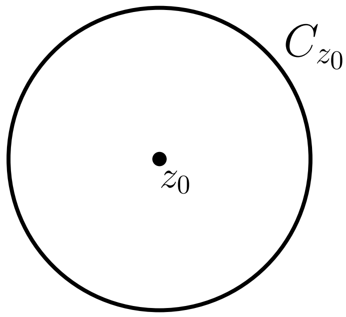
Section C: Open Mapping Theorem

OPEN MAPPING THEOREM: *Every nonconstant holomorphic function defined on a connected open subset of* $\mathbb{C}$ *is an open mapping.*

Brute force proof: Let $f(z_0) = w_0$. Since the zeros of $f(z) = w_0$ are isolated, there exists $\delta > 0$ such that in the closed disc

$$D_{z_0} = \left\{ z \middle| |z - z_0| \leq \delta \right\}$$

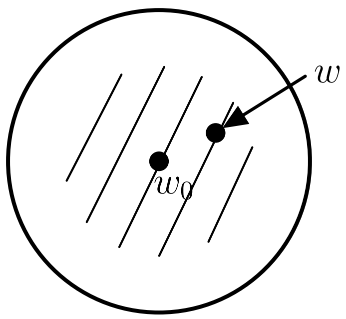
$f(z)$ is never equal to w_0 except at the center. In particular, on the circle $C_{z_0} = \partial D_{z_0}$ the continuous function $|f(z) - w_0|$ is positive. Since C_{z_0} is closed and bounded, there exists a positive lower bound $\varepsilon > 0$:



$$|f(z) - w_0| \geq \varepsilon \quad \text{for all } z \in C_{z_0}.$$

Now we shall prove that for any w such that

$$|w - w_0| < \frac{\varepsilon}{2},$$



$w = f(z)$ for some $z \in D_{z_0}$. This will finish the proof of the theorem. To this end define $g(z) = f(z) - w$, a nonconstant holomorphic function defined on an open set containing D_{z_0}. Two observations:

1. for $z \in C_{z_0}$, $|g(z)| > \dfrac{\varepsilon}{2}$.

(Proof:

$$\begin{aligned}|g(z)| &= |f(z) - w_0 + (w_0 - w)| \\ &\geq |f(z) - w_0| - |w_0 - w| \\ &\geq \varepsilon - |w_0 - w| \\ &> \varepsilon - \frac{\varepsilon}{2} \\ &= \frac{\varepsilon}{2}. \quad)\end{aligned}$$

2. $|g(z_0)| = |w_0 - w| < \dfrac{\varepsilon}{2}$.

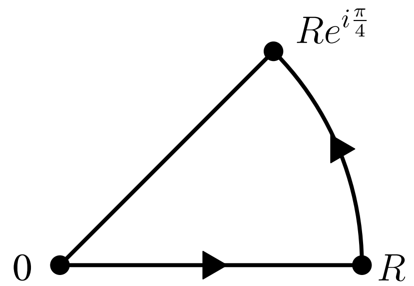
Since D_{z_0} is closed and bounded, the continuous function $|g(z)|$ attains its minimum value at some point of D_{z_0}. The combination of (1) and (2) shows that this minimum is not attained at any point of C_{z_0}, and we conclude that it is attained at a point z in the interior of D_{z_0}: $|z - z_0| < \delta$. Therefore, the minimum principle $\Rightarrow g(z) = 0$. That is, $f(z) = w$.

QED

PROBLEM 5-3 For this problem assume the Gaussian integral from vector calculus:

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$

Apply the Cauchy integral theorem to the function e^{-z^2} and the path



a. Take great care in showing that the line integral $\int e^{-z^2} dz$ along the circular arc tends to 0 as $R \to \infty$.

b. Conclude the **Fresnel** formulas

$$\int_0^{\infty} \cos(x^2) dx = \int_0^{\infty} \sin(x^2) dx = \sqrt{\frac{\pi}{8}}.$$

OPEN MAPPING THEOREM bis

We continue discussing the terrific theorem, but now we give an

Elegant proof: We begin this proof with the same setup as we just did, but now we use the counting theorem instead of the maximum principle as our main tool. We have $f(z) = w_0$, but now we let $N =$ the number of times $f(z) = w_0$ at $z = z_0$. That is,

$$N = \text{ the order of the zero of } f(z) - w_0 \text{ at } z_0.$$

Just for clarification, this means that the Taylor series of $f(z)$ at z_0 has the form

$$f(z) = w_0 + c_N(z - z_0)^N + c_{N+1}(z - z_0)^{N+1} + \dots,$$

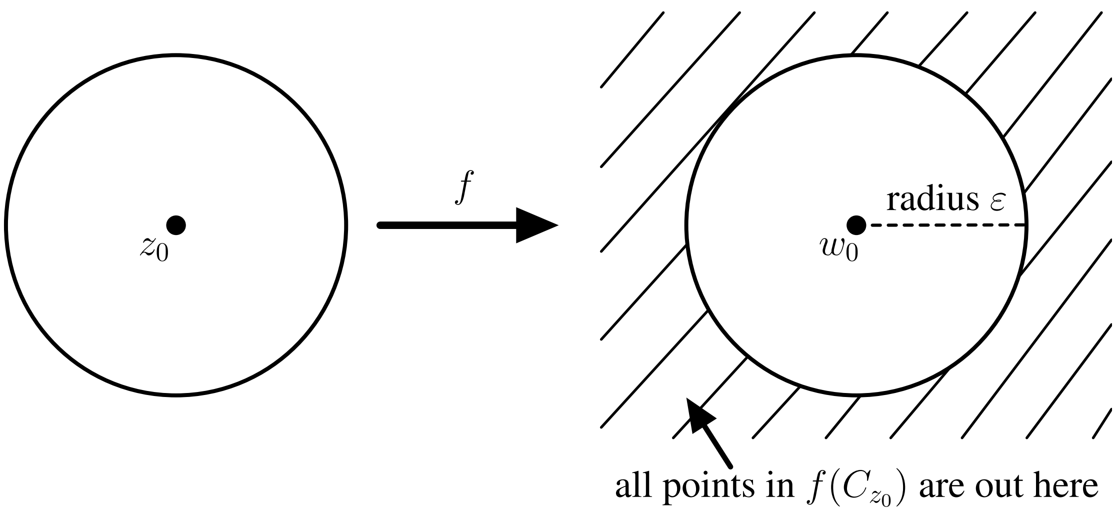
where $c_N \neq 0$. In other words,

$$\begin{cases} f(z_0) = w_0, \\ f^{(k)}(z_0) \text{ for } 1 \leq k \leq N - 1, \\ f^{(N)}(z_0) \neq 0. \end{cases}$$

The counting theorem (page 108)$\Rightarrow$

$$\frac{1}{2\pi i} \int_{C_{z_0}} \frac{f'(z)}{f(z) - w_0} dz = N.$$

(We've applied the theorem to $f - w_0$ rather than to f.)



Now we notice that if $|w - w_0| < \varepsilon$, then $f(z) - w$ is never 0 on C_{z_0}, so the counting theorem again implies that

$$\frac{1}{2\pi i}\int_{C_{z_0}} \frac{f'(z)}{f(z)-w}dz = \begin{array}{c}\text{the number of times}\\ f(z) = w \text{ for}\\ |z - z_0| < \delta.\end{array}$$

The left side of this equation is a continuous function of w for $|w - w_0| < \varepsilon$, since the denominator is never 0 for $z \in C_{z_0}$. But it's an integer! Therefore, it is constant! And thus equal to its value N at z_0.

CONCLUSION: for $|w - w_0| < \varepsilon$, the equation $f(z) = w$ has exactly N solutions for $|z - z_0| < \delta$ (counting according to multiplicity).

QED bis

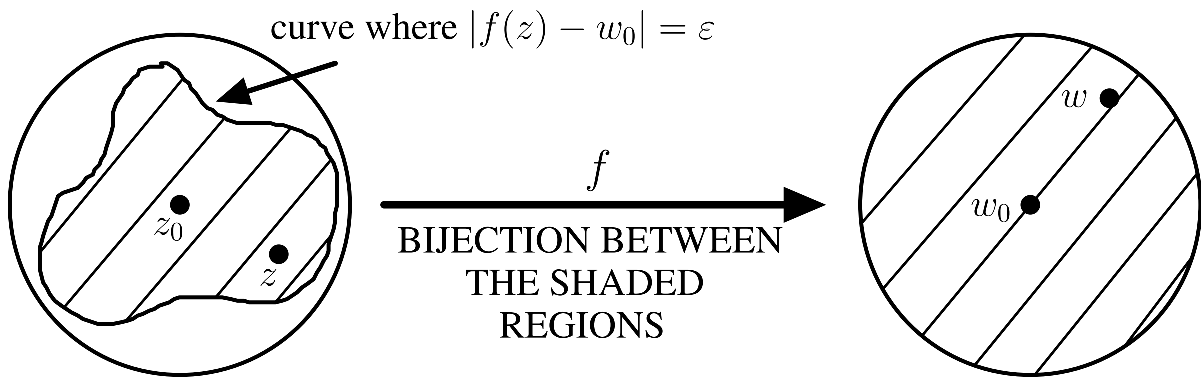
Notice how much better this result is than what we previously knew! Before we needed to assume $|w - w_0| < \frac{\varepsilon}{2}$, but now only that $|w - w_0| < \varepsilon$. But the better aspect by far is that we now know the number of solutions of $f(z) = w$. Not merely "at least one," but now exactly N.

Section D: Inverse Functions

Now we focus all our attention on the case in which $f(z_0) = w_0$ just one time, or $N = 1$. Then for $|w - w_0| < \varepsilon$ (in our notation), there is one and only one z such that

$$f(z) = w \text{ and } |z - z_0| < \delta.$$

Rough sketch



Thus f is locally a bijection between a neighborhood of z_0 and a neighborhood (a disc) of w_0. In this situation we may say that there is an inverse function f^{-1}

defined near w_0, giving points z near z_0:

$$f(z) = w \leftrightarrow z = f^{-1}(w).$$

Thus we have easily established an <u>inverse function</u> theorem for holomorphic functions! We are now going to analyze this f^{-1}.

First, suppose g is a holomorphic function defined near z_0, and try to apply the residue theorem to the holomorphic function

$$\frac{g(z)f'(z)}{f(z) - w}.$$

So we assume g is holomorphic in a neighborhood of the closed disc D_{z_0} and that $|w - w_0| < \varepsilon$. Then the function we are considering has just one singularity in D_{z_0} and it is the point (unique) where $f(z) - w = 0$. This is a simple pole, so we calculate the residue to be

$$\frac{g(z)f'(z)}{f'(z)} = g(z).$$

The residue theorem gives the result

$$\frac{1}{2\pi i}\int_{C_{z_0}} \frac{g(\zeta)f'(\zeta)}{f(\zeta) - w}d\zeta = g(z).$$

In particular, when $g(z) = z$ we obtain

$$\frac{1}{2\pi i}\int_{C_{z_0}} \frac{\zeta f'(\zeta)}{f(\zeta) - w}d\zeta = z.$$

That gives us an "explicit" formula for f^{-1}:

$$\boxed{f^{-1}(w) = \frac{1}{2\pi i}\int_{C_{z_0}} \frac{\zeta f'(\zeta)}{f(\zeta) - w}d\zeta}$$

This formula alone shows that f^{-1} is holomorphic!

More about this later, but now we obtain the Taylor series for f^{-1}, centered at w_0. We simply note that for $\zeta \in C_{z_0}$,

$$|f(\zeta) - w_0| > \varepsilon,$$

so that for $|w - w_0| < \varepsilon$,

$$\begin{aligned} f(\zeta) - w &= f(\zeta) - w_0 - (w - w_0) \\ &= (f(\zeta) - w_0)\left(1 - \frac{w - w_0}{f(\zeta) - w_0}\right) \end{aligned}$$

and we have a geometric series expansion

$$\frac{1}{f(\zeta) - w} = \sum_{n=0}^{\infty} \frac{(w - w_0)^n}{(f(\zeta) - w_0)^{n+1}}.$$

Therefore,

$$f^{-1}(w) = \sum_{n=0}^{\infty} c_n (w - w_0)^n,$$

where the Taylor coefficients are

$$\begin{aligned} c_n &= \frac{1}{2\pi i} \int_{C_{z_0}} \frac{\zeta f'(\zeta)}{(f(\zeta) - w_0)^{n+1}} d\zeta \\ &= \operatorname{Res}\left(\frac{z f'(z)}{(f(z) - w_0)^{n+1}}, z_0\right). \end{aligned}$$

Notice that, of course, $c_0 = z_0$.

PROBLEM 5-4

1. Apply the Lagrange-Bürmann expansion theorem (it's coming soon) to the function
$$f(z) = 1 - e^{-z},$$
with $f(0) = 0$.

   Also solve the equation explicitly near $z = 0$, $w = 0$, and write down the Taylor series for $f^{-1}(w)$.

   Compare these results to compute for all $n \geq 1$
$$\operatorname{Res}\left(\frac{1}{(1 - e^{-z})^n}, 0\right).$$

2. From page 91 in Chapter 4 we have the formula
$$\int_0^{\infty} \frac{x^{\alpha-1}}{x+1} dx = \frac{\pi}{\sin \alpha\pi} \qquad \text{for } 0 < \alpha < 1.$$

Manipulate this formula using real change of integration variable to find a formula for

$$\int_{-\infty}^{\infty} \frac{e^{\beta x}}{\cosh x} dx \qquad \text{for } -1 < \beta < 1.$$

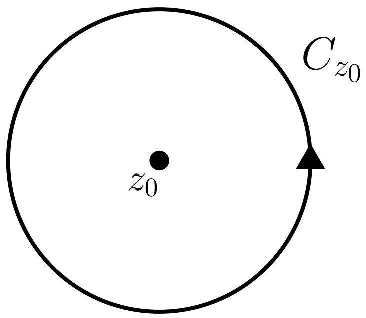
Express your result elegantly as $\frac{\pi}{??}$.

<u>Continuation of the formula for f^{-1}</u>

We are still in the situation where f is holomorphic in a neighborhood of z_0 and $f(z_0) = w_0$ and $f'(z_0) \neq 0$. We've seen that f has a holomorphic inverse f^{-1} mapping w_0 to z_0 and defined in a neighborhood of w_0.

Moreover, if g is holomorphic in a neighborhood of z_0, then

$$g(f^{-1}(w)) = \frac{1}{2\pi i} \int_{C_{z_0}} \frac{g(\zeta) f'(\zeta)}{f(\zeta) - w} d\zeta.$$



By applying geometric series to expand $\frac{1}{f(\zeta) - w}$ in a power series centered at w_0, we then obtain

$$g(f^{-1}(w)) = \sum_{n=0}^{\infty} c_n (w - w_0)^n,$$

where

$$c_n = \text{Res}\left(\frac{g(z) f'(z)}{(f(z) - w_0)^{n+1}}, z_0 \right).$$

We now make a technical adjustment in this formula by noting that

$$c_0 = g(z_0)$$

and for $n \geq 1$

$$\frac{f'}{(f - w_0)^{n+1}} = -\frac{d}{dz} \frac{1}{n(f - w_0)^n},$$

so that we can apply our so-called "integration by parts" residue formula from page 91, Chapter 4 to write

$$c_n = \frac{1}{n} \text{Res}\left(\frac{g'(z)}{(f(z) - w_0)^n}, z_0 \right).$$

The result we have obtained goes under the name of the

LAGRANGE-BÜRMANN EXPANSION

$$g(f^{-1}(w)) = g(z_0) + \sum_{n=1}^{\infty} c_n (w - w_0)^n,$$

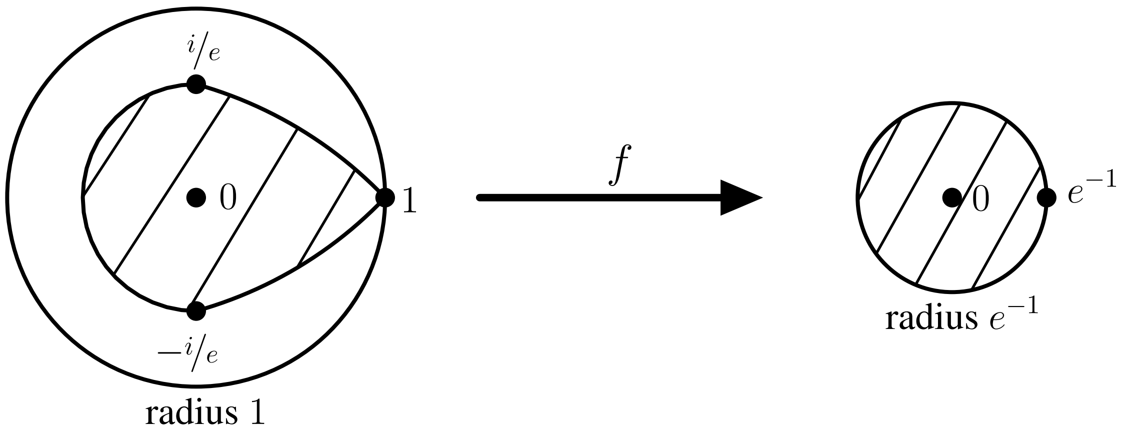
where

$$c_n = \frac{1}{n} \operatorname{Res}\left(\frac{g'(z)}{(f(z) - w_0)^n}, z_o \right).$$

<u>EXAMPLE</u>: $f(z) = ze^{-z}$, $f(0) = 0$
As $f'(0) = 1$, f is indeed one-to-one near 0, and our results apply.

(Preliminary: $f'(z) = (1 - z)e^{-z}$, so $f'(1) = 0$, so f is <u>not</u> one-to-one in any neighborhood of 1. So, we'll try $\delta = 1$ for the radius of the circle in the z-plane.)

For $|z| = 1$ we have $|f(z)| = |e^{-z}| = e^{\operatorname{Re} z} \geq e^{-1}$, so we can take any $0 < \varepsilon < e^{-1}$ for our radius in the w-plane.



Then the Lagrange-Bürmann coefficients for $g(z) = z$ and $n \geq 1$ are

$$\begin{aligned} c_n &= \frac{1}{n} \operatorname{Res}\left(\frac{1}{(ze^{-z})^n}, 0 \right) \\ &= \frac{1}{n} \operatorname{Res}\left(\frac{e^{nz}}{z^n}, 0 \right) \\ &= \frac{1}{n} \cdot (\text{coefficient of } z^{n-1} \text{ in the Maclaurin expansion of } e^{nz}) \\ &= \frac{n^{n-1}}{n(n-1)!}. \end{aligned}$$

Thus, $|w| < \dfrac{1}{e} \Rightarrow$

$$f^{-1}(w) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} w^n.$$

First few terms:

$$f^{-1}(w) = w + w^2 + \frac{3}{2}w^3 + \frac{8}{3}w^4 + \frac{125}{24}w^5 + \dots.$$

We fully expect this series to have radius of convergence e^{-1}. We can verify this directly by using the ratio test:

$$\begin{aligned}
\lim_{n\to\infty} \frac{c_n}{c_{n+1}} &= \lim_{n\to\infty} \frac{\frac{n^{n-1}}{n!}}{\frac{(n+1)^n}{(n+1)!}} \\
&= \lim_{n\to\infty} \frac{(n+1)n^{n-1}}{(n+1)^n} \\
&= \lim_{n\to\infty} \frac{n^{n-1}}{(n+1)^{n-1}} \\
&= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n-1}} \\
&= \frac{1}{e}.
\end{aligned}$$

REMARK: It appears that the above result is valid for $|w| < e^{-1}$, but that it also should apply in the limit with $w \in \mathbb{R}$ increasing to e^{-1}, and giving the expected $z = 1$. That is,

$$\boxed{1 = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} e^{-n}}$$

It is rather easy to verify this result, but

CHALLENGE: Verify this equation directly. (<u>I don't know how!</u>)

PROBLEM 5-5 Let $1 < a < \infty$ be a fixed real number. This problem is concerned with trying to solve the equation

$$wz^a - z + 1 = 0$$

for z, where the complex number w is small. For $w = 0$ we obtain the unique solution $z = 1$, and we want our solution to remain close to 1 for small $|w|$.

To fit the Lagrange-Bürmann framework we define

$$f(z) = \frac{z-1}{z^a} \qquad \text{for } z \text{ near } 1,$$

where we use the determination of z^a given by

$$z^a = \exp(a \log z), \qquad \text{with } -\pi < \arg z < \pi.$$

1. Prove that for $|w| < \dfrac{(a-1)^{a-1}}{a^a}$,

$$f^{-1}(w) = 1 + \sum_{n=1}^{\infty} \frac{1}{n} \binom{an}{n-1} w^n.$$

2. For $|w|$ sufficiently small, $f^{-1}(w)$ is close to 1, so we can define $\log(f^{-1}(w))$ close to 0. Then calculate all the c_n's in the Maclaurin series

$$\log(f^{-1}(w)) = \sum_{n=1}^{\infty} c_n w^n.$$

Section E: Infinite Series and Infinite Products

Now we continue our theme of using the residue theorem in various ways, first by a discussion of evaluating certain infinite series. We shall discover a prominent role played by the function $\boldsymbol{\pi \cot \pi z}$.

This function is holomorphic on $\mathbb{C}$ except for its poles at the points where $\sin \pi z$... namely the integers $n \in \mathbb{Z}$. At each such n we have

$$\begin{aligned}
\operatorname{Res}(\pi \cot \pi z, n) &= \operatorname{Res}\left(\pi \frac{\cos \pi z}{\sin \pi z}, n\right) \\
&= \frac{\pi \cos \pi n}{\pi \cos \pi n} \\
&= 1.
\end{aligned}$$

We're also going to require a couple of important estimates. We'll use the standard coordinate representation $z = x + iy$.

$\operatorname{Re}(z) =$ integer $+ 1/2$: Since $\cot \pi z$ is periodic with period 1, we may as well assume $z = \dfrac{1}{2} + iy$. Then

$$\begin{aligned}\cot \pi z &= \cot\left(\frac{\pi}{2} + \pi i y\right)\\ &= -\tan(\pi i y)\\ &= -\frac{\sin \pi i y}{\cos \pi i y}\\ &= -\frac{i \sinh \pi y}{\cosh \pi y},\end{aligned}$$

so

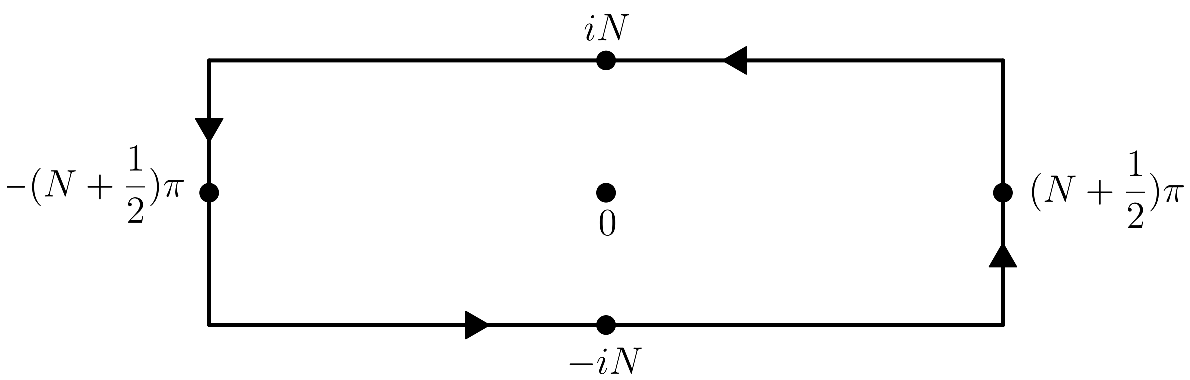
$$|\cot \pi z| = \frac{|\sinh \pi y|}{\cosh \pi y} < 1.$$

$\mathrm{Im}\, z = y$ with $|y|$ large: From Problem 1-7 on page 7 we have

$$\begin{aligned}|\cot \pi z|^2 &= \frac{|\cos(\pi x + \pi i y)|^2}{|\sin(\pi x + \pi i y)|^2}\\ &= \frac{|\cosh(i\pi x - \pi y)|^2}{|\sinh(i\pi x - \pi y)|^2}\\ &= \frac{\sinh^2 \pi y + \cos^2 \pi x}{\sinh^2 \pi y + \sin^2 \pi x}\\ &\le \frac{\sinh^2 \pi y + 1}{\sinh^2 \pi y}\\ &= 1 + \frac{1}{\sinh^2 \pi y}\\ &< 2 \quad \text{(say)},\end{aligned}$$

since $\sinh^2 \pi y \to \infty$ as $|y| \to \infty$.

We are going to be interested in this rectangular path, which we call γ_N:



For large N we have for all $z \in \gamma_N$ that

$$|\cot \pi z| < \sqrt{2},$$

so that

$$|\pi \cot \pi z| < 5.$$

Now consider any holomorphic function of f on $\mathbb{C}$ with finitely many isolated singularities, which we'll designate generically by ζ. Furthermore, suppose that $f(z) \to 0$ as $z \to \infty$ at a rate at least as fast as $|z|^{-2}$:

$$|f(z)| \leq \frac{C}{|z|^2} \text{ for large } |z|.$$

We then apply the residue theorem to the product

$$f(z)\pi \cot \pi z$$

inside γ_N for large N, large enough to contain all the singularities of f. We obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\gamma_N} f(z)\pi \cot \pi z \, dz \\
&\qquad = \sum_{z \text{ inside } \gamma_N} \operatorname{Res}(f(z)\pi \cot \pi z, z) \\
&\qquad = \sum_{n=-N}^{N} \operatorname{Res}(f(z)\pi \cot \pi z, n) + \sum_{\zeta \notin \mathbb{Z}} \operatorname{Res}(f(z)\pi \cot \pi z, \zeta).
\end{aligned}$$

On the left side of the equation we have an estimate

$$\begin{aligned}
\left|\int_{\gamma_N} f(z)\pi \cot \pi z \, dz\right| &\leq \frac{C}{N^2} \cdot 5 \cdot \text{length of } \gamma_N \\
&= \frac{5C}{N^2} \cdot ((2N+1)\pi + 2N) \\
&\to 0 \text{ as } N \to \infty.
\end{aligned}$$

Thus, when $N \to \infty$ we obtain

$$0 = \lim_{N\to\infty} \sum_{n=-N}^{N} \operatorname{Res}(f(z)\pi \cot \pi z, n) + \sum_{\zeta \notin \mathbb{Z}} \operatorname{Res}(f(z)\pi \cot \pi z, \zeta)$$

Therefore, we write briefly

$$\bigstar \qquad \boxed{\sum_{n=-\infty}^{\infty} \operatorname{Res}(f(z)\pi \cot \pi z, n) = -\sum_{\zeta \notin \mathbb{Z}} \operatorname{Res}(f(z)\pi \cot \pi z, \zeta)}$$

EXAMPLE: $f(z) = \dfrac{1}{z^2+a^2}$, where we assume $ia \notin \mathbb{Z}$. Our conditions are met, so we obtain immediately

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} &= -\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2+a^2}, ia\right) - \operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2+a^2}, -ia\right) \\ &= -\frac{\pi \cot \pi ia}{2ia} - \frac{\pi \cot(-\pi ia)}{-2ia} \\ &= -\frac{\pi \cot \pi ia}{ia} \\ &= -\pi \frac{\cos \pi ia}{ia \sin \pi ia} \\ &= -\pi \frac{\cosh \pi a}{iai \sinh \pi a} \\ &= \frac{\pi \cosh \pi a}{a \sinh \pi a}.\end{aligned}$$

Thus,

$$\boxed{\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi \coth \pi a}{a}}$$

P.S. We can let $a \to 0$ to obtain

$$\begin{aligned}2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} &= \frac{\pi \coth \pi a}{a} - \frac{1}{a^2} \\ &= \frac{\pi \cosh \pi a}{a \sinh \pi a} - \frac{1}{a^2} \\ &= \frac{\pi a \cosh \pi a - \sinh \pi a}{a^2 \sinh \pi a} \\ &= \frac{\pi a \cosh \pi a - \sinh \pi a}{\pi a^3} \cdot \frac{\pi a^3}{a^2 \sinh \pi a},\end{aligned}$$

and l'Hôpital's rule gives progressive fractions for $a \to 0$:

$$\frac{\pi a \cosh \pi a - \sinh \pi a}{\pi a^3},$$
$$\frac{\pi \cosh \pi a + \pi^2 a \sinh \pi a - \pi \cosh \pi a}{3\pi a^2},$$
$$\frac{\pi^2 \sinh \pi a}{3\pi a},$$
$$\frac{\pi^3 \cosh \pi a}{3\pi},$$
$$\frac{\pi^2}{3}.$$

Thus, we find for $a \to 0$,

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$$

REMARK: We'll soon obtain the last result more easily, and <u>at the same time</u>

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90},$$
$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945},$$

and <u>all</u> the rest:

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}}!$$

Now we apply ★ to the holomorphic function

$$f(z) = \frac{1}{z^{2k}},$$

where k is any positive integer. Thus, we obtain immediately

$$\sum_{n=-\infty}^{\infty} \operatorname{Res}\left(\frac{\pi \cot \pi z}{z^{2k}}, n\right) = 0.$$

For $n \neq 0$ we have simple poles, so we obtain

$$\sum_{n\neq 0} \frac{1}{n^{2k}} = -\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^{2k}}, 0\right).$$

Since $2k$ is even, we can also write

$$\sum_{n=1} \frac{1}{n^{2k}} = -\frac{1}{2}\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^{2k}}, 0\right).$$

The function in question has a pole of order $2k+1$ at 0, which would make it difficult to compute the residue. However, we have a formula already for the Laurent series of $\cot z$! It's form Problem 3-4:

$$\cot z = \sum_{k=0}^{\infty} \frac{B_{2k}}{(2k)!}(-1)^k 2^{2k} z^{2k-1}. \qquad \text{(Bernoulli numbers)}$$

Thus, the right side of the formula we have obtained equals

$$\begin{aligned} -\frac{1}{2} \cdot \text{coefficient of } \frac{1}{z} \text{ in } \frac{\pi \cot \pi z}{z^{2k}} & \\ &= -\frac{1}{2} \cdot \text{coefficient of } z^{2k-1} \text{ in } \pi \cot \pi z \\ &= -\frac{1}{2} \cdot \pi \frac{B_{2k}}{(2k)!}(-1)^k 2^{2k} \pi^{2k-1}. \end{aligned}$$

Thus, we have derived

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \left(\frac{B_{2k}(-1)^{k-1}}{(2k)!} 2^{2k-1} \right) \pi^{2k}}$$

(Notice that this shows in another way that $B_{2k}(-1)^{k-1} > 0$ for $k \geq 1$.)

DEFINITION: the infinite series on the left side are special values of the **Riemann zeta function**:

$$\boxed{\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}} \qquad \text{converges absolutely if } \operatorname{Re} z > 1.$$

(In this formula $n^{-z} = \exp(-z \ln n)$ and we use $\ln n \in \mathbb{R}$.)

SOME VALUES: from page 111 we compute

$$\zeta(2) = \frac{\pi^2}{6}$$
$$\zeta(4) = \frac{\pi^4}{90}$$
$$\zeta(6) = \frac{\pi^6}{945}$$
$$\zeta(8) = \frac{\pi^8}{9450}$$
$$\zeta(10) = \frac{\pi^{10}}{93555}$$

REMARK: The numbers $\zeta(3)$, $\zeta(5)$, etc. are not well understood. A startling result was proved by Roger Apéry in 1979:

$$\zeta(3) \text{ is irrational.}$$

In that regard

$$\sum_{n=0}^{\infty} \frac{1}{(2k+1)^3} = \frac{7}{8}\zeta(3),$$

but we can actually compute

$$\sum_{n=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}. \quad \text{(we'll prove this soon)}$$

<u>Another calculation</u>: We now apply our formula to the function

$$f(z) = \frac{1}{z(z-a)},$$

where a is any complex number other than an integer. From ★ we obtain immediately

$$\sum_{n=-\infty}^{\infty} \operatorname{Res}\left(\frac{\pi\cot\pi z}{z(z-a)}, n\right) = -\operatorname{Res}\left(\frac{\pi\cot\pi z}{z(z-a)}, a\right) = -\frac{\pi\cot\pi a}{a}.$$

Therefore,

$$\sum_{n\neq 0} \frac{1}{n(n-a)} + \operatorname{Res}\left(\frac{\pi\cot\pi z}{z(z-a)}, 0\right) = -\frac{\pi\cot\pi a}{a}.$$

This remaining residue has a pole of order 2, but we can write

$$\frac{1}{z(z-a)} = \left(\frac{1}{z-a} - \frac{1}{z}\right)\frac{1}{a}$$

so, we obtain

$$\begin{aligned}\operatorname{Res}\left(\frac{\pi\cot\pi z}{z(z-a)}, 0\right) &= \frac{1}{a}\left[\operatorname{Res}\left(\frac{\pi\cot\pi z}{z-a}, 0\right) - \operatorname{Res}\left(\frac{\pi\cot\pi z}{z}, 0\right)\right] \\ &= \frac{1}{a}\left[-\frac{1}{a} - 0\right] \qquad \text{(0, because it's an \underline{even} function)} \\ &= -\frac{1}{a^2}\end{aligned}$$

So, we have

$$\sum_{n\neq 0} \frac{1}{n(n-a)} - \frac{1}{a^2} = -\frac{\pi\cot\pi a}{a}.$$

DIFFERENT VIEWPOINT: regard this as a formula for $\cot$. Thus

$$\pi\cot\pi a = \frac{1}{a} + \sum_{n\neq 0} \frac{a}{n(a-n)}.$$

Adjustment:

$$\begin{aligned}\pi\cot\pi a &= \frac{1}{a} + \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{a}{n(a-n)} \\ &= \frac{1}{a} + \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \left(\frac{1}{n} + \frac{1}{a-n}\right) \\ &= \frac{1}{a} + \lim_{N\to\infty} \sum_{n=-N}^{N} \left(\frac{1}{a-n}\right) \\ &= \lim_{N\to\infty} \sum_{n=-N}^{N} \left(\frac{1}{a-n}\right).\end{aligned}$$

We rewrite this formula as

$$\boxed{\pi \cot \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{z-n}}$$

with the assumption that $z \neq$ integer and the doubly infinite series is taken in a "principle value" sense

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \left(\frac{1}{z-n}\right).$$

This formula is a dramatic display of two things about $\pi \cot \pi z$:

1. simple pole at each n with residue 1,
2. periodic with period 1.

(If we knew nothing about cot we would be tempted to concoct the infinite series as a function satisfying these two conditions!)

<u>PRODUCT REPRESENTATION OF SINE</u>:
We begin with the preceding formula

$$\pi \cot \pi z = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z-n}.$$

This is valid for all $z \in \mathbb{C} \setminus \mathbb{Z}$. First, we rewrite it as

$$\begin{aligned}
\pi \cot \pi z - \frac{1}{z} &= \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} \frac{1}{z-n} \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{z-n} + \sum_{n=1}^{N} \frac{1}{z+n} && \text{(changed } n \text{ to } -n\text{)} \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n}\right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n-z}\right).
\end{aligned}$$

We prepare to integrate by first noting that the left side equals

$$\begin{aligned}
\frac{\pi \cos \pi z}{\sin \pi z} - \frac{1}{z} &= \frac{d}{dz}(\log \sin \pi z - \log z) \\
&= \frac{d}{dz} \log \frac{\sin \pi z}{z} \\
&= \frac{d}{dz} \log \frac{\sin \pi z}{\pi z};
\end{aligned}$$

we've supplied an extra π so that not only is the quotient $\dfrac{\sin \pi z}{\pi z}$ holomorphic near 0 (removable singularity), but also equals 1 at $z = 0$. We now integrate from 0 to z, avoiding all integers except 0. We obtain

$$\log \frac{\sin \pi z}{\pi z} = \sum_{n=1}^{\infty} \left[\log\left(1 + \frac{z}{n}\right) + \log\left(1 - \frac{z}{n}\right) \right].$$

There's a great deal of ambiguity in this "equation," all having to do with the fact that the choice of logarithm involves additive constants $2N\pi i$ for integers N. For $z = 0$ both sides are 0 (to within $2N\pi i$).

Notice, however, that for large n we have terms for which the principle value of $\log$ may be used, and the corresponding Maclaurin expansions then give

$$\begin{aligned}\log\left(1 + \frac{z}{n}\right) &= \frac{z}{n} - \frac{z^2}{2n^2} + \ldots, \\ \log\left(1 - \frac{z}{n}\right) &= -\frac{z}{n} - \frac{z^2}{2n^2} + \ldots,\end{aligned}$$

so that the sum equals

$$0 - \frac{z^2}{n^2} + \ldots$$

and the series converges.

Now we exponentiate, thus wiping away all ambiguity, since $e^{2N\pi i} = 1$. Thus, we obtain

$$\begin{aligned}\frac{\sin \pi z}{\pi z} &= \lim_{N\to\infty} \prod_{n=1}^{N} \left(1 + \frac{z}{n}\right)\left(1 - \frac{z}{n}\right) \\ &= \lim_{N\to\infty} \prod_{n=1}^{N} \left(1 - \frac{z^2}{n^2}\right) \\ &=: \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).\end{aligned}$$

Therefore,

$$\boxed{\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)}$$

Euler product for sine

This amazing equation displays elegantly that $\sin \pi z$ equals 0 precisely when $z \in \mathbb{Z}$, and "factors" $\sin \pi z$ as if it were a polynomial divisible by $z - n$ for all $n \in \mathbb{Z}$.

When we let $z = \dfrac{1}{2}$ we obtain the equations

$$1 = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right),$$

so that

$$\begin{aligned} \frac{2}{\pi} &= \prod_{n=1}^{\infty} \left(\frac{4n^2 - 1}{4n^2}\right) \\ &= \prod_{n=1}^{\infty} \left(\frac{(2n-1)(2n+2)}{2n \cdot 2n}\right), \end{aligned}$$

or we might write

$$\frac{2}{\pi} = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdot \ldots,$$

a formula that has the name **Wallis' product** (1655).

A brief treatment of infinite products in general.
For given complex numbers $c_1, c_2, \ldots$ we want to define

$$\prod_{n=1}^{\infty} c_n = \lim_{N \to \infty} \prod_{n=1}^{N} c_n.$$

We must take reasonable care in discussing this situation.

Case 1: Assume that for all n, $c_n \neq 0$. We then institute the requirement that the limit we have given be nonzero. It then necessarily follows that

$$\lim_{n \to \infty} c_n = 1.$$

(Converse is false: consider $c_n = \dfrac{n}{n+1}$ so that $\displaystyle\prod_{n=1}^{N} c_n = \dfrac{1}{N+1} \to 0$.)

We can then compute that

$$\log \prod_{n=1}^{N} c_n = \sum_{n=1}^{N} \log c_n$$

and can assume that for large n we use the principle determination for $\log c_n$. Then we also have the

<u>Proposition</u>: *Assume $c_n \neq 0$ for all n and that $c_n \to 1$. Then $\prod_{n=1}^{\infty} c_n$ exists and is not $0 \Leftrightarrow \sum_{n=1}^{\infty} \log c_n$ converges.*

<u>General case</u>: Assume only that

$$\lim_{n\to\infty} c_n = 1$$

Then for sufficiently large n_0, $c_n \neq 0$ for $n_0 < n < \infty$, and we can use <u>Case 1</u>. So, we say that the infinite product

$$\prod_{n=1}^{\infty} c_n$$

<u>converges</u> if

$$\prod_{n=n_0+1}^{\infty} c_n \neq 0,$$

and we define

$$\prod_{n=1}^{\infty} c_n = \left(\prod_{n=1}^{n_0} c_n\right)\left(\prod_{n=n_0+1}^{\infty} c_n\right).$$

(thus, $\prod_{n=1}^{\infty} c_n = 0 \Leftrightarrow$ some $c_n = 0$. Therefore, convergent infinite products have very obvious zeroes!)

PROBLEM 5-6 The technique described at the beginning of this section, page 132, can be applied with <u>$\pi \cot \pi z$</u> replaced by <u>$\pi \csc \pi z$</u>. Same assumptions on f. It's easy to show that $\pi \csc \pi z$ is bounded on γ_N so the line integral on γ_N tends to 0 as $N \to \infty$. You need not prove those facts.

Apply all of that to the function $f(z) = \dfrac{1}{(z-a)^3}$, where a is not an integer, and this find the sum of the infinite series

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n-a)^3}.$$

Now set $a = \dfrac{1}{2}$ and thus calculate

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}.$$

PROBLEM 5-7

1. Calculate $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$.

2. Calculate $\prod_{n=2}^{\infty}\left(\frac{n^3+1}{n^3-1}\right)$.

3. For $|z|<1$ calculate $\prod_{n=0}^{\infty}\left(1+z^{2^n}\right)$.

4. Prove that for all $z \in \mathbb{C}$

$$\cos \pi z = \prod_{n=0}^{\infty}\left(1-\frac{z^2}{\left(n+\frac{1}{2}\right)^2}\right)$$

HINT: <u>easy</u> if you are clever!

Chapter 6

The Gamma Function

This chapter introduces one of the most important "special functions" in all of mathematics. It is always called by the capital Greek letter gamma, just as a historical accident.

Section A: Development

Here's what I'll call a basic definition: Γ is the function defined by

(1) $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \quad$ for $\underline{\operatorname{Re} z > 0}$.

In this definition we use the power t^{z-1} with $\ln t \in \mathbb{R}$. The restriction on the real part of z is to insure that the integral is (absolutely) convergent near $t = 0$, as

$$\int_0^1 \left|t^{z-1}\right| dt = \int_0^1 t^{\operatorname{Re} z - 1} dt < \infty \Leftrightarrow \operatorname{Re} z > 0.$$

(The convergence for large t is assured, as e^{-t} dominates any power of t.)

(2) Recursion $\boxed{\Gamma(z+1) = z\Gamma(z)}$ The proof is an easy exercise in integration by parts:

$$\begin{aligned}
\Gamma(z+1) &= \int_0^\infty t^z d(-e^{-t}) \\
&= -t^z e^{-t}\Big|_0^\infty + \int_0^\infty e^{-t} d(t^z) \\
&= 0 + \int_0^\infty e^{-t} z t^{z-1} dt \\
&= z\Gamma(z).
\end{aligned}$$

(3) $\boxed{\Gamma(n+1) = n! \text{ for } n = 0, 1, 2, \ldots}$ Follows from (1), since

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1.$$

(4) Analytic continuation: The recursion enables us to define $\Gamma(z)$ for all $z \in \mathbb{C}$ except $0, -1, -2, \ldots$. For we can define

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \text{ for } \operatorname{Re} z > -1, \text{ etc.}$$

And then (2) continues to hold for all z except $0, -1, -2, \ldots$.

(5) Poles: Thus, for example we see that Γ has a pole at 0, and $\operatorname{Res}(\Gamma, 0) = 1$. Thus Γ is a holomorphic function on all of $\mathbb{C}$ except for poles at $0, -1, -2, -3, \ldots$. It's given by the integral (1) only for $\operatorname{Re} z > 0$.

(6) Another method for analytic continuation: For $\operatorname{Re} z > 0$ we have the formula

$$\Gamma(z) = \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt.$$

The second integral is actually an entire holomorphic function, since $t \geq 1$ allows any value for z. The first integral can be rewritten, still for $\operatorname{Re} z > 0$, as

$$\begin{aligned}
\int_0^1 e^{-t} t^{z-1} dt &= \int_0^1 \sum_{n=0}^\infty \frac{(-1)^n t^n}{n!} t^{z-1} dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 t^{z+n-1} dt \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{z+n} t^{z+n} \Big|_0^1 \\
&= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{z+n}.
\end{aligned}$$

We have used $\operatorname{Re} z > 0$ throughout the computation, but the expression on the last line needs no such restriction! Therefore, analytic continuation gives

$$\Gamma(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{z+n} + \int_1^\infty e^{-t} t^{z-1} dt,$$

valid for all $z \in \mathbb{C}$ except $0, -1, -2, \ldots$. And we immediately read off the residues at these (simple) poles:

$$\boxed{\operatorname{Res}(\Gamma, -n) = \frac{(-1)^n}{n!}}$$

for all $n = 0, 1, 2, \ldots$.

Section B: The Beta Function

(7) DEFINITION: For $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$ we define

$$B(a,b) = 2\int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta \cos^{2b-1}\theta d\theta.$$

Change variable $\sin\theta = \sqrt{t}$ to get also

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt.$$

Note that $B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$.

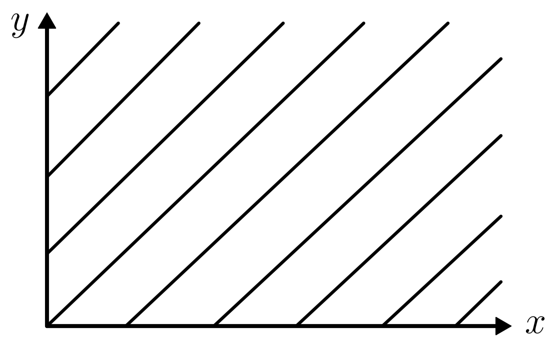
(8) Beta in terms of gamma
In the integral formula for Γ replace t by x^2 to get

$$\Gamma(z) = 2\int_0^\infty e^{-x^2}x^{2z-1}dx \qquad \text{for } \operatorname{Re} z > 0.$$

Then we multiply

$$\begin{aligned}\Gamma(a)\Gamma(b) &= 4\int_0^\infty e^{-x^2}x^{2a-1}dx \cdot \int_0^\infty e^{-x^2}x^{2b-1}dx \\ &= 4\int_0^\infty e^{-y^2}y^{2a-1}dy \cdot \int_0^\infty e^{-x^2}x^{2b-1}dx \\ &\overset{\text{Fubini}}{=} 4\int_0^\infty \int_0^\infty e^{-x^2-y^2}y^{2a-1}x^{2b-1}dxdy.\end{aligned}$$

Now employ polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, and the calculus formula $dxdy = rdrd\theta$, to get



$$\begin{aligned}\Gamma(a)\Gamma(b) &= 4\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}r^{2a-1}\sin^{2a-1}\theta\, r^{2b-1}\cos^{2b-1}\theta\, rdrd\theta \\ &= 2\int_0^{\frac{\pi}{2}}\sin^{2a-1}\theta\cos^{2b-1}\theta\, d\theta \cdot 2\int_0^\infty e^{-r^2}r^{2(a+b)-1}dr \\ &= B(a,b)\Gamma(a+b).\end{aligned}$$

Thus,

$$\boxed{B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}}$$

(9) Gaussian integral: Let $a = b = \frac{1}{2}$ to get $\pi = \Gamma\left(\frac{1}{2}\right)^2$. That is,

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

That is, from (8),

$$2\int_0^\infty e^{-x^2}dx = \sqrt{\pi},$$

or in other words,

$$\boxed{\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}}$$

(REMARK: The technique in (8) comes from the standard multivariate calculus method for evaluating the Gaussian integral, so it's no surprise that (9) follows from (8).)

(10) Another amazing formula: Now we assume $0 < a < 1$ and we take $b = 1 - a$. Then

$$\begin{aligned}\Gamma(a)\Gamma(1-a) &= B(a, 1-a) \\ &= 2\int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta \cos^{1-2a}\theta \, d\theta \\ &= 2\int_0^{\frac{\pi}{2}} \tan^{2a-1}\theta \, d\theta.\end{aligned}$$

Now change variables:

$$t = \tan\theta, \quad \text{so } dt = \sec^2\theta \, d\theta = (1+\tan^2\theta)d\theta = (1+t^2)d\theta,$$

and we find

$$\begin{aligned}\Gamma(a)\Gamma(1-a) &= 2\int_0^\infty t^{2a-1}\frac{dt}{t^2+1} \\ &\overset{t=\sqrt{x}}{=} 2\int_0^\infty \frac{x^{a-\frac{1}{2}}}{x+1}\cdot\frac{1}{2\sqrt{x}}dx \\ &= \int_0^\infty \frac{x^{a-1}}{x+1}dx \\ &\overset{!}{=} \frac{\pi}{\sin \pi a} \qquad \text{(page 91, Ch. 4)}\end{aligned}$$

All functions in sight are holomorphic, so we obtain this formula not just for real a, $0 < a < 1$, but also

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}}$$

for all $z \in \mathbb{C}$, $z \neq$ integer. Notice that the poles of $\dfrac{\pi}{\sin \pi z}$ at $z = 1, 2, 3, \ldots$ come from the factor $\Gamma(1-z)$, and its poles at $z = 0, -1, -2, \ldots$ come from the factor $\Gamma(z)$. We also retrieve the residue calculation for $z = -n$, where $n = 0, 1, 2, \ldots$:

$$\begin{aligned}
\operatorname{Res}(\Gamma(z)\Gamma(1-z), -n) &= \operatorname{Res}\left(\frac{\pi}{\sin \pi z}, -n\right); \\
\operatorname{Res}(\Gamma(z), -n) \cdot \Gamma(1+n) &= \frac{1}{\cos \pi n} = (-1)^n; \\
\operatorname{Res}(\Gamma, -n) &= \frac{(-1)^n}{n!}, \text{ as in (5).}
\end{aligned}$$

(11) $\underline{\Gamma(z) \text{ is never } 0}$: (follows from (9)) Therefore,

$$\boxed{\begin{array}{l} \dfrac{1}{\Gamma(z)} \text{ is an entire holomorphic function and has zeros} \\ \text{only at } 0, -1, -2, \ldots, \text{ all of which are simple.} \end{array}}$$

Section C: Infinite Product Representation

We begin with the original definition for $\operatorname{Re} z > 0$:

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

We use the basic calculus fact

$$e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n.$$

We skip the (rather easy) verification of passing the limit across the integral sign, so here's what we find:

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt.$$

Let n be fixed and change variable with $t = ns$:

$$\begin{aligned}
\int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt &= \int_0^1 (1-s)^n (ns)^{z-1} n ds \\
&= n^z \int_0^1 (1-s)^n s^{z-1} ds \\
&= n^z B(n+1, z) \\
&= n^z \frac{\Gamma(n+1)\Gamma(z)}{\Gamma(n+1+z)} \\
&= n^z n! \frac{\Gamma(z)}{\Gamma(n+1+z)} \\
&= n^z n! \frac{1}{(n+z)(n-1+z)\cdots(1+z)z}.
\end{aligned}$$

Therefore, we conclude

$$\boxed{\Gamma(z) = \lim_{n\to\infty} \frac{n^z n!}{z(1+z)\cdots(n+z)}}$$

This formula has been derived under the assumption that $\operatorname{Re} z > 0$. However, the left side is holomorphic for all z except $0, -1, -2, \ldots$, and it is possible to prove the same is true for the right side. Therefore, the principle of analytic continuation implies that formula is actually valid for all $z \in \mathbb{C}$ except the nonpositive integers.

This formula is not quite an infinite product, and we now show how to arrange it as such a product. So that we don't have to continue to worry about the poles, let's rewrite it this way:

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= \lim_{n\to\infty} \frac{z(1+z)\cdots(n+z)}{n^z n!} \\
&= \lim_{n\to\infty} n^{-z} \cdot z \cdot \frac{1+z}{1} \cdot \frac{2+z}{2} \cdot \ldots \cdot \frac{n+z}{n} \\
&= \lim_{n\to\infty} n^{-z} z \prod_{k=1}^{n} \left(1 + \frac{z}{k}\right).
\end{aligned}$$

Almost there! However, n^{-z} has no limit as $n \to \infty$, nor does that finite product. In fact,

$$\prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)$$

diverges, because the infinite series

$$\sum_{k=1}^{\infty} \log\left(1 + \frac{z}{k}\right)$$

diverges. This is essentially because $\log\left(1+\frac{z}{k}\right) \approx \frac{z}{k}$ for large k, and the harmonic series $\sum\limits_{k=1}^{\infty} \frac{1}{k}$ diverges.

(Example: $\prod\limits_{k=1}^{n}\left(1+\frac{1}{k}\right) = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n} = n+1 \to \infty$. But it's interesting that alternating signs give

$$\prod_{k=1}^{\infty}\left(1+\frac{(-1)^{k-1}}{k}\right) = 2 \cdot \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} \cdot \frac{5}{6} \cdots$$

so that

$$\prod_{k=1}^{\infty}\left(1+\frac{(-1)^{k-1}}{k}\right) = 1.)$$

This divergence can be fixed by multiplying the k^{th} factor by $e^{-\frac{z}{k}}$. For we have the Maclaurin series $(1+\varepsilon)e^{-\varepsilon} = 1 - \frac{\varepsilon^2}{2} + \ldots$, so that $\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}} = 1 - \frac{z^2}{2k^2} + \ldots$ and the series $\sum\limits_{k=1}^{\infty} \frac{z^2}{2k^2}$ converges.

Thus we obtain

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= z \lim_{n\to\infty} n^{-z} \cdot \prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}} \cdot \prod_{k=1}^{n} e^{\frac{z}{k}} \\
&= z \lim_{n\to\infty} e^{-z\ln n} \cdot \prod_{k=1}^{n} e^{\frac{z}{k}} \cdot \prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}} \\
&= z \lim_{n\to\infty} e^{z\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right)} \cdot \prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}.
\end{aligned}$$

Now we use the classical "Euler constant," also called the "Euler-Mascheroni constant,"

$$\gamma := \lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\ln n\right).$$

We've therefore derived the infinite product we were seeking:

$$\boxed{\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}}$$

Personal note: I first learned about $\Gamma(z)$ around 1960, from studying the masterful *A Course of Modern Analysis* by Whittaker and Watson, 1902. Their Chapter XII starts with the formula we've just derived as their definition of $\Gamma(z)$.

P.S. $\gamma = 0.5772157\ldots$ and no one knows whether γ is a <u>rational</u> number! Its decimal expansion has been calculated to over 29 billion digits!

PROBLEM 6-1

1. Prove that $\Gamma'(1) = \int_0^\infty e^{-t}\ln t dt$.
2. Prove that $\Gamma'(1) = -\gamma$.
3. Prove that for all nonzero real y, $|\Gamma(iy)|^2 = \dfrac{\pi}{y\sinh \pi y}$.

Section D: Gauss' Multiplication Formula

We begin with the infinite product representation of $\dfrac{1}{\Gamma(z)}$ given at the end of the preceding section:

$$\frac{1}{\Gamma(z)} = ze^{yz}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}.$$

Then compute $\log$ of both sides, producing

$$-\log\Gamma(z) = \log z + \gamma z + \sum_{n=1}^{\infty}\left(\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right);$$

we need not specify which values of $\log$ we are using (except that the convergence of the series requires $\log\left(1+\dfrac{z}{n}\right)$ to have limit 0 as $n\to\infty$). The reason is we now differentiate with respect to z, removing all additive constants:

$$-(\log\Gamma(z))' = \frac{1}{z} + \gamma + \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right).$$

One more derivative produces

$$\begin{aligned}(\log\Gamma(z))'' &= \frac{1}{z^2} + \sum_{n=1}^{\infty}\frac{1}{(n+z)^2}\\ &= \sum_{n=0}^{\infty}\frac{1}{(n+z)^2}. \qquad \bigstar\end{aligned}$$

Now let N be a fixed integer ≥ 2. We obtain easily

$$(\log\Gamma(Nz))'' = \sum_{n=0}^{\infty}\frac{N^2}{(n+Nz)^2}.$$

Rewrite this as

$$(\log\Gamma(Nz))'' = \sum_{n=0}^{\infty} \frac{1}{\left(\frac{n}{N}+z\right)^2}. \qquad \bigstar\bigstar$$

We also obtain from $\bigstar$ that for any integer $0 \le k \le N-1$

$$\left(\log\Gamma\left(z+\frac{k}{N}\right)\right)'' = \sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{k}{N}+z\right)^2}.$$

Therefore,

$$\left(\log\Gamma\left(z+\frac{k}{N}\right)\right)'' = \sum_{m=0}^{\infty} \frac{1}{\left(\frac{mN+k}{N}+z\right)^2}. \qquad \bigstar\bigstar\bigstar$$

Observation: the integers $mN+k$ for $m \ge 0$ and $0 \le k \le N-1$ are precisely the integers $n \ge 0$ counted exactly one time. We conclude that

$$(\log\Gamma(Nz))'' = \sum_{m=0}^{N-1} \left(\log\left(\Gamma\left(z+\frac{k}{N}\right)\right)\right)''.$$

Integrate twice to obtain

$$\log\Gamma(Nz) = \sum_{k=0}^{N-1} \log\left(\Gamma\left(z+\frac{k}{N}\right)\right) + c_1 z + c_2$$

for some c_1 and c_2 both independent of z.

Now you may finish the development:

PROBLEM 6-2

1. Prove that $\Gamma(Nz) = ce^{c_1 z} \prod_{k=0}^{N-1} \Gamma\left(z+\frac{k}{N}\right)$.

2. Now replace z by $z+1$ and then divide the two equations. Conclude that
$$e^{c_1} = N^N.$$
$$\Gamma(Nz) = cN^{Nz} \prod_{k=0}^{N-1} \Gamma\left(z+\frac{k}{N}\right).$$

3. Now let $z = \dfrac{1}{N}$ to obtain
$$1 = cN \prod_{j=1}^{N} \Gamma\left(\frac{j}{N}\right).$$

4. Rewrite in the form

$$1 = cN \prod_{j=0}^{N-1} \Gamma\left(1 - \frac{j}{N}\right).$$

5. Multiply to obtain

$$1 = c^2 N^2 \prod_{j=1}^{N-1} \Gamma\left(\frac{j}{N}\right) \Gamma\left(1 - \frac{j}{N}\right).$$

6. Conclude that

$$1 = c^2 N^2 \frac{\pi^{N-1}}{\prod_{j=1}^{N-1} \sin \frac{\pi j}{N}}.$$

7. Therefore, show that

$$1 = c^2 N (2\pi)^{N-1}.$$

8. Therefore, conclude

$$\boxed{\Gamma(Nz) = (2\pi)^{\frac{1-N}{2}} N^{Nz - \frac{1}{2}} \prod_{k=0}^{N-1} \Gamma\left(z + \frac{k}{N}\right)}$$

This is known as the ***Gauss Multiplication Formula***.

PROBLEM 6-3 When $N = 2$ we call that result the *Gauss duplication formula*

$$\Gamma(2z) = \frac{1}{\sqrt{2\pi}} 2^{2z - \frac{1}{2}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right).$$

Show that if z is a positive integer, then this formula is elementary if we know $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

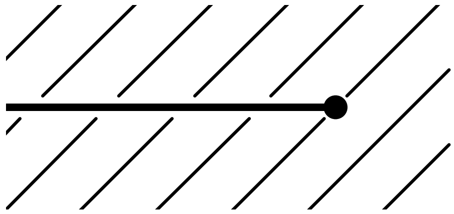
Section E: Another Representation of Γ

Again, we work using the fact that $\dfrac{1}{\Gamma(z)}$ is an entire function of z. Hermann Hankel, a contemporary of Riemann, found an integral representation of $\dfrac{1}{\Gamma}$ valid for all z. Here is his method.

Let $z \in \mathbb{C}$ be fixed. Consider the holomorphic function of w given by

$$f(w) = e^w w^{-z}, \quad \text{where } -\pi < \arg w < \pi.$$

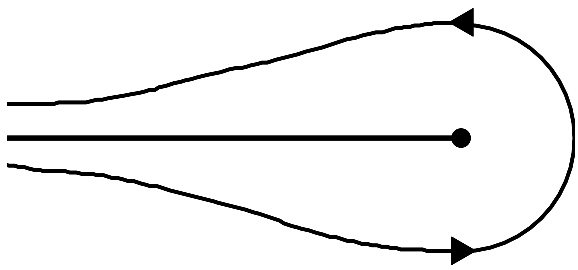
That is, we are using the principle value of $\arg w$. Then



$$|f(w)| = e^{\mathrm{Re}(w)} |w^{-z}|,$$

so f tends to 0 exponentially as $\mathrm{Re}(w) \to -\infty$ and $\mathrm{Im}(w)$ remains bounded.

As a result, if γ is any curve of the following sort



then we can perform the integral

$$I(z) := \frac{1}{2\pi i} \int_\gamma e^w w^{-z} dw.$$

Here is the result that we are going to prove:

THEOREM:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_\gamma e^w w^{-z} dw$$

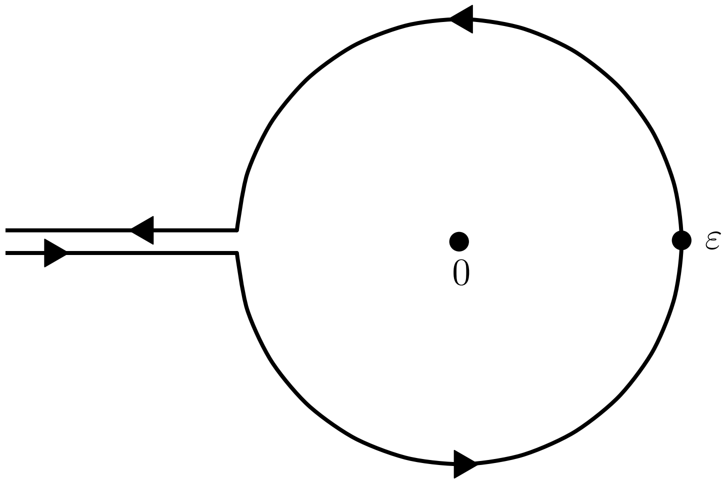
for any such curve γ.

REMARK: This terrific result displays the entire function $\dfrac{1}{\Gamma(z)}$ in terms of the entire functions

$$w^{-z} = e^{-(\log w)z}.$$

Thus $I(z)$ and $\dfrac{1}{\Gamma(z)}$ are both entire functions.

Proof: It suffices to give the proof for $\mathrm{Re}(z) < 1$, thanks to analytic continuation. Cauchy's theorem and the exponential decay of e^w as $\mathrm{Re}(w) \to \infty$ show that the integral $I(z)$ is <u>independent</u> of the choice of γ. Therefore, it's up to us to choose a convenient γ. Let $\varepsilon > 0$ and chooce $\gamma = \gamma_\varepsilon$ as shown:



Then

$$2\pi i I(z) = \int_{\substack{-\infty \\ \arg w = -\pi}}^{-\varepsilon} \dots dw + \int_{|w|=\varepsilon} \dots dw + \int_{\substack{-\varepsilon \\ \arg w = \pi}}^{-\infty} \dots dw.$$

The middle of these three integrals is

$$\int_{|w|=z} e^w w^{-z} dw.$$

We now show this has limit 0 as $\varepsilon \to 0$: first, e^w certainly has limit 1. The crucial term is

$$\begin{aligned} w^{-z} &= e^{-z \log w} \\ &= e^{-(x+iy)(\log \varepsilon + i \log w)}, \end{aligned}$$

so

$$\begin{aligned} |w^{-z}| &= e^{-x \log \varepsilon + y \arg w} \\ &= \varepsilon^{-x} e^{y \arg w} \\ &\leq \varepsilon^{-x} e^{|y|\pi}. \end{aligned}$$

Thus, this middle integral is bounded by a constant times $\varepsilon^{-x} \cdot 2\pi\varepsilon \to 0$ (since $x < 1$).

Therefore, as $I(z)$ doesn't actually depend on ε, we can let $\varepsilon \to 0$ to find that

$$2\pi i I(z) = \int_{\substack{-\infty \\ \arg w = -\pi}}^{0} e^w w^{-z} dw + \int_{\substack{0 \\ \arg w = \pi}}^{-\infty} e^2 w^{-z} dw.$$

Let $t = -w$. Then

$$\begin{aligned}
2\pi i I(z) &= \int\limits_{\substack{\infty \\ \arg w=-\pi}}^{0} e^{-t}e^{-z\log w}(-dt) + \int\limits_{\substack{0 \\ \arg w=\pi}}^{\infty} e^{-t}e^{-z\log w}(-dt) \\
&= \int_{\infty}^{0} e^{-t}e^{-z(\log t-i\pi)}(-dt) + \int_{0}^{\infty} e^{-t}e^{-z(\log t+i\pi)}(-dt) \\
&= \int_{0}^{\infty} e^{-t}e^{-z\log t}\left(e^{i\pi z} - e^{-i\pi z}\right)dt \\
&= 2i\sin\pi z\int_{0}^{\infty} e^{-t}t^{-z}dt \\
&= 2i\sin\pi z\Gamma(1-z) \qquad\qquad (\text{since } \mathrm{Re}(1-z) > 0).
\end{aligned}$$

Therefore,

$$\begin{aligned}
I(z) &= \frac{\sin\pi z}{\pi}\Gamma(1-z) \\
&= \frac{1}{\Gamma(z)}
\end{aligned}$$

QED

Section F: The Bohr-Mollerup theorem

This relatively recent (1922) theorem is the work of two mathematicians at the University of Copenhagen, Harald Bohr and Johannes Mollerup. It was published in Laerebog i Kompleks Analyse vol. III. Harald Bohr was a younger brother of the famous physicist Niels Bohr.

I first heard about this theorem in the splendid 39-page book by Emil Artin, *The Gamma Function*, originally published in 1931 as Einführung in die Theorie der Gammafunktion.

The theorem deals with the concept of positive functions on $\mathbb{R}$ which are log convex; f is log convex if $\ln f(x)$ is convex.

THEOREM: $\Gamma(x)$ is log convex for $0 < x < \infty$.

Proof: Γ is a C^∞ function on $\mathbb{R}$, so it suffices to prove that $(\ln\Gamma)'' \geq 0$. Now

$$(\ln\Gamma)' = \frac{\Gamma'}{\Gamma},$$

so we need to show

$$\left(\frac{\Gamma'}{\Gamma}\right)' = \frac{\Gamma\Gamma'' - (\Gamma')^2}{\Gamma^2} \geq 0.$$

That is,

$$(\Gamma')^2 \leq \Gamma\Gamma''.$$

Now

$$\begin{aligned}
\Gamma(x) &= \int_0^\infty e^{it} t^{x-1} dt, \\
\Gamma'(x) &= \int_0^\infty e^{-t} t^{x-1} \ln t dt, \\
\Gamma''(x) &= \int_0^\infty e^{-t} t^{x-1} (\ln t)^2 \, dt.
\end{aligned}$$

This leads one to introduce an <u>inner product</u> for real valued functions on $(0, \infty)$:

$$\langle u, v \rangle := \int_0^\infty e^{-t} t^{x-1} u(t) v(t) dt.$$

Thus we should prove

$$\langle 1, \ln t \rangle^2 \leq \langle 1, 1 \rangle \langle \ln t, \ln t \rangle.$$

Hah! This is the Cauchy-Schwarz inequality!

QED

Here, then, is the

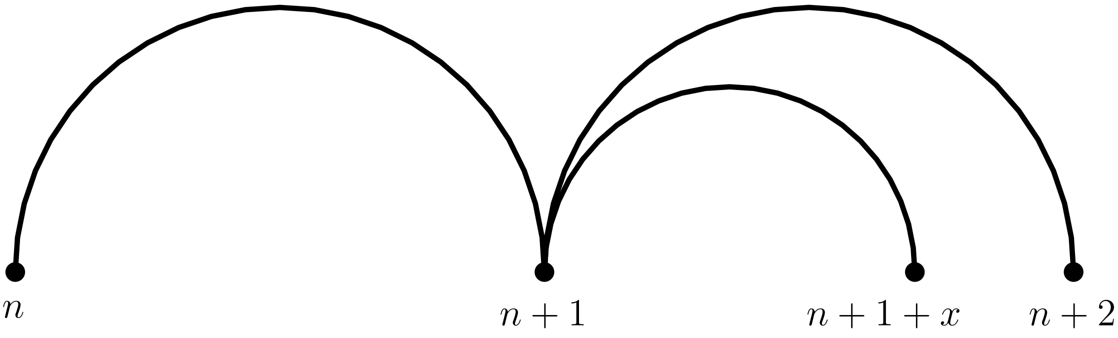
Bohr-Mollerup theorem: Let $(0, \infty) \xrightarrow{f} (0, \infty)$ be a function with three properties:

$$\begin{aligned}
f(x+1) &= x f(x) \qquad \text{for } 0 < x < \infty, \\
f(1) &= 1, \\
f &\text{ is log convex.}
\end{aligned}$$

Then $f = \Gamma$.

Proof: We'll use another characterization of convex functions: the secant lines of their graphs increase in slope as they move from left to right.

Let $0 < x \leq 1$ be fixed, and consider integers $n \geq 1$. Then examine these three intervals:



Then $\ln f(x)$ is convex, so

$$\begin{aligned}\frac{\ln f(n+1)-\ln f(n)}{1} &\leq \frac{\ln f(n+1+x)-\ln f(n+1)}{x}\\ &\leq \frac{\ln f(n+2)-\ln f(n+1)}{1}.\end{aligned}$$

Therefore,

$$\ln n \leq \frac{\ln f(n+1+x)-\ln f(n+1)}{x} \leq \ln(n+1). \qquad (\bigstar)$$

We also have

$$f(n+1)=n!$$

Therefore, multiply $\bigstar$ by x to obtain

$$x\ln n \leq \ln f(n+1+x)-\ln n! \leq x\ln(n+1).$$

Add $\ln n!$ to obtain

$$x\ln n+\ln n! \leq \ln f(n+1+x) \leq x\ln(n+1)+\ln n!$$

Exponentiate:

$$n^x n! \leq f(n+1+x) \leq (n+1)^x n!$$

Since

$$\begin{aligned}f(n+1+x) &= (n+x)f(n+x) = \dots\\ &= (n+x)\dots(1+x)xf(x),\end{aligned}$$

we have

$$\frac{n^x n!}{x(x+1)\dots(x+n)} \leq f(x) \leq \frac{(n+1)^x n!}{x(x+1)\dots(x+n)}.$$

Rewrite:

$$\frac{n^x n!}{x(x+1)\dots(x+n)} \le f(x) \le \frac{n^x n!}{x(x+1)\dots(x+n)} \cdot \left(1+\frac{1}{n}\right)^x.$$

Now we take the limit as $n \to \infty$. Since the extra factor $\left(1+\frac{1}{n}\right)^x$ has limit 1, the two inequalities show that the limits exist as $n \to \infty$ and are equal to $f(x)$ for any $0 < x \le 1$:

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\dots(x+n)}. \qquad (\bigstar\bigstar)$$

And since Γ itself is log convex, we have

$$f(x) = \Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\dots(x+n)} \qquad \text{for } 0 < x \le 1.$$

It's very easy to check that if the limit in $\bigstar\bigstar$ exists for x, then it also exists for $x+1$, and equals x times the limit for x. Thus, it satisfies exactly the property $f(x+1) = xf(x)$. Therefore $\bigstar\bigstar$ holds for all $x > 0$. Therefore, we see that

$$\boxed{f(x) = \Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\dots(x+n)} \qquad \text{for } 0 < x < 1.}$$

QED

That formula is very familiar to us, as it appears in Section C. But we did not use it in the proof of Bohr-Mollerup. Instead, we derived the expression for the log convex f and then used the log convexity of Γ.

Here is an illustration of the use of Bohr-Mollerop:

PROBLEM 6-2 outlines a proof of the Gauss multiplication formula. Here's another proof which starts with the function

$$f(z) := N^z \prod_{k=0}^{N-1} \Gamma\left(\frac{z+k}{N}\right).$$

We restrict attention to $z = x \in \mathbb{R}$, and observe two things:

1.

$$\begin{aligned} f(x+1) &= N^{x+1} \prod_{k=0}^{N-1} \Gamma\left(\frac{x+1+k}{N}\right) \\ &= N^{x+1} \prod_{j=1}^{N} \Gamma\left(\frac{x+j}{N}\right), \end{aligned}$$

so we see

$$\begin{aligned}\frac{f(x+1)}{f(x)} &= N\frac{\Gamma\left(\frac{x+N}{N}\right)}{\Gamma\left(\frac{x}{N}\right)}\\ &= N\frac{\Gamma\left(\frac{x}{N}+1\right)}{\Gamma\left(\frac{x}{N}\right)}\\ &= N\frac{x}{N}\\ &= x;\end{aligned}$$

thus $\underline{f(x+1) = xf(x)}$.

2.

$$\ln f(x) = x\ln N + \sum_{k=0}^{N-1} \ln\Gamma\left(\frac{x+k}{N}\right),$$

a sum of convex functions of x. Thus, $\underline{f \text{ is log convex}}$.

$\therefore$ Bohr-Mollorup $\Rightarrow \dfrac{f}{f(1)} = \Gamma$. I.e.,

$$f(x) = f(1)\Gamma(x).$$

Finally, we need to calculate $f(1)$. By definition,

$$\begin{aligned}f(1) &= N\prod_{k=0}^{N-1}\Gamma\left(\frac{1+k}{N}\right)\\ &= N\prod_{j=1}^{N}\Gamma\left(\frac{j}{N}\right) = N\prod_{j=1}^{N-1}\Gamma\left(\frac{j}{N}\right)\\ &\overset{j\to N-j}{=} N\prod_{j=1}^{N-1}\Gamma\left(1-\frac{j}{N}\right).\end{aligned}$$

$\therefore$

$$f(1)^2 = N^2\prod_{j=1}^{N-1}\Gamma\left(\frac{j}{N}\right)\Gamma\left(1-\frac{j}{N}\right).$$

We use

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

to get

$$\begin{aligned}f(1)^2 &= N^2\pi^{N-1}\prod_{j=1}^{N-1}\frac{1}{\sin\frac{\pi j}{N}}\\ &= \frac{N^2\pi^{N-1}}{\frac{N}{2^{N-1}}}\end{aligned}$$

by PROBLEM 1-9 (5). So we find

$$f(1)^2 = N\pi^{N-1}2^{N-1}$$

and

$$f(1) = N^{\frac{1}{2}}(2\pi)^{\frac{N-1}{2}}.$$

$\therefore$

$$f(x) = N^{\frac{1}{2}}(2\pi)^{\frac{N-1}{2}}\Gamma(x).$$

$\therefore$

$$\Gamma(x) = N^{-\frac{1}{2}}(2\pi)^{\frac{1-N}{2}}N^x\prod_{k=0}^{N-1}\Gamma\left(\frac{x+k}{N}\right).$$

Finally, replace x by Nx:

$$\Gamma(Nx) = (2\pi)^{\frac{1-N}{2}}N^{Nx-\frac{1}{2}}\prod_{k=0}^{N-1}\Gamma\left(x+\frac{k}{N}\right).$$

QED

Chapter 7

Conformal Transformations

We introduced this topic in Section D of Chapter 2, but now we want to cover it in much more detail. In order to make it clear, here is the

DEFINITION: Let D_1 and D_2 be open subsets of $\mathbb{C}$, and consider a function

$$D_1 \xrightarrow{f} D_2.$$

Then f is a conformal transformation if f is a bijection from D_1 to D_2 $\big($thus f is an injection and $f(D_1) = D_2\big)$ such that f is holomorphic and its inverse

$$D_2 \xrightarrow{f^{-1}} D_1$$

is also holomorphic.

As we noted in Chapter 3, f is angle preserving — hence the adjective conformal.

We begin with a very reasonable result, but whose proof seems quite involved:

Section A: Conformal Transformations of $\mathbb{C}$

THEOREM: The conformal transformations $\mathbb{C} \xrightarrow{f} \mathbb{C}$ are precisely the functions of the form $f(z) = az + b$, where $a \neq 0$ and b are constants.

Proof:

Step 1: It must be true that $\lim\limits_{z\to\infty} f(z) = \infty$. To show this, let A be any (large) positive number. Since f^{-1} is continuous, then it must be bounded when restricted to the closed disc $|w| \le A$. (Heine-Borel theorem) Thus, there exists a $B > 0$ such that

$$|w| \le A \Rightarrow \left|f^{-1}(w)\right| \le B.$$

Replace w by $f(z)$:

$$|f(z)| \le A \Rightarrow |z| \le B.$$

The contrapositive statement is

$$|z| > B \Rightarrow |f(z)| > A.$$

As A is arbitrary, Step 1 is established.

<u>Step 2</u>: Define

$$g(z) = \frac{1}{f\left(\frac{1}{z}\right)} \quad \text{for small } |z|.$$

Then g is holomorphic for small $z \ne 0$, and from Step 1

$$\lim_{z \to 0} g(z) = 0.$$

Thus the isolated singularity of g at 0 is removed by defining $g(0) = 0$. (Riemann's removable singularity theorem)

<u>Step 3</u>: Since f is injective, so is g. Therefore,

$$g'(0) \ne 0.$$

Therefore, the Maclaurin series for g has the form

$$g(z) = Cz + \dots \quad \text{and} \quad C \ne 0.$$

Therefore,

$$|g(z)| \ge D|z| \quad \text{for all small } z, \text{ where } D > 0.$$

Therefore,

$$|f(z)| = \frac{1}{\left|g\left(\frac{1}{z}\right)\right|} \le \frac{|z|}{D} \quad \text{for large } |z|.$$

<u>Step 4</u>: We apply the Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z} d\zeta \quad \text{for all } |z| < R.$$

Differentiate twice:

$$f''(z) = \frac{2!}{2\pi i} \int_{|z|=R} \frac{f(\zeta)}{(\zeta - z)^3} d\zeta.$$

Estimate:

$$\begin{aligned} |f''(z)| &\le \frac{2}{2\pi} \int_{|\zeta|=R} \frac{|f(\zeta)|}{|\zeta - z|^3} |d\zeta| \\ &\le \frac{1}{\pi} \int_{|\zeta|=R} \frac{|\zeta||d\zeta|}{D(|\zeta| - |z|)^3} \\ &= \frac{1}{D\pi} \int_{|\zeta|=R} \frac{R|d\zeta|}{(R - |z|)^3} \\ &= \frac{2R^2}{D(R - |z|)^3}. \end{aligned}$$

Let $R \to \infty$ to conclude

$$|f''(z)| = 0.$$

Finally, we obtain that $f(z) = az + b$. (Of course, $a \neq 0$.)

QED

Section B: The Cayley Transformation

This example comes from PROBLEM 3-1. I'm going first to introduce some notation:

$$\mathbb{D} = \text{ the open unit disc } |z| < 1$$

$$\mathbb{U} = \text{ the open upper half plane } \operatorname{Im}(z) > 0$$

The Cayley transform $\mathbb{U} \xrightarrow{f} \mathbb{D}$ is given by the Möbius transformation

$$f(z) = \frac{z - i}{z + i}. \qquad \begin{cases} i \to 0 \\ \infty \to 1 \\ 0 \to -1 \end{cases}$$

Its inverse is

$$f^{-1}(w) = i\frac{1 + w}{1 - w}. \qquad \begin{cases} -1 \to 0 \\ \infty \to -1 \\ 0 \to i \end{cases}$$

Section C: The Schwarz "Lemma"

For historical reasons this extremely important theorem is called a lemma:

Schwarz Lemma: Let f be a holomorphic function defined on $\mathbb{D}$ which satisfies

$$\begin{cases} |f(z)| \leq 1 & \text{for all } z \in \mathbb{D}, \\ f(0) = 0. \end{cases}$$

Then

$$\begin{cases} |f(z)| \leq |z| & \text{for all } z \in \mathbb{D}, \\ |f'(0)| \leq 1. \end{cases}$$

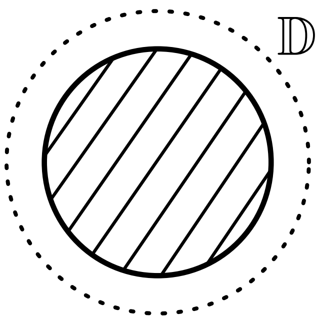
Moreover, if $|f(z)| = |z|$ for some $z \neq 0$, or $|f'(0)| = 1$, then $f(z) = az$, where $|a| = 1$.

Proof: This proof almost works itself, provided we start by defining

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & \text{for } 0 < |z| < 1, \\ f'(0) & \text{for } z = 0. \end{cases}$$

Clearly, g is holomorphic for $0 < |z| < 1$ and continuous for all $|z| < 1$. The Riemann removable singularity theorem implies that g is holomorphic for $|z| < 1$.

We want to use the maximum modulus principle, but we need to insert a technicality: let $0 < a < 1$ and restrict z to $|z| \leq a$. Then the maximum modulus principle implies that for $|z| \leq a$,



$$\begin{aligned} |g(z)| &\leq \sup_{|\zeta|=a} |g(\zeta)| \\ &= \sup_{|\zeta|=a} \frac{|f(\zeta)|}{a} \\ &\leq \frac{1}{a}. \end{aligned}$$

Therefore,

$$\begin{aligned} 0 < |z| \leq a &\Rightarrow |f(z)| \leq \frac{|z|}{a}; \\ z = 0 &\Rightarrow |f'(0)| \leq \frac{1}{a}. \end{aligned}$$

This argument required $0 < a < 1$, but now for any fixed $|z| < 1$, we let $a \to 1$ and find that

$$|f(z)| \le |z|;$$
$$|f'(0)| \le 1.$$

Finally, in the two cases of equality, $|g(z)| = 1$ for some $|z| < 1$. As $|g(z)| \le 1$ for all $|z| < 1$, we conclude by the maximum principle that $g(z) =$ constant. Write $g(z) = a$, where of course $|a| = 1$. Thus $f(z) = az$.

QED

COROLLARY: Let $\mathbb{D} \xrightarrow{f} \mathbb{D}$ be holomorphic with $f(0) = 0$. Then f is a conformal transformation of $\mathbb{D}$ onto itself $\Leftrightarrow f(z) = az$, where $|a| = 1$.

Proof: $\Leftarrow$ is clear, so we need to prove $\Rightarrow$. The Schwarz lemma implies that $|f'(0)| \le 1$. But we can also apply the Schwarz lemma to f^{-1}, so that $\left|\left(f^{-1}\right)'(0)\right| \le 1$. Aha!

$$f\left(f^{-1}(z)\right) = z,$$

so the chain rule implies

$$f'\left(f^{-1}(z)\right)\left(f^{-1}\right)'(z) = 1.$$

Let $z = 0$:

$$f'(0)\left(f^{-1}\right)'(0) = 1.$$

As both factors in this equation are complex numbers with moduli ≤ 1, we conclude

$$|f'(0)| = 1.$$

So the Schwarz lemma again implies that $f(z) = az$.

QED

Section D: Conformal Transformations of $\mathbb{D}$

I'll begin by defining some very special Möbius transformations. For any complex number a with $|a| < 1$, let ϕ_a be this function:

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}.$$

Notice that

$$\phi_a^{-1} = \frac{z + a}{1 + \bar{a}z},$$

so

$$\phi_a^{-1} = \phi_{-a}.$$

A simple calculation:

$$\begin{aligned}
|\text{den}|^2 - |\text{num}|^2 &= |1 - \overline{a}z|^2 - |z - a|^2 \\
&= \left(1 - 2\,\mathrm{Re}(\overline{a}z) + |\overline{a}z|^2\right) - \left(|z|^2 - 2\,\mathrm{Re}(\overline{a}z) + |a|^2\right) \\
&= 1 + |a|^2|z|^2 - |z|^2 - |a|^2 \\
&= \left(1 - |z|^2\right)\left(1 - |a|^2\right).
\end{aligned}$$

As $|a| < 1$, we see that

$$\begin{aligned}
|z| < 1 &\Leftrightarrow |\phi_a(z)| < 1, \\
|z| = 1 &\Leftrightarrow |\phi_a(z)| = 1, \\
|z| > 1 &\Leftrightarrow |\phi_a(z)| > 1.
\end{aligned}$$

$\therefore$ ϕ_a is a conformal transformation of $\mathbb{D}$ on $\mathbb{D}$, and ϕ_a preserves the unit circle. (Note that $\phi_a\left(\frac{1}{\overline{a}}\right) = \infty$.)

THEOREM: The conformal transformation of $\mathbb{D}$ onto $\mathbb{D}$ are precisely the functions $\omega\phi_a$, where the constant ω satisfies $|\omega| = 1$ and $a \in \mathbb{D}$.

Proof: Let $\mathbb{D} \xrightarrow{f} \mathbb{D}$ be such a transformation. Let $a \in \mathbb{D}$ be the point satisfying $f(a) = 0$. Clearly, we should look at the composition $f \circ \phi_a^{-1}$. This function is a conformal transformation of $\mathbb{D}$ onto $\mathbb{D}$ which sends $0 \to 0$:

$$\left(f \circ \phi_a^{-1}\right)(0) = f\left(\phi_a^{-1}(0)\right) = f(a) = 0.$$

The **Schwarz lemma** shows that

$$\left(f \circ \phi_a^{-1}\right)(z) = \omega z,$$

where $|\omega| = 1$. Therefore,

$$f(z) = \omega\phi_a(z).$$

QED

No wonder that the Schwarz lemma is so important!

Section E: Statement of the Riemann mapping theorem

This amazing and important theorem is beyond the scope of this book, beyond what I think of as computational complex analysis. However, it is very important to know what this theorem says, and to be able to compute many illustrations of it.

DEFINITION: In the development of the subject called topology, a space is said to be simply connected if it is path connected and any closed curve in the space can be continuously shrunk to a point while remaining in the space. The idea is very clear for an open set $D \subset \mathbb{C}$ which is path connected—it is simply connected if it "contains" no holes.

Simply connected D:
a disk, a convex region, a region whose complement is connected.

Not simply connected examples:
an annulus, $\mathbb{C} \setminus$ closed disk, $\mathbb{C} \setminus$ one point.

RIEMANN MAPPING THEOREM: Let $D \subset \mathbb{C}$ be a simply connected open set which is not the space $\mathbb{C}$ itself. Then there exists a conformal transformation of D onto the open unit disc $\mathbb{D}$.

The conformal transformation in the statement of the theorem is called a Riemann mapping of D onto $\mathbb{D}$.

Here is a relatively easy fact:

THEOREM: A Riemann mapping of D onto $\mathbb{D}$ is "essentially unique." That is, if $z_0 \in D$, then there exists one and only one $D \xrightarrow{f} \mathbb{D}$ satisfying the properties:

- f is a conformal transformation of D onto $\mathbb{D}$,
- $f(z_0) = 0$,
- $f'(z_0)$ is a positive real number.

Proof: Suppose that f and g both satisfy these conditions. Then consider the composition

$$f \circ g^{-1} : \mathbb{D} \to \mathbb{D}.$$

This is a conformal transformation of $\mathbb{D}$ onto $\mathbb{D}$ which maps 0 to 0:

$$\left(f \circ g^{-1}\right)(0) = f\left(g^{-1}(0)\right) = f(z_0) = 0.$$

The **Schwarz lemma** then shows that

$$\left(f \circ g^{-1}\right)(z) = \omega z, \qquad \text{where } |\omega| = 1.$$

That is,

$$f(z) = \omega g(z).$$

Then

$$f'(z_0) = \omega g'(z_0),$$

so ω is a positive real number. Thus $|\omega| = 1 \Rightarrow \omega = 1$. This proves <u>uniqueness</u>. To prove <u>existence</u> we appeal to the Riemann mapping theorem to get a Riemann mapping h of D onto $\mathbb{D}$. Then $h(z_0) \in D$. We then first form the composition

$$h_1 = \phi_{h(z_0)} \circ h.$$

Then

$$h_1(z_0) = 0.$$

As h_1 is conformal, $h_1'(z_0) \neq 0$. Then $h_1'(z_0) = |h_1'(z_0)|\, e^{i\alpha}$ for some real α. Then

$$h_2 = e^{-i\alpha} h_1$$

gives

$$h_2'(z_0) = |h_1'(z_0)| > 0.$$

QED

REMARK: The disc $\mathbb{D}$ acts here as a "model" for simply connected open sets D. Sometimes it seems more natural to use the upper half space $\mathbb{U}$ as a model. We can simply move between $\mathbb{D}$ and $\mathbb{U}$ by using the Cayley transform. We would then speak of a Riemann mapping of D onto the upper half plane.

We could also make such a Riemann mapping to be unique by requiring (say)

$$f(z_0) = i,$$
$$f'(z_0) > 0.$$

Chapter 8

Introduction To Elliptic Functions

Section A: Doubly Periodic Functions on $\mathbb{C}$

"Introduction," because this is a vast subject, in which there is still great interest in further research.

Of course you know about periodic functions. Here are a few together with their periods:

$$\begin{array}{cc} e^z & 2\pi i \\ \sin z & 2\pi \\ \tan z & \pi \\ \cosh z & -2\pi i \\ \sin z & 6\pi \\ \tan z & -13\pi \\ |\sin z| & \pi \end{array}$$

What you have probably never thought of is functions with two significantly different periods. Such a function defined $f : \mathbb{C} \to \hat{\mathbb{C}}$ would need to satisfy

$$f(z + w_1) = f(z) \quad \text{and} \quad f(z + w_2) = f(z).$$

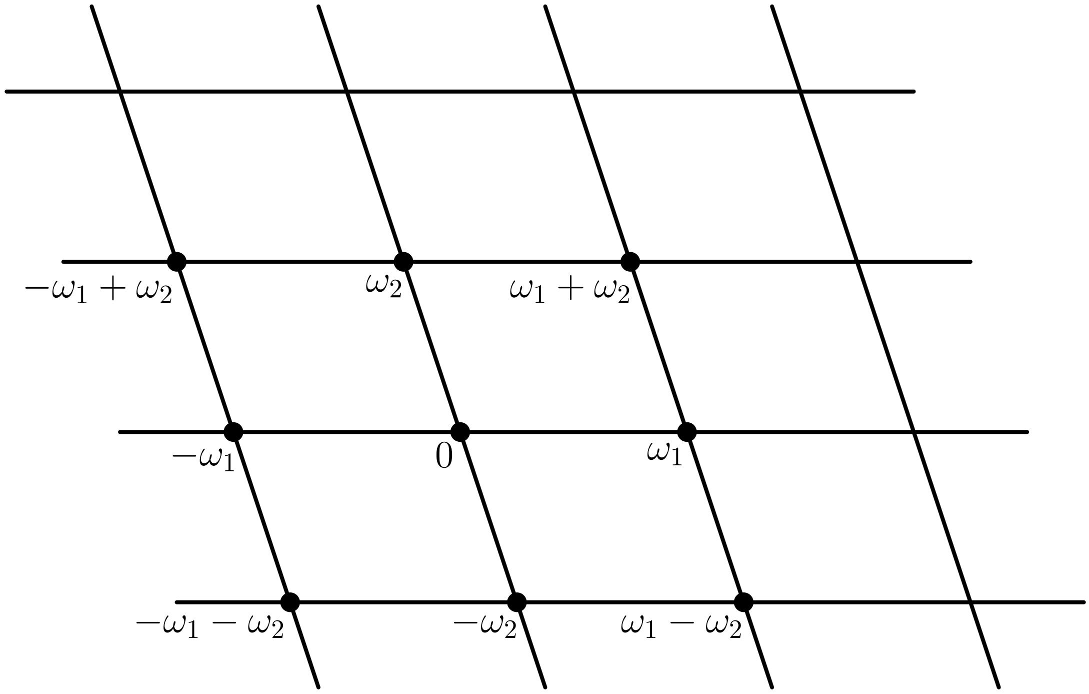
We'll insist that the ratio $\dfrac{\omega_1}{\omega_2}$ is not real. (It's not hard to show that if f is continuous then f is either periodic with one period or constant.)

Since f has periods ω_1 and ω_2, it also satisfies

$$f(z + m\omega_1 + n\omega_2) = f(z) \qquad \text{for all } m, n \in \mathbb{Z}.$$

We say that f is doubly periodic.

It's a good idea to have a geometric description in mind, such as this:

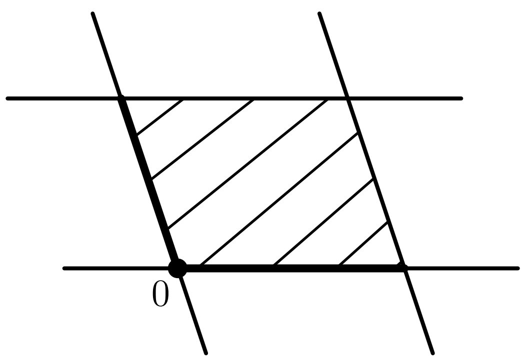


The points $m\omega_1 + n\omega_2$ compose what's called a lattice:

$$\Lambda = \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}.$$

This is a subgroup of $\mathbb{C}$.

Given this lattice there is what's called its fundamental parallelogram:



$$P_0 = \{a\omega_1 + b\omega_2 | 0 \le a < 1, 0 \le b < 1\}.$$

The lattice and P_0 produce a tiling of $\mathbb{C}$,

$$\begin{aligned}\mathbb{C} &= \bigcup_{m,n\in\mathbb{Z}} (m\omega_1 + n\omega_2 + P_0) \\ &= \text{a disjoint union of translates of } P_0.\end{aligned}$$

Finally, notice that all the information about a doubly periodic function is found in any translate $w + P_0$, where $w \in \mathbb{C}$.

Immediately we are confronted with a very easy

OBSERVATION: If f is a doubly periodic function which is an entire holomorphic function, then $f =$ constant.

Proof: f is bounded in $\overline{P_0}$. By periodicity, f is bounded on $\mathbb{C}$. Liouville's theorem thus implies that f is constant.

QED

Section B: Elliptic functions

Because of that observation we must work with doubly periodic functions which are holomorphic except for isolated poles:

DEFINITION: An elliptic function is a nonconstant holomorphic function whose isolated singularities are poles.

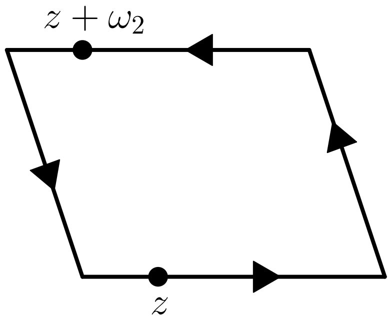
We now derive a few significant properties of elliptic functions, even though we do not yet know even one example!

THEOREM: The number of poles of an elliptic function in any translate of P_0 is atleast 2 (counted according to multiplicity).

Proof: Let P be a translate of P_0 chosen so that none of f's poles are on ∂P. (Easy enough, as the poles of f are isolated.) Then the residue theorem $\Rightarrow$

$$\int_{\partial P} f(z)dz = 2\pi i \sum_{\zeta\in P} \operatorname{Res}(f,\zeta).$$

But periodicity $\Rightarrow \int\limits_{\partial P} f(z)dz = 0.$



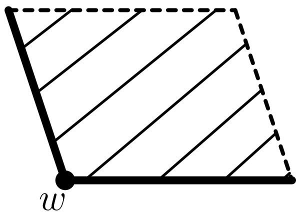
$$f(z+\omega_2)=f(z)$$

$\therefore$ the sum of the residues of f in P equals 0, implying that f cannot have only one simple pole.

QED

DEFINITION: The <u>order</u> of an elliptic function is the number of poles it has in any translate of P_0. The order is therefore an integer ≥ 2.

REMINDER: The translates of P_0 literally look like this:



Thus no two points in P differ by an element of the lattice. Thus the way of counting is accurate.

> **THEOREM**: Let f be an elliptic function of order m. Let P be any translate of the fundamental parallelogram. Then for any point $\zeta \in \hat{\mathbb{C}}$,
>
> $f = \zeta$ exactly m times in P.

<u>Proof</u>: We can look a the elliptic function $f-\zeta$; therefore, we may as well assume $\zeta = 0$. The counting theorem gives (Chapter 5, Section A) for a choice of P with no zeros or poles of f in ∂P,

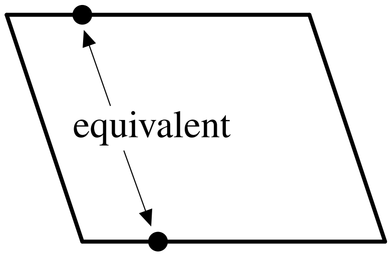
$$\int_{\partial P} \frac{f'}{f} dz = 2\pi i \cdot (\#\text{Zeros in } P - \#\text{Poles in } P)\,,$$

and the periodicity of $\dfrac{f'}{f}$ again implies the integral equals 0.

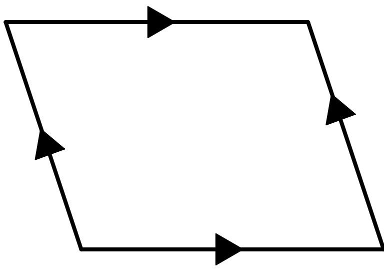
QED

It's a very good idea to think of the lattice Λ in a more geometric way. This is to regard two complex numbers z and z' to be equivalent modulo Λ if and only if $z - z' \in \Lambda$. Then we can look at any translate of P_0 in the following way

P



Then think of "gluing" these two sides of P together, and likewise the other pair of sides:



Thus when we look at a point z of $\mathbb{C}$, we're really looking at all points of $z \in \Lambda$ at the same time. Then we conclude that there are really no "special points" (such as 0) of $\mathbb{C}$ in this context.

Now at least it's time to produce our first example of an elliptic function:

Section C: The Weierstrass $\wp$ function

Knowing that every elliptic function must have order ≥ 2, we seek a function of order 2. We try to make it as simple as possible by requiring it to have exactly one pole of order 2. In doing our calculations we try to make it look as simple as possible by placing this pole at the origin, with Laurent expansion at that point having the form

$$\frac{1}{z^2} + \frac{0}{z} + \dots.$$

An idea leaps to our attention. This function should have Laurent expansion at any $\omega \in \Lambda$ of the form

$$\frac{1}{(z-\omega)^2} + \frac{0}{z-\omega} + \dots.$$

So why not define this function to be

$$\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^2} \quad ?$$

Here's why not: this infinite series diverges.

Remedy: there are two logical things to try, which we now discuss:

1. Form finite sums such as

$$\sum_{\substack{\omega\in\Lambda \\ |\omega|<R}} \frac{1}{(z-\omega)^2}$$

and hope enough cancellation occurs to form a set of principle value

$$\lim_{R\to\infty} \sum_{\substack{\omega\in\Lambda \\ |\omega|<R}} \frac{1}{(z-\omega)^2}.$$

Something like this definitely works, but another remedy seems to be easier:

2. Compute the derivative of the above function, and then let $R \to \infty$ (this series converges!) to get the derivative of the function we want:

$$\sum_{\omega\in\Lambda} \frac{-2}{(z-\omega)^3}.$$

That is definitely an elliptic function of order 3. But then we have to try to integrate this function.

3. Now we modify (2) by defining

$$\wp(z) = \sum_{\omega\in\Lambda} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right);$$

but that's no good since $0 \in \Lambda$ and $-\dfrac{1}{0^2}$ is not allowed. Therefore, we actually define

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \qquad \bigstar$$

We'll now prove that this makes sense. However, it's tricky to prove it's periodic. So we'll instead differentiate $\wp(z)$ term by term to achieve

$$\begin{aligned}\wp'(z) &= -\frac{2}{z^2} - 2\sum_{\substack{\omega\in\Lambda\\ \omega\neq 0}} \frac{1}{(z-\omega)^3}\\ &= -2\sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3}. \qquad \bigstar\bigstar\end{aligned}$$

That's clearly doubly periodic and we'll then be able to prove $\wp(z)$ is also doubly periodic.

We put aside for the time being the analysis of the convergence question.

Theorem: $\wp$ is doubly periodic.

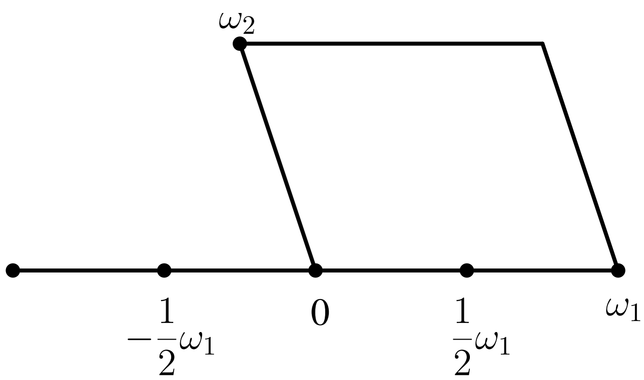
Proof: Define

$$f(z) = \wp(z+\omega_1) - \wp(z).$$

We want to prove $f = 0$. First differentiate:

$$f'(z) = \wp'(z+\omega_1) - \wp'(z) = 0$$

since $\wp'$ is doubly periodic. Therefore, $f(z)$ is constant. Therefore,



$$\wp(z+\omega_1) - \wp(z) = C.$$

We notice from the definition that $\wp(z)$ is <u>even</u>:

$$\begin{aligned}\wp(-z) &= \sum_{\omega\in\Lambda}\left(\frac{1}{(-z-\omega)^2}-\frac{1}{\omega^2}\right)\\ &= \sum_{\omega\in\Lambda}\left(\frac{1}{(z+\omega)^2}-\frac{1}{\omega^2}\right)\\ &= \sum_{\omega\to-\omega}\left(\frac{1}{(z-\omega)^2}-\frac{1}{(-\omega)^2}\right)\\ &= \sum_{\omega\in\Lambda}\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right)\\ &= \wp(z).\end{aligned}$$

Now let $z = -\frac{1}{2}\omega_1$ above:

$$\wp\left(\frac{1}{2}\omega_1\right) - \wp\left(-\frac{1}{2}\omega_1\right) = C,$$

and therefore $C = 0$.

In the same way, we see that $\wp(z+\omega_2) = \wp(z)$.

QED

DISCUSSION OF CONVERGENCE: First, we compare $|m\omega_1 + n\omega_2|$. It's a fact that there exists a positive number C depending only on Λ such that

$$\frac{1}{C} \le \frac{|m\omega_1+n\omega_2|}{\sqrt{m^2+n^2}} \le C$$

for all $\omega = m\omega_1 + n\omega_2 \in \Lambda$, $\omega \ne 0$. The upper bound is clear, as

$$\begin{aligned}|m\omega_1+n\omega_2| &\le |m\omega_1| + |n\omega_2|\\ &\le \max\left(|\omega_1|,|\omega_2|\right)\left(|m|+|n|\right)\\ &\le \max\left(|\omega_1|,|\omega_2|\right)\sqrt{2}\sqrt{m^2+n^2}.\end{aligned}$$

The lower bound is a little trickier. We rely on the hypothesis that $\frac{\omega_1}{\omega_2}$ is not real. Therefore

$$\frac{\omega_1}{\omega_2} = \left|\frac{\omega_1}{\omega_2}\right| e^{i\alpha}$$

and $\mathrm{Re}\left(e^{i\alpha}\right) = \cos\alpha$, $|\cos\alpha| < 1$. Then

$$\begin{aligned}
|m\omega_1 + n\omega_2|^2 &= |m\omega_1|^2 + 2\,\mathrm{Re}\left(m\omega_1\overline{n\omega_2}\right) + |n\omega_2|^2 \\
&= |m\omega_1|^2 + 2mn\,\mathrm{Re}\left(\frac{\omega_1}{\omega_2}|\omega_2|^2\right) + |n\omega_2|^2 \\
&= |m\omega_1|^2 + 2mn|\omega_2|^2\left|\frac{\omega_1}{\omega_2}\right|\cos\alpha + |n\omega_2|^2 \\
&\geq |m\omega_1|^2 - 2|\cos\alpha|\;|mn\omega_1\omega_2| + |n\omega_2|^2.
\end{aligned}$$

Now the simple inequality for real numbers,

$$2|xy| \leq x^2 + y^2,$$

implies

$$\begin{aligned}
|m\omega_1 + n\omega_2|^2 &\geq |m\omega_1|^2 - |\cos\alpha|\left(|m\omega_1|^2 + |n\omega_2|^2\right) + |n\omega_1|^2 \\
&= \left(1 - |\cos\alpha|\right)\left(|m\omega_1|^2 + |n\omega_2|^2\right) \\
&\geq \left(1 - |\cos\alpha|\right)\left(\min\left(|\omega_1|, |\omega_2|\right)\right)^2\left(m^2 + n^2\right),
\end{aligned}$$

so that

$$|m\omega_1 + n\omega_2| \geq \sqrt{1 - |\cos x|}\,\min\left(|\omega_1|, |\omega_2|\right)\sqrt{m^2 + n^2}.$$

Therefore, the convergence of

$$\sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{|w|^t}$$

is equivalent to the convergence of

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{\left(\sqrt{m^2+n^2}\right)^t}.$$

Finally, the convergence of this latter double sum is equivalent to the convergence of the double integral

$$\iint_{x^2+y^2\geq 1} \frac{1}{\left(\sqrt{x^2+y^2}\right)^t}dxdy.$$

This can be determined by using polar coordinates in $\mathbb{R}^2$: the double integral equals

$$\int_0^{2\pi}\int_1^{\infty} \frac{rdrd\theta}{r^t} = 2\pi\int_1^{\infty}\frac{dr}{r^{\theta-1}}$$

and this is finite $\Leftrightarrow t > 2$. We conclude that

$$\sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{|w|^t} < \infty \Leftrightarrow t > 2.$$

As a result, the series defining $\wp'(z)$ is absolutely convergent since the corresponding $t = 3$. And the terms in the series defining $\wp(z)$ have modulus

$$\begin{aligned}
\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| &= \left|\frac{\omega^2 - (z-\omega)^2}{(z-\omega)^2\omega^2}\right| \\
&= \left|\frac{2z\omega - z^2}{(z-\omega)^2\omega^2}\right| \\
&\le \frac{|2z\omega| + |z|^2}{|\omega|^2\left(|\omega| - |z|\right)^2},
\end{aligned}$$

and if $|\omega| > 2|z|$, this is

$$\begin{aligned}
&\le \frac{|2z||\omega| + |z|^2}{|\omega|^2\left(\frac{1}{2}|\omega|\right)^2} \\
&= 4\frac{|2z||\omega| + \frac{1}{2}|z||\omega|}{|\omega|^4} \\
&= \frac{10|z|}{|\omega|^3}.
\end{aligned}$$

Again, absolute convergence for all $z \notin \Lambda$.

This finishes our discussion of convergence. The result is

$$\boxed{\begin{array}{l}\wp \text{ is an even elliptic function of order } 2,\\ \wp' \text{ is an odd function of order } 3.\end{array}}$$

Section D: Basic properties of $\wp$

1. $\wp'$ has order 3. It has a <u>triple pole</u> at 0. We can also locate its zeros. For $\wp'$ is odd:

$$\wp'(-z) = -\wp'(z).$$

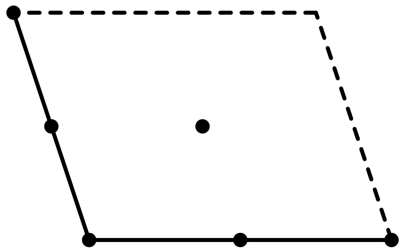
$\therefore$

$$\wp'\left(\frac{1}{2}\omega_1\right) \overset{\text{periodic}}{=} \wp'\left(-\frac{1}{2}\omega_2\right) \overset{\text{odd}}{=} -\wp'\left(\frac{1}{2}\omega_1\right).$$

$\therefore$

$$\wp'\left(\frac{1}{2}\omega_1\right) = 0.$$

Likewise for $\frac{1}{2}\omega_2$ and $\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2$:



2. At a zero of $\wp'$ we have a double value of $\wp$. These are often denoted this way:

$$\begin{aligned} e_1 &= \wp\left(\frac{1}{2}\omega_1\right), \\ e_2 &= \wp\left(\frac{1}{2}\omega_2\right), \\ e_3 &= \wp\left(\frac{\omega_1 + \omega_2}{2}\right). \end{aligned}$$

3. **$\wp$ satisfies a nonlinear ordinary differential equation of order 1**

This is a startling result!

We begin by defining the elliptic function

$$f(z) = (\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3).$$

$f(z) = 0$ only at points in Λ equivalent to $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, and $\frac{\omega_1 + \omega_2}{2}$.

Each of these is a <u>double</u> zero, as we've just noticed. Thus f is an elliptic function of order 6.

Also $\wp'(z)^2$ has double zeros at these points, and only there. Therefore, the doubly periodic function

$$\frac{\wp'(z)^2}{f(z)}$$

has no zeros at all! (It also has no poles: $(\wp')^2$ has no poles of order 6 at points of Λ, and f also has no poles of order 6 at the same points.) The quotient is therefore constant: it's never 0, so it would have to be elliptic of "order 0." Also, it is entire holomorphic and therefore constant by Liouville's theorem.

We can calculate this constant by examining the function near 0:

$$\begin{aligned}\wp(z) &\sim \frac{1}{z^2};\\ f(z) &\sim \frac{1}{z^6};\\ \wp'(z) &\sim -\frac{2}{z^3};\\ \wp'(z)^2 &\sim \frac{4}{z^6}.\end{aligned}$$

$\therefore$

$$\frac{(\wp')^2}{f} \sim 4 \quad \text{near } 0.$$

$\therefore$

$$\boxed{\wp'(z)^2 = 4\left(\wp(z)-e_1\right)\left(\wp(z)-e_2\right)\left(\wp(z)-e_3\right).}$$

4. **General elliptic functions for the lattice Λ**

Suppose f is an elliptic function of order M. We are going to represent it by using $\wp$ itself.

First, suppose f is also even: $f(-z) = f(z)$.

Then consider a zero of $f : f(a) = 0$. Then $f(-a) = 0$ as well. There are two cases: (1) a and $-a$ are distinct modulo Λ. I.e., $2a \notin \Lambda$. In this case, if a is of order k, then $-a$ is also a zero of order k. Then a and $-a$ are equal modulo Λ. I.e., $2a \in \Lambda$. Typically there are three cases: $a = \frac{\omega_1}{2}$ or $\frac{\omega_2}{2}$ or $\frac{\omega_1+\omega_2}{2}$. Then the zero a is a zero of even order.

Therefore, the zeros of f can be written $a_1, \ldots, a_M$, and this list accounts for all the zeros of f according to their multiplicities. And M is even.

Then we notice that each a_j yields a zero of $\wp(z) - \wp(a_j)$, and the points $\frac{\omega_1}{2}$, $\frac{\omega_2}{2}$, and $\frac{\omega_1 + \omega_2}{2}$ are also double zeros for $\wp(z) - \wp\left(\frac{\omega_1}{2}\right)$ etc.

Thus, the function

$$\prod_{j=1}^{M} \left(\wp(z) - \wp(a_j)\right)$$

has exactly the same zero structure as $f(z)$.

In the same way, if the poles of f are written as $b_1, \ldots, b_M$, then

$$\prod_{j=1}^{M} \frac{1}{\wp(z) - \wp(b_j)}$$

has the same pole structure as $f(z)$.

Therefore,

$$\prod_{j=1}^{M} \frac{\wp(z) - \wp(a_j)}{\wp(z) - \wp(b_j)}$$

has the identical structure for its zeros and poles as $f(z)$. This shows that $f(z)$ divided by that product has no zeros and no poles. Being doubly periodic, Liouville's theorem implies that it is constant! Therefore

$$f(z) = \text{constant} \cdot \prod_{j=1}^{M} \frac{\wp(z) - \wp(a_j)}{\wp(z) - \wp(b_j)}.$$

<u>Second</u>, suppose f is odd: $f(-z) = -f(z)$. Then, since $\wp'$ is odd,

$$\frac{f(z)}{\wp'(z)}$$

is an even elliptic function. Therefore, it also has a representation of the form

$$\prod_{j=1}^{N} \frac{\wp(z) - \wp(c_j)}{\wp(z) - \wp(d_j)}.$$

THEOREM: Every elliptic function for the lattice Λ is a rational function of $\wp$ and $\wp'$. In fact, if f is elliptic, then

$$f(z) = g(z) + \wp'(z)h(z),$$

where g and h are rational functions of $\wp$.

Proof: The equation

$$f(z) = \frac{f(z)+f(-z)}{2} + \frac{f(z)-f(-z)}{2}$$

presents f as an even elliptic function plus an odd elliptic function.

QED

5. **The Laurent expansion of $\wp$ centered at 0**

Before writing this expansion we define the following lattice sums:

$$G_n = G_n(\Lambda) = \sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{\omega^n}, \quad n = 3, 4, 5, \ldots.$$

These series converge absolutely since $n \geq 3$. And since $-\Lambda = \Lambda$, it follows that

$$G_n = (-1)^n G_n.$$

$\therefore G_n = 0$ for all odd n.

We'll immediately need the first two:

$$G_4 = \sum_{\Lambda\setminus\{0\}} \frac{1}{\omega^4},$$
$$G_6 = \sum_{\Lambda\setminus\{0\}} \frac{1}{\omega^6}.$$

To work out the Laurent series we first write the geometric series

$$\begin{aligned} -\frac{1}{z-\omega} &= \frac{1}{\omega - z} \\ &= \frac{1}{\omega}\frac{1}{1-\frac{z}{\omega}} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{\omega^{n+1}} \end{aligned}$$

for small z. Differentiate:

$$\begin{aligned} \frac{1}{(z-\omega)^2} &= \sum_{n=1}^{\infty} \frac{nz^{n-1}}{\omega^{n+1}}; \\ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} &= \sum_{n=1}^{\infty} \frac{(n+1)z^n}{\omega^{n+2}}. \end{aligned}$$

Thus

$$\begin{aligned}
\wp &= \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) \\
&= \frac{1}{z^2} + \sum_{\omega \neq 0} \sum_{n=1}^{\infty} \frac{(n+1)z^n}{\omega^{n+2}} \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty} \sum_{\omega \neq 0} \frac{(n+1)z^n}{\omega^{n+2}} \\
&= \frac{1}{z^2} + \sum_{n=1}^{\infty} (n+1) G_{n+2} z^n.
\end{aligned}$$

Drop the odd terms:

Laurent series:

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2} z^{2n}, \quad \text{near } z = 0.$$

Next we give another method for producing the differential equation satisfied by $\wp(z)$.