SOME APPLICATIONS OF COMBINATORIAL DIFFERENTIAL TOPOLOGY

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0. Introduction

The title of this paper is perhaps a bit confusing, since it is not at all clear what is meant by the term "combinatorial differential topology". We use this expression to refer to the application of the standard concepts of differential topology, such as vector fields and their corresponding flows, to the study of combinatorial spaces such as simplicial complexes. This paper will be a survey of some of the author's work in this direction (see [11-18]). In order to aid us in our discussion, we will take as the unifying idea the notion of a combinatorial dynamical system.

The notion of a *dynamical system* has become one of the grand unifying ideas in all of science over the last century. The study of combinatorial dynamical systems is also rather classical. Indeed, the subject of *symbolic dynamics* is precisely that (see, e.g. [43], or [54] for a look at some of the numerous directions in which the theory has been applied). The main goal in this paper is to present a class of very topological combinatorial dynamical systems, the study of which allows one to recreate, in the context of combinatorial spaces, many phenomena which one might normally think of as being associated only to smooth manifolds.

We will be exploring a number of applications of these ideas to topology, geometry, mathematical physics and computer science. We will also be presenting some potential future applications which at this point are just "fantasy." While much of this conference has focussed on combinatorial aspects of "quantum mathematics", most

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of this paper will be concerned with combinatorial aspects of "classical mathematics". However, the examples discussed in sections 5 and 8 are closely related to issues discussed by others.

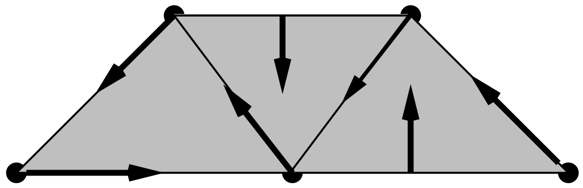
1. Discrete Vector Fields and Dynamical Systems

In this section we introduce what will be the central notions of this paper, the idea of a discrete vector field on a cell complex, and the dynamics of the corresponding flow. Most of the content of this section appeared first in [17].

Let K be a finite regular CW complex. (See Chapter III of [38] for definitions). The regularity of K is merely for technical convenience. If the reader is unfamilaiar with such general cell complexes, then one may think of the special case of polyhedra or simplicial complexes.

Definition 1.1. *A discrete vector field V is a set of pairs of cells of the form $\{\alpha \prec \beta\}$, where the notation $\alpha \prec \beta$, or $\beta \succ \sigma$, indicates that α is a "maximal face" of β (i.e. $\alpha \subset \overline{\beta}$ and $\dim(\alpha) + 1 = \dim(\beta)$) , and such that no cell is in more than one pair.*

It is useful to picture a discrete vector field by drawing arrows on K (see Figure 1.2). If $\{\alpha \prec \beta\}$ is a pair in V, then we draw an arrow from α to β (i.e. an arrow whose tail is in α and whose head is in β).



A discrete vector field.

Figure 1.2.

Since no cell is in more than one pair in V, every cell α of K satisfies exactly one of the following conditions:

(i) α is the tail of exactly one arrow

(ii) α is the tail of exactly one arrow

(iii) α is neither the head nor the tail of any arrow.

A *zero* (or *rest point* or *critical point*) of V is a cell which satisfies condition (iii), i.e. a cell which is not in any pair of V.

We will show in the course of this paper that using this notion of a discrete vector field one can recover in a combinatorial setting many aspects of smooth vector fields and their dynamics. At the start, however, we should point out one essential difference between the smooth and combinatorial categories. There is an obvious involution on the space of smooth vector fields on a smooth manifold, sending any vector field V to $-V$. On the other hand, in the general combinatorial setting we have presented, if V is a discrete vector field on a cell complex K then there is no notion of "$-V$". One can recover this basic operation if K is a combinatorial manifold. In this case $-V$ does exist as a discrete vector field on the dual cell complex K^*. Namely, $-V$ is the collection of all pairs $\{\beta^* \prec \alpha^*\}$ such that $\{\alpha \prec \beta\}$ is a pair in V (where α^* denotes the cell in K^* which is dual to the cell α in K, and β^* is defined similarly.) This is closely related to the fact that the homology of a combinatorial manifold satisfies Poincaré duality, but there is no such symmetry in the homology of a general cell complex. We return to this issue again in section 4 and in part III of section 7.

Our goal is to investigate the topological properties of the dynamics of V. We can begin by observing the following simple combinatorial analogue of the Poincaré-Hopf Theorem.

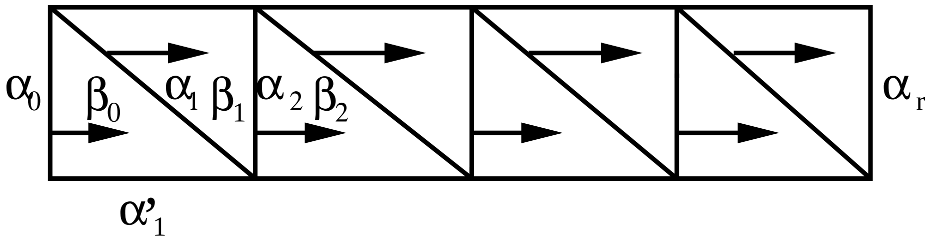
Theorem 1.3. *The Euler characteristic of K, $\chi(K)$, is equal to $\sum_{zeros\ \alpha}(-1)^{dim(\alpha)}$.*

This theorem follows from the simple observation that the cells which are not zeros are paired, with each cell paired with a cell whose dimension differs by one, and hence has opposite parity.

To proceed further, let us begin our discussion of the dynamics of V. We define a $V-$*path* to be a sequence γ of simplices

$$\gamma = \alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)}, \ldots, \beta_{r-1}^{(p+1)}, \alpha_r^{(p)} \tag{1.1}$$

such that for each $i = 1, \dots r$, $\{\alpha_i \prec \beta_i\}$ is in V, and $\beta_i \succ \alpha_{i+1} \neq \alpha_i$. We draw such a path in Figure 1.4. We say a V-path γ as in (1.1) is a (non-trivial) closed path if $r > 0$ and $\alpha_0 = \alpha_r$. We say that γ has length r and index p.



A V-path from α_0 to α_r.

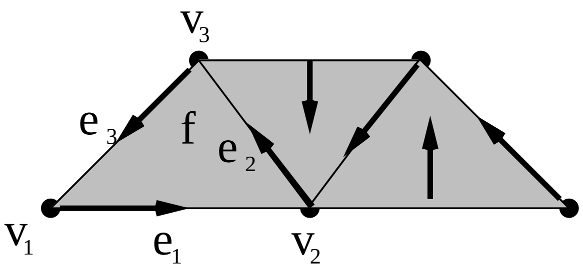
Figure 1.4.

Note that a V-path is not determined by its starting cell. In Figure 1.4, the cells $\alpha_0, \beta_0, \alpha_1'$ also form a V-path.

The main idea of this section is that the topology of K is "carried" by the closed orbits and rest points of V. Let us now make this precise. Define the *chain-recurrent set* of V to be the set of cells which are either rest points of V or are contained in some non-trivial closed path of V. Here we see an important simplification that occurs when moving from the smooth to the combinatorial categories. In the smooth setting, the chain recurrent set also includes points which "almost" lie on closed orbits. In our discrete setting, there are no "almost closed orbits".

We partition the chain recurrent set into *basic sets*, where α_1 and α_2 are in the same basic set if and only if there is a non-trivial closed V-path which contains both α_1 and α_2. For example, in Figure 1.5 we again show the discrete vector field of Figure 1.2, but this time we have labelled the cells in the chain recurrent set. For this discrete vector field the chain-recurrent set $\mathcal{R}$ consists of two basic sets.

$$\mathcal{R} = \{v_1, e_1, v_2, e_2, v_3, e_3\} \cup \{f\}$$



The chain recurrent set

Figure 1.5.

In general, as in this example, basic sets come in 2 types. If Λ is a basic set then either

$$\Lambda = \{\alpha^{(p)}\} \quad \text{where } \alpha \text{ is a zero of } V \tag{1.2}$$

or

$$\Lambda = \{\alpha_i^{(p)}, \beta_j^{(p+1)}\} \quad \begin{array}{l} i = 1, 2 \dots \\ j = 1, 2, \dots \end{array} \tag{1.3}$$

is the union of closed paths of index p. For any basic set Λ, let

$$\dot{\Lambda} = \overline{\Lambda} - \Lambda.$$

We observe that if Λ consists of a single zero of V as in (1.2), then for any coefficient field $\mathbb{F}$,

$$H_i(\overline{\Lambda}, \dot{\Lambda}, \mathbb{F}) \cong \begin{cases} \mathbb{F} & \text{if } i = p \\ 0 & \text{if } i \neq p \end{cases}$$

While if Λ is a union of closed orbits as in (1.3), then our answer is not as precise, but we can say that

$$H_i(\overline{\Lambda}, \dot{\Lambda}, \mathbb{F}) \cong 0 \quad \text{if } i \neq p, p+1$$

We are now prepared to state the Morse inequalities for a discrete vector field. Define the Morse numbers m_i by

$$m_i = \sum_{\text{basic sets}\Lambda} \dim H_i(\overline{\Lambda}, \dot{\Lambda}, \mathbb{F})$$

where $\mathbb{F}$ is any fixed coefficient field. Let b_i denote the Betti numbers of K

$$b_i = \dim H_i(K, \mathbb{F}).$$

Theorem 1.6 (Strong Morse Inequalities)**.**

$$\begin{aligned} m_0 &\geq b_0 \\ m_1 - m_0 &\geq b_1 - b_0 \\ m_2 - m_1 + m_0 &\geq b_2 - b_1 + b_0 \\ &\cdots \\ m_n - m_{n-1} + \cdots \pm m_0 &= b_n - b_{n-1} + \cdots \pm b_0 = \chi(K) \end{aligned}$$

where $n = dim(K)$*.*

The Strong Morse Inequalities simply the following weaker, but simpler inequalities.

Corollary 1.7 (Weak Morse Inequalities)**.**

$$m_i \geq b_i \quad \textit{for each } i$$

This is precisely the sort of result we were looking for. The complex K can have no more topology than that carried by the closed orbits and rest points of V. In the next section we indicate how one proves these theorems.

2. The Proof of the Morse Inequalities

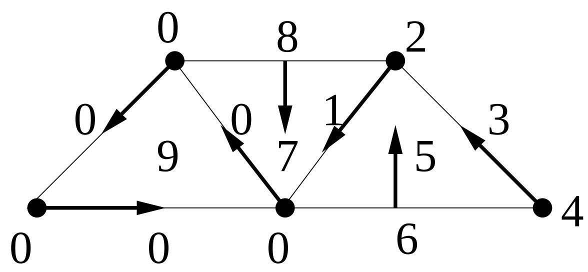
The main ingredient in the proof of the Morse inequalities is the idea of *Lyapunov function.* A Lyapunov function for V is a real-valued function on the set of cells of K with the property that it is monotone non-increasing along all V paths, and it is strictly decreasing off the chain recurrent set. More precisely,

Definition 2.1. *A function*

$$f : \{cells\ of\ K\} \rightarrow \mathbb{R}$$

is a Lyapunov function for V if for any V path $\alpha_0, \beta_0, \alpha_1$, we have $f(\alpha_0) \geq f(\beta_0) \geq f(\alpha_1)$ and $f(\alpha_0) = f(\alpha_1)$ if and only if α_0 and α_1 are in the same basic set (which implies that β_0 is also in the same basic set).

In Figure 2.2 we illustrate a Lyapunov function for the discrete vector field shown in Figure 1.2.



A Lyapunov Function for the discrete vector field shown in Figure 1.2

Figure 2.2.

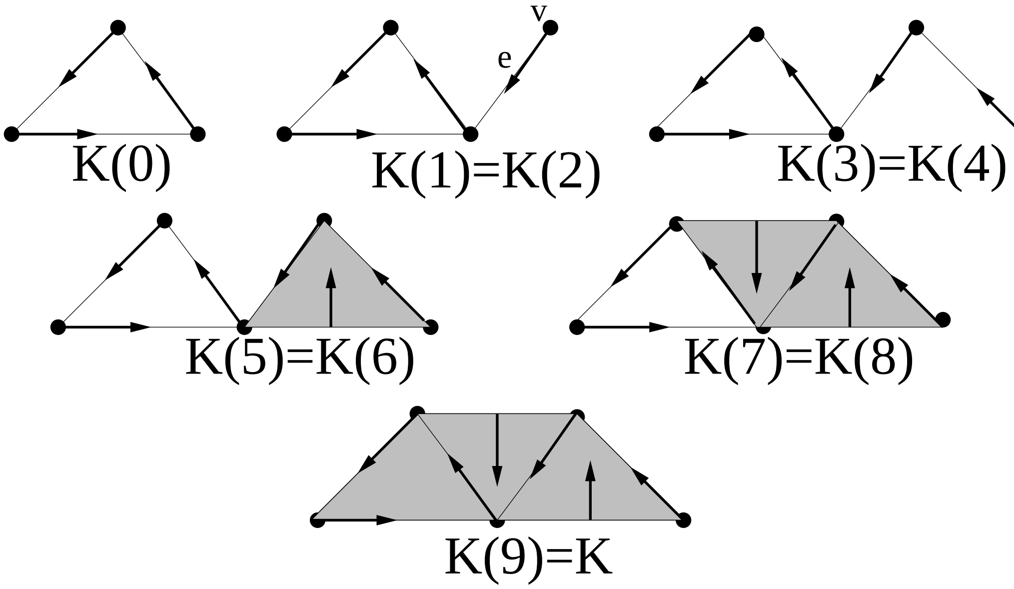
In fact, we have the following theorem (see Theorem 2.4 of [17]).

Theorem 2.3. *For every regular cell complex K and every discrete vector field V on K, there is a Lyapunov function for V.*

We can now indicate the proof of the Morse inequalities. The idea is to build K, beginning with the empty set, by adding the cells in the order prescribed by the Lyapunov function f. More precisely, for any real number c, define the level subcomplex $K(c)$ to be the union of all of the cells β of K satisfying $f(\beta) \leq c$, along with all of their faces. That is,

$$K(c) = \cup_{f(\beta)\leq c} \cup_{\alpha \leq \beta} \alpha.$$

In Figure 2.4 we illustrate all of the level subcomplexes of the example shown in Figure 2.2.



The level subcomplexes of the Lyapunov function shown in Figure 2.2

Figure 2.4.

Note that the discrete vector field V on K induces a discrete vector field on each of these level subcomplexes. Namely, for any pair of cells $\{\alpha \prec \beta\}$ in V, and any level subcomplex K', it is easy to see that α is in K' if and only if β is. We draw an arrow from α to β in any subcomplex K' in which α and β appear.

We can now prove the Morse inequalities by induction, showing that they are true for each simplicial complex appearing in this list with respect to the induced discrete vector fields (beginning the induction with $K(-1) = \emptyset$, for which the inequalities are trivially valid). One must show that if the Morse inequalities are true at one step then they are necessarily true at the next step. There are two cases, determined by whether or not we are adding any cells in the chain-recurrent set (we are assuming, merely for simplicity, that the Lyapunov function is as 1-1 as possible, i.e. with the exception of being constant on each basic set, and this can always be arranged). Consider first the case when the added cells are not chain-recurrent, see for example the transition from $K(0)$ to $K(1)$ in Figure 2.4. In this case the Morse numbers of the complex do not change, so we must see that the Betti numbers also stay the same. The edge $e = f^{-1}(1)$ is not in the chain recurrent set, and as a result, it has an endpoint v which is assigned a larger value. Thus, when we move from $K(0)$ to $K(1)$ and add the edge e, we must also add v. As a consequence, in the complex $K(1)$ the vertex v is a "free face" of e (i.e. a face which is not the face of any other simplex). We can deformation retract $K(1)$ onto $K(0)$ by "pushing in" the edge e starting at the vertex v. This special sort of deformation retract is called a "collapse". Not only can we say that the homotopy type of the level subcomplex does not change when a cell which is not in the chain-recurrent set is added (so that, in particular, the Betti numbers do not change), but the new complex collapses onto the old complex. We note that the equivalence relation generated by collapsing is called *simple-homotopy*.

Now consider the case in which we add a basic set, e.g. in the transition from $K(8)$ to $K(9)$. In this case one can check that the entire boundary of the basic set is in the original complex, and the new complex is built by simply gluing the basic set in along its boundary. The weak Morse inequalities follow from the fact that under this operation, the Betti numbers can grow by at most the homology of the basic set relative to its boundary. The Stong Morse Inequalities follow from an examination of the long exact sequence corresponding to the pair (in this case) $(K(9), K(8))$, and the excision property. The details can be found on page 21 of [41].

3. A Refinement: Counting Closed Orbits According to their Homotopy Class

In this section we discuss one way in which to make our counting of closed orbits more precise. Namely, we can count the closed orbits according to their free homotopy class. We will also indicate some possible connections to Seiberg-Witten theory, and symplectic geometry. However, these ideas are, at the moment, very much in the realm of fantasy. As the content of this section is somewhat independent of the rest of the paper, our presentation will be brief, and some terms will be left undefined. For details, see [17] and [20].

Let $\rho : \pi_1(K) \to O(n)$ be a representation. Consider the zeta function

$$\zeta_V(z,\rho)=\exp\left[\sum_{k=1}^{\infty}\frac{z^k}{k}\sum_{p=0}^{\dim K}(-1)^p\sum_{\gamma\in\mathcal{C}_k^p}m(\gamma)trace(\rho(\gamma))\right]$$

where $\mathcal{C}_k^p$ = closed V paths of length k and index p. The multiplicity $m(\gamma)$ of a closed V-path γ is $\pm$, depending on whether the orientation of a p-simplex in γ is preserved or reversed as one "slides" it along γ.

Theorem 3.1. *For any fixed K, V and ρ, the function $\zeta_V(z,\rho)$ has a positive radius of convergence and can be analytically continued to a meromorphic function on all of $\mathbb{C}$.*

Our main result is that a special value of the zeta function is related to the Reidemeister torsion $T(K,\rho)$ of K, a topological invariant of K. This is a slight complication, as one must also include a term built from the zeros of V.

Theorem 3.2. *If $H_*(\overline{\Lambda},\dot{\Lambda},\rho)\neq 0$ for each non-trivial basic set Λ, then $\zeta_V(z,\rho)$ is analytic at $z=1$. If, in addition, V has no zeros then*

$$T(K,\rho)=\zeta_V(1,\rho).$$

In the general case in which V may have zeros, then

$$T(K,\rho)=\mathrm{tor}(\mathcal{M}^*(\rho))\zeta_V(1,\rho).$$

where $\mathcal{M}^*(\rho)$ indicates the Morse complex of K (a differential complex built from the zeros of V and the gradient paths that connect them - see section7), and $\mathrm{tor}(\mathcal{M}^*(\rho))$ *means the torsion of this complex.*

The main ingredient in the proof of Theorem 3.1 will be presented in section 10. One proof of Theorem 3.2, based on the Witten deformation of the Laplace operator, will be discussed in section 8. Theorem 3.2 should be compared to the theorems of Fried [22],[23], Hutchings-Lee [29] [30], and Pazhitnov [45], where similar results are proved for the closed orbits of classes of smooth vector fields.

In the case of a smooth 3-manifold and a vector field dual to a closed 1-form, the identity takes on particular significance, since in this case the result is conjecturally related to the Seiberg-Witten invariant of the product of the 3-manifold and a circle (see [29] for a precise statement of these ideas). A goal of much ongoing research in gauge theory is a combinatorial formula for Seiberg-Witten invariants. Perhaps our Theorem 3.2 can be useful for understanding the combinatorial nature of Seiberg-Witten at least in the case of a 4-manifold of the form $M^3 \times S^1$.

Taubes has demonstrated a very deep connection between Seiberg-Witten theory and the symplectic topology of a 4-manifold [53]. Perhaps one could begin an investigation of combinatorial Seiberg-Witten theory by developing a theory of combinatorial symplectic geometry (which would, of course, be extremely interesting in its own right). It is not clear to me what one might expect of such a theory.

Perhaps the hint of a starting point of such a theory can be found in [19], in which we present a combinatorial version of Novikov's generalization of Morse theory. Given a smooth manifold and a closed 1-form on the manifold, Novikov shows that one can relate the zeros of the 1-form to certain generalized Betti numbers. Standard Morse theory results from the case in which the 1-form is exact. In [19],we present a discrete version of Novikov's theory. In particular, we present a notion of a combinatorial differential form on a CW complex which is, in some sense, dual to the notion of discrete vector field which we have been discussing. Perhaps this notion of combinatorial differential form can serve as a starting point for the investigation of other facets of "combinatorial differential topology and geometry".

Lastly, a question: Which smooth vector fields can be "triangulated"? That is, given a smooth manifold and a smooth vector field, when can the manifold be triangulated so that there exists a discrete vector field with the "same" dynamics. This can be done for gradient vector fields, and somewhat more generally (as will be shown in a later paper) but it is no doubt true for larger classes of vector fields. Dennis Sullivan has suggested that there might be a close relationship between a vector field being stable and being triangulable, and there seems to be much truth in this.

4. Discrete Morse Theory

In this section, we begin our examination of an important special case of our discussion. Namely, the case of discrete vector fields with no closed orbits. We refer to the theory in this case as *discrete Morse theory* ([15]).

Definition 4.1. *We say that a discrete vector field V is a gradient vector field if the are no closed V-paths. In this case, we call a Lyapunov function for V a Morse function.*

Let us recall the main points in our proof of the Morse inequalities in section 2. We showed that when passing from one level subcomplex to the next, if what is added is not in the chain recurrent set then the larger subcomplex collapses onto the smaller. When we add a basic set it is glued in along its entire boundary. In the case of a gradient vector field, each basic set consists of a single cell, a zero of V. From this we can easily deduce the main theorem of discrete Morse theory.

Theorem 4.2. *Let V be a gradient vector field on a finite regular CW complex K. Then K is (simple-)homotopy equivalent to a CW complex with exactly one cell of dimension p for each zero of V of dimension p.*

An important special case is when the gradient vector field has exactly one zero. We state this case separately for future reference.

Theorem 4.3. *Let K be a regular CW complex with a gradient vector field with exactly one critical point. Then the critical point must be a vertex, and K collapses to that vertex.*

There have been other adaptations of Morse theory that are suitable for the study of combinatorial spaces. For example, a Morse theory of piecewise linear functions appears in [36] and the very powerful "Stratified Morse Theory" was developed by Goresky and MacPherson [27],[28]. These theories, especially the latter, have been successfully applied to prove some very striking results.

One strength of the theory we present here is, as we will indicate in this paper, we are able to recreate in the combinatroial setting just about every facet of smooth

Morse theory. Moreover, as a result of its purely discrete nature, it can be applied to any simplicial complex, not just those with some special structure. There is a growing list of references in which Theorem 4.2 has been used to investigate the topology of some explicit, but complicated, simplicial complexes which arise in combinatorics. See, for example, the papers [2],[3], [9], [12], [34], [35], [37], [49]. These papers contain some beautiful mathematics in which the authors construct "by hand" gradient vector fields on the complexes under consideration.

So far, all of our results have been stated for general cell complexes. If the cell complexes under consideration have more structure, then we have additional tools at our disposal. For example, recall from section 1 that if K is a combinatorial manifold and V is a discrete vector field on K, then $-V$ is a discrete vector field on K^*. It is rather easy to see that $-V$ is a discrete gradient vector field if and only if V is. From this observation (along the Morse complex presented in part I of section 7, one can give a proof of Poincaré duality for combinatorial manifolds. Moreover, for combinatorial manifolds one can sometimes strengthen the conclusion of Theorem 4.2 from a statement about homotopy equivalence to a statement about combinatorial equivalence. This idea rests on some beautiful work of J.H.C. Whitehead [55]. We present here a special case of the main theorem of [55].

Theorem 4.4. *Let K be a combinatorial manifold with boundary which simplicially collapses to a vertex. Then K is a combinatorial ball.*

As one example of how this theorem map be applied, suppose that K is a cell complex with a gradient vector field with precisely two critical cells. Then Theorem 4.2 implies that K is homotopy equivalent to a sphere. Whitehead's theorem implies the following refinement.

Theorem 4.5. *Let K be a combinatorial manifold with a gradient vector field with exactly two critical points. Then K is a combinatorial sphere.*

The proof of this corollary is quite simple (given Whitehead's wonderful theorem). Suppose that K is an n-dimensional combinatorial manifold with a gradient vector field with exactly two critical points. Then it is easy to see that one of the critical cells must be an n-cell and the other a 0-cell (if $n = 0$ then they are both 0-cells). Let α denote the critical n-cell. Then $K - \alpha$ has only a single critical cell, so by Theorem 4.3 it collapses. By Whitehead's theorem, $K - \alpha$ must be a combinatorial ball, which implies that K is a combinatorial sphere. □

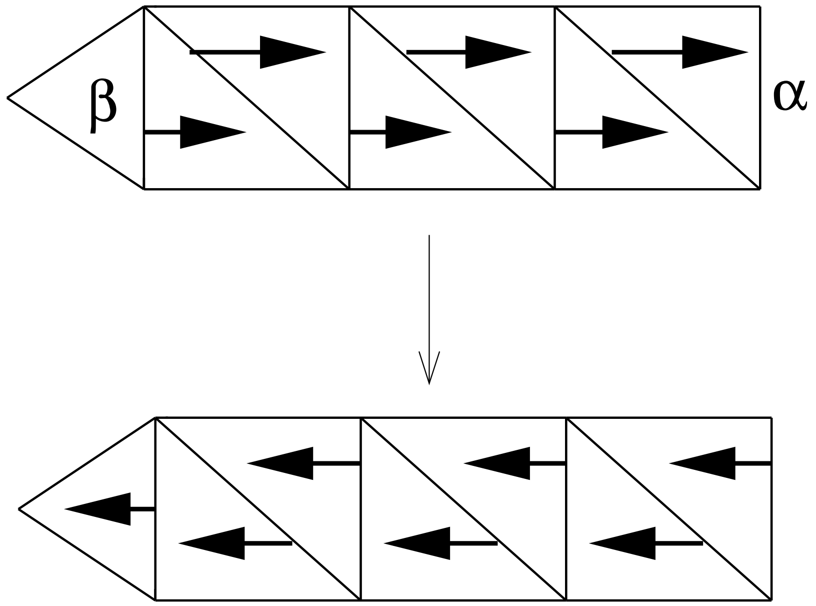
Given this sphere theorem, it is natural to wonder if this discrete Morse theory can be use to prove the Poincaré Conjecture. In fact, one can use the discrete Morse theory to give a proof of the higher dimensional Poincaré conjecture in the PL category (and its far-reaching generalization the PL s-cobordism theorem) exactly along the lines of the Morse theoretic proof of the h-cobordism theorem presented in [41] in the smooth category.

Very loosely speaking, to prove the Poincaré conjecture in dimensions higher than 4 (following Milnor's presentation in [41]) one begins with a combinatorial n-manifold M, $n \geq 5$, which is a homotopy n-sphere. Endow M with a gradient vector field V (in the combinatorial setting one may begin with the empty vector field consisting of no pairs at all). The goal is to show that it is possible to "cancel out" the critical points of V until one is left with a gradient vector field with exactly two critical simplices. It must then be a combinatorial sphere, by Theorem 4.5. The following is a combinatorial version of the "First Cancellation Theorem" of [41] (see Lemma 2.1).

Theorem 4.6. *Let V be a gradient vector field. Suppose $\alpha^{(p)}$ and $\beta^{(p+1)}$ are zeros of V and there is exactly one V-path from the boundary of β to α. Then there is a gradient vector field W such that*

$$zeros(W) = zeros(V) - \{\alpha, \beta\}$$

The proof is again quite simple. Suppose that α and β are as in the top half of Figure 4.7, where we have drawn the unique gradient path from the boundary of β to α. Create a new vector field by "reversing the arrows" along this gradient path, while not changing any of the other arrows (see the bottom of Figure 4.7). It is easy to see that the hypothesis that the gradient path from β to α was unique implies that our construction has not resulted in the creation of any closed paths, and hence the new vector field is a gradient vector field. Moreover, it is also clear that neither α nor β is a critical cell of the new vector field, and we have not changed the criticality of any other cell. This completes the proof.



Cancelling the critical simplices α and β

Figure 4.7.

It is interesting, I think, to compare the proof of this combinatorial version to that of the classical smooth version, either as proved by Morse in [42] or Milnor [41]. The proof is essentially the same in both versions, in that one simply reverses the vector field along the distinguished path. However, when working in the smooth category one has more work to do, since the result must be a smooth vector field. Thus one must also turn all nearby vectors, making certain that the result is a gradient vector field, so that, for example, one must make certain that no closed orbits are introduced. All such difficulties are avoided in the combinatorial setting, because there are no "nearby" vectors.

This is a rather extreme case of the sort of simplification that one sees quite often when working in the combinatorial category. To be honest, while presenting some technical difficulties, the proof of the cancellation theorem is not one of the most challenging steps in the proof of the higher-dimensional Poincaré conjecture. As far as I know, the transition from the smooth category to the combinatorial category does not introduce any other significant simplifications in the proof of the higher dimensional Poincaré conjecture. In addition, the "difficulties" that occur in dimensions 3 and 4 in the smooth category definitely persist in the combinatorial category.

The cancelation theorem can be used to show that every connected combinatorial n-manifold (resp. n-manifold with nonempty boundary) has a gradient vector field with exactly one critical vertex, and one critical n-cell (resp. no critical n-cells). Combining this result with the Morse inequlaities yields the following general result, which we will apply in section 9.

Theorem 4.8. *Let M be a connected combinatorial 3-manifold with boundary, satisfying $\chi(M) = 1$. Then M is collapsible if and ony if M has a discrete gradient vector field with no critical edges.*

This result can be used to simplify some existing proofs in the literature (e.g. [10]).

5. A Simple Example

In this section we demonstrate some of the ideas of the previous section with a simple example. Fix a positive integer n, and consider the following $(n-1)$-dimensional simplicial complex, which we denote M_n. Starting with the following expression

$$(x_0 x_1 x_2 \dots x_n)$$

consider all ways of adding legal pairs of parentheses. An expression resulting from adding $p+1$ pairs of parentheses will be a p-simplex in our complex. The faces of this p-simplex are all expressions that result from removing corresponding pairs of parentheses.

For example, consider the case $n = 3$. The vertices of M_3 are the expressions

$$v_1 = ((x_0x_1)x_2x_3), v_2 = ((x_0x_1x_2)x_3), v_3 = (x_0(x_1x_2)x_3),$$

$$v_4 = (x_0(x_1x_2x_3)), v_5 = (x_0x_1(x_2x_3)),$$

and the edges are the expressions

$$e_1 = (((x_0x_1)x_2)x_3), e_2 = ((x_0(x_1x_2))x_3), e_3 = (x_0((x_1x_2)x_3)),$$

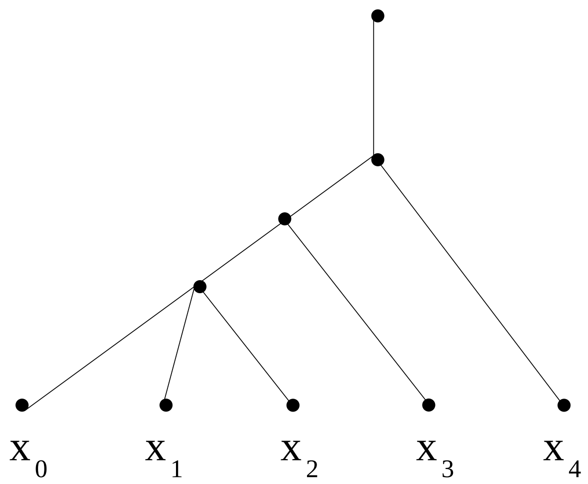
$$e_4 = (x_0(x_1(x_2x_3))), e_5 = ((x_0x_1)(x_2x_3)).$$

One can easily check the relations

$$e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}e_3 = \{v_3, v_4\}, e_4 = \{v_4, v_5\}, e_5 = \{v_5, v_1\},$$

so that M_3 is a circle triangulated with 5 edges and 5 vertices.

These complexes arise in a number of different settings. For example, in (quantum) algebra, given a set and a binary multiplication map, one might investigate how associative the multiplication rule is. In that case, M_2, for example, arises from studying all ways of multiplying 3 elements, with $(x_0x_1x_2)$ representing a homotopy between $((x_0x_1)x_2)$ and $(x_0(x_1x_2))$. Such complexes also arise in the study of planar rooted trees. For example, the edge $(((x_0x_1x_2)x_3)x_4)$ of M_4 can naturally be associated with the planar rooted tree shown in Figure 5.1. Moreover, the complexes M_n arise in Teichmüller theory, as they are closely related to the simplicial complex of subdivisions of an $(n + 1)$-gon into subpolygons.



The planar rooted tree corresponding to $(((x_0x_1x_2)x_3)x_4)$

Figure 5.1.

The main goal of this section is to use discrete Morse theory to give a simple proof of the following result.

Theorem 5.2. *The complex* M_n *is homotopy equivalent to an* $(n-1)$*-sphere.*

This result is well known, and it is only our proof that is new. We will prove this theorem by showing that one can easily construct a discrete gradient vector field on M_n which has presisely two critical simplices, namely one critical vertex and one critical $(n-1)$-simplex. The theorem then follows from Theorem 4.2. In fact, although we will not demonstrate this, it is not difficult to see that M_n is a combinatorial manifold. Therefore we can apply Theorem 4.5 to learn the following stronger result.

Theorem 5.3. *The complex* M_n *is a combinatorial* $(n-1)$*-sphere.*

Let us now describe the construction of the desired gradient vector field V. Let s be a simplex of M_n. Suppose that there is not a pair of parenthesis around x_0 and x_1. If it is possible to legally add a pair of parenthesis around x_0 and x_1 do so and call the resulting simplex t. We then add the pair $\{s \prec t\}$ to V. For example, in M_4 the expression $((x_0x_1x_2)(x_3x_4))$ is paired with $(((x_0x_1)x_2)(x_3x_4))$. After this step, the expressions which have not been paired with any other expression are those that have at least one parenthesis between x_0 and x_1, and it is simple to see that any such parenthesis must be a left parenthesis. There is one additional unpaired expression, namely the expression $s^* = ((x_0x_1)x_2x_3 \dots x_n)$. According to our rule, this should be paired with the original expression $(x_0, x_1 \dots x_n)$ with no added parentheses, but this is not permitted.

If s is any expression other than s^* that is currently unpaired, and a pair of parentheses can legally be added around the elements x_1 and x_2, do so and call the resulting simplex t. We then add the pair $\{s \prec t\}$ to V. After this step, the expressions which have not been paired with any other expression are s^* and those that have at least one left parenthesis between x_0 and x_1, and at least one left parenthesis between x_1 and x_2. Pair such an expression with the one resulting from adding a pair of parenthese around x_2 and x_3 if possible. Continue this process as long as possible. When it has terminated, the only expressions that have not been paired up with any other expression are s^* and the one that has a left parenthesis between every consecutive pair x_1 and x_{i+1} for $i = 0, 1, \dots, n-1$, i.e. the expression $t^*(x_0(x_1(x_2(\dots(x_{n-2}(x_{n-1}x_n))))\dots)$. Note that t^* is an $(n-1)$-simplex of the complex M_n.

This completes our construction of the vector field V. All that needs to be checked is there are no closed V-paths. Denote by V_k the discrete vector field that has been constructed after the k^{th} step in the construction, i.e. after consideration of the pair x_{k-1}, x_k. Let $s_0^{(p)}, t_0^{(p+1)}, s_1(p)$ denote a V-path. This requires that s_0 and t_0 be paied in V. Suppose that s_0 and t_0 are paired in V_k. The reader can check that this implies that either s_1 is the head of an arrow in V_k (and hence the V-path cannot be continued) or s_1 is paired in V_{k-1}. Thus, by induction, there can be no closed V-paths.

6. An Application to Computer Science

We now present an application of discrete Morse theory to computer science. That such an application exists should not be surprising, since many algorithms can be naturally interpreted as a discrete dynamical system of the sort we have introduced. One idea of this work is that a typical "impossibility result" in computer science shows that a problem is impossible to solve by showing that any algorithm will fail to solve the problem in at least one case. In some settings a failure of the algorithm can be identified with a zero or closed orbit of a discrete vector field. Hence the ideas we have presented, in particular the Morse inequalities of Theorem 1.7, can be used to give a topological explanation for the necessary failure of any algorithm, as well as quantitative lower bounds on the number of failures. In this section we present a single example of this phenomenon. Hopefully we will be able to explore this topic further in a later paper. For a more complete treatment of the content of this section, see [19].

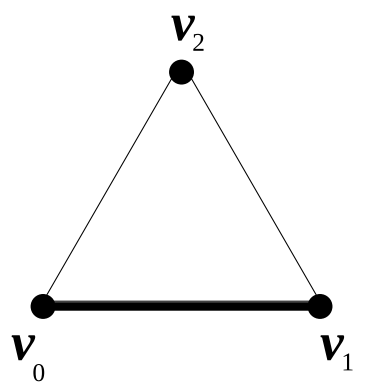
The problem we study is a topological version of a standard type of "search problem". The generalized version that we will present first appeared in [47]. Let S be an n-dimensional simplex, with vertices $v_0, v_1, \ldots, v_n$, and K a subcomplex of S which is known to you. Let σ be a face of S which is not known to you. Your goal is to determine if σ is in K. In particular, you need not determine the face σ, just whether or not it is in K. You are permitted to ask questions of the form "Is v_i in σ?". You may use the answers to the questions you have already asked in determining which vertex to ask about next. Of course, you can determine if σ is in K by asking $n+1$ questions, since by asking about all $n+1$ vertices you can completely determine σ. You win this game if you answer the given question after asking fewer that $n+1$ questions.

Say that K is *nonevasive* if there is a winning strategy for this game, i.e there is a question algorithm that determines whether or not $\sigma \in K$ in fewer than $n+1$ questions, no matter what σ is. Say K is *evasive* otherwise.

Kahn, Saks and Sturtevant proved the following relationship between the evasiveness of K and its algebraic topology.

Theorem 6.1. *If $\tilde{H}_*(K) \neq 0$, where $\tilde{H}_*(K)$ denotes the reduced homology of K, then K is evasive.*

In fact, they proved something stronger, and we will come back to this point later. We illustrate the previous theorem with a simple example. Let S be the 2-simplex shown in Figure 6.2, spanned by the vertices v_0, v_1 and v_2, with K the subcomplex consisting of the edge $[v_0, v_1]$ together with the vertex v_2.



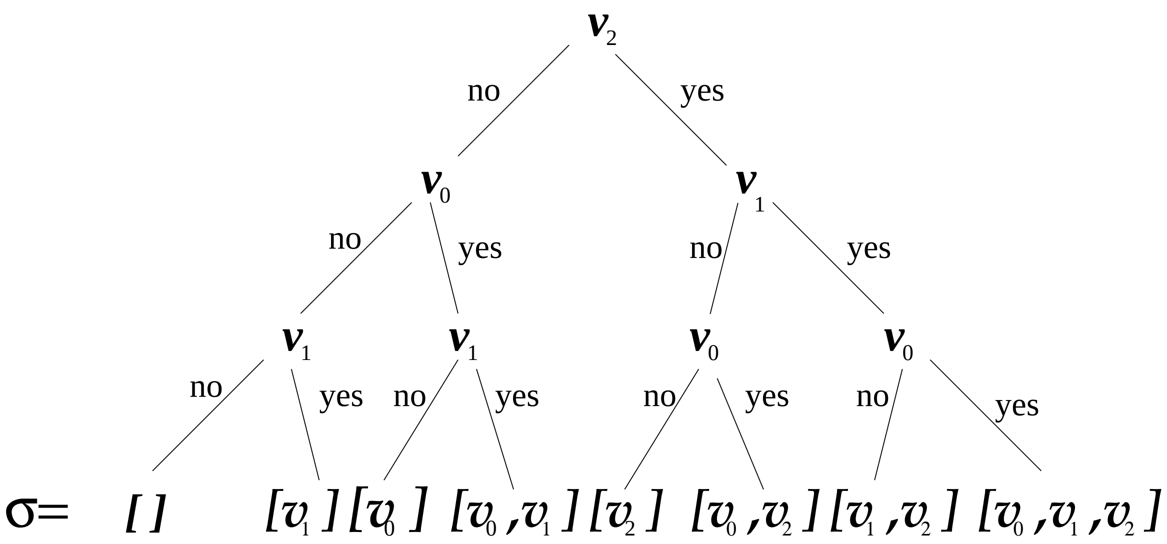
An example of an evasive subcomplex of the 2-dimensional simplex.

Figure 6.2.

A possible guessing algorithm is shown in Figure 6.3 (where we indicate which vertex to ask about at each stage, given the answers of the previous questions). Define an *evader* of a guessing algorithm to be a face σ of S with the property that when questions are asked in the order determined by the algorithm one must ask all three questions before it is known whether or not σ is in K. In particular, the evaders of the illustrated guessing algorithm are:

$$\sigma = [v_2], \quad [v_0, v_2]$$

Note that the subcomplex K has nonzero reduced homology, so the theorem of Kahn, Saks and Sturtevant guarantees that every guessing algorithm has some evaders.



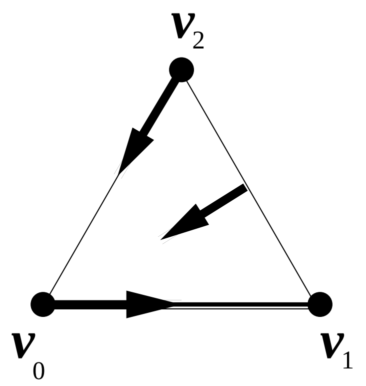
A guessing algorithm

Figure 6.3.

Discrete Morse theory comes to the fore when one observes that a guessing algorithm induces a discrete vector field on S. For example, the guessing algorithm shown in Figure 6.3 induces the vector field

$$\begin{aligned} V = \quad & \{\emptyset < [v_1]\}\{[v_0] < [v_0, v_1]\} \\ & \{[v_2] < [v_0, v_2]\}\{[v_1, v_2] < [v_0, v_1, v_2]\} \end{aligned}$$

That is, V consists of those pairs of faces of S which are not distinguished by the guessing algorithm until the last question. There is slight subtlety here in that a guessing algorithm pairs a vertex with the empty simplex $\emptyset$, which is not permitted in our original definition a discrete vector field. Thus, to get a true discrete vector field, we must remove this pair from V. (It is precisely this subtle point that results in the reduced homology of K being the relevant measure of topological complexity, rather than the nonreduced homology.) However, for simplicity, from now on we will simply ignore this technical point.



The vector field induced by the guessing algorithm shown in Figure 6.3.

Figure 6.4.

The reader can easily check that the discrete vector field illustrated in Figure 6.4 is a gradient vector field. This is true in general.

Theorem 6.5. *The discrete vector field induced by any guessing algorithm is a gradient vector field.*

We will postpone the proof of this result until the end of this section.

Now restrict V to K (by taking only those pairs in V s.t. both simplices are in K). For example, in our example, this results in the vector field

$$V_K = \{\{[v_0] < [v_0, v_1]\}\}.$$

From the Theorem 6.5, V has no closed orbits. Any discrete vector field consisting of a subset of the pairs of V has fewer paths, and hence also has no close orbits. Therefore, V_K is a gradient vector field on K. Note that V pairs every face of S with another face, and hence there are no critical simplices (we are continuing to ignore for now the simplex which is paired with the emptyset). Thus, the critical simplices of V_K are precisely the simplices of K which are paired in V with a face of S which is not in K. These are precisely the simplices of K which are the evaders of the guessing algorithm.

The Morse inequalities imply that the number of evaders in K is at least $\dim \widetilde{H}_*(K)$. Evaders occur in pairs, with each pair having one face of K and one face not in K. This yields the following quantitative refinement of Theorem 6.1.

Theorem 6.6. *For any guessing algorithm*

$$\#\ \textit{of evaders}\ \geq 2\dim \widetilde{H}_*(K)$$

Suppose that K is nonevasive. Then there is some guessing algorithm which has no evaders. From our above discussion we seen that this implies that K has a gradient vector field with no critical simplices. Actually, this is not quite true. The gradient vector field must have a critical vertex - the vertex that is paired with the empty set - this is that minor technicality that we have been ignoring. Applying Lemma 4.3 yields the following strengthening of Theorem 6.1.

Theorem 6.7. *If K is nonevasive, then K simplicially collapses to a point.*

This theorem appears in [33]. The interested reader can consult [19] for some additional refinements of this theorem.

We end this section with a proof of Theorem 6.5. Fix a subcomplex K of an n-simplex S, and a guessing algorithm. Associate to each p-simplex α of S the sequence of integers

$$n(\alpha) = n_0(\alpha) < n_1(\alpha) < \cdots < n_p(\alpha)$$

where the $n_i(\alpha)$'s are the numbers of the questions answered "yes" if $\sigma = \alpha$.

If V is the vector field induced by the guessing algorithm and

$$\alpha_0^{(p)},\quad \beta_0^{(p+1)},\quad \alpha_1^{(p)}$$

is a V-path, then $\{\alpha_0, \beta_0\}$ is in V, which means that α_0 and β_0 are not distinguished until the $(n+1)^{st}$ question. Thus,

$$n(\beta_0) = n_0(\alpha_0) < n_1(\alpha_0) < \cdots < n_p(\alpha_0) < n+1.$$

We now observe that the vertices of a_1 are a subset of the vertices of b_0. Suppose the vertex of β_0 which is not in α_1 is the vertex tested in question $n_i(\beta_0)$. Then we must have $i \neq n+1$ (since $\alpha_0 \neq \alpha_1$). This demonstrates that

$$n(\alpha_1) = n_0(\alpha_0) < n_1(\alpha_0) < \cdots < n_{i-1}(\alpha_0) < n_i(\alpha_1) < \ldots$$

for some $i < n+1$, and such that $n_i(\alpha_1) > n_i(\alpha_0)$. Thus $n(\alpha_1) > n(\alpha_0)$ in the lexicographic order, which is sufficient to prove that there are no closed orbits. QED

7. Topological Invariants via counting Gradient Paths

The First Cancellation Theorem (see Theorem 4.6) states that if these is a unique gradient path between two critical points, then the critical points are not essential and can be cancelled out. This hints at the fact that if we have extra information about the gradient paths travelling between critical points, then we may be able to strengthen our conclusions. In what follows we present three examples of this phenomenon.

I. The Morse Complex

In the first, and fundamental example, we will show how a complete knowledge of the gradient paths of a gradient vector field V on a simplicial complex K allows one to calculate the homology of K exactly.

Let $\mathcal{M}_p \subseteq C_p(K, \mathbb{Z})$ denote the span of the critical p-cells (we call $\mathcal{M}_*(K, \mathbb{Z})$ the space of Morse chains). From Theorem 4.2 it follows that there is a chain differential $\tilde{\partial}$, so that the homology of the complex

$$\mathcal{M}_* : 0 \longrightarrow \mathcal{M}^n \xrightarrow{\tilde{\partial}} \mathcal{M}_{n-1} \xrightarrow{\tilde{\partial}} \mathcal{M}_{n-2} \xrightarrow{\tilde{\partial}} \ldots \xrightarrow{\tilde{\partial}} \mathcal{M}_0 \longrightarrow 0 \tag{7.1}$$

is isomorphic to the homology of K.

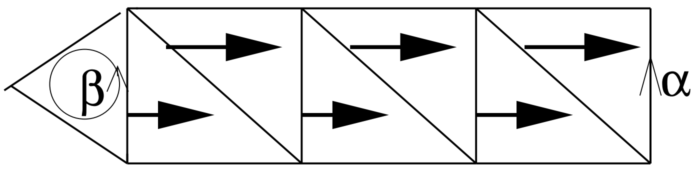
Our main goal is to give an explicit description of this differential in terms of the gradient paths of V. Choose an orientation for each critical simplex. For any critical $(p+1)$-simplex β, set

$$\tilde{\partial}\beta = \sum_{\text{critical } \alpha^{(p)}} c_{\alpha\beta}\alpha \tag{7.2}$$

where

$$c_{\alpha\beta} = \sum_{\gamma\in\Gamma(\beta,\alpha)} m(\gamma). \tag{7.3}$$

The sum is over all $\Gamma(\beta,\alpha)$, the set of gradient paths from some maximal face of β to α. Given such a gradient path γ, an orientation on β induces an orientation on α. Rather than giving a precise definition here, we will simply note that the intuitive idea is that one "slides" the orientation along γ. For example, in Figure 7.1 we illustrate a single gradient path $\gamma \in \Gamma(\beta,\alpha)$. In this case, the indicated orientation on β induces the indicated orientation on α. We define the multiplicity of γ, $m(\gamma)$ to ± 1 depending on whether or not the chosen orientation on β indices the chosen orientation on α.



Counting such gradient paths leads to the differential of the Morse complex.

Figure 7.1.

We can now present the main theorem.

Theorem 7.2. *The operator $\tilde\partial$ defined in (7.2) and (7.3) satisfies $\tilde\partial\circ\tilde\partial = 0$. With this differential, the differential complex (7.1) has the same homology as the underlying simplicial complex K.*

We refer to this complex as the *Morse complex* (although a number of different names appear in the litereature). The main ingredient of the proof of Theorem 7.2 is presented in section 10. In the smooth setting this complex is studied in great detail in [48].

Here we see another important simplification that arises in the transistion from the smooth to the combinatorial category. In the smooth setting, one must take care to ensure that gradient paths only travel from a critical point to critical points of smaller index. Moreover, the standard definitions of multiplicity in this setting requires that all gradient paths be "nondegenerate" in a certain sense. Gradient vector fields which satisfy these conditions are called *Morse-Smale gradient vector fields*. Such gradient vector fields are generic in the space of all gradient vector fields. In the combinatorial setting, however, such issues simply never arise.

Although we presented the Morse complex from the point of view of homology, we could equally well have presented a Morse cocomplex which calculates the cohomology of K. In this case, the complex would have the form

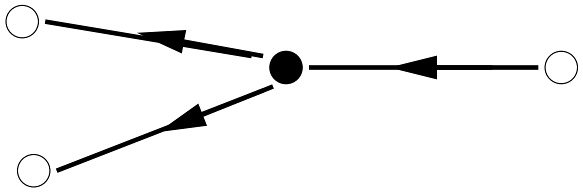
$$\mathcal{M}^* : 0 \longrightarrow \mathcal{M}^0 \xrightarrow{\tilde{\delta}} \mathcal{M}^1 \xrightarrow{\tilde{\delta}} \mathcal{M}^2 \xrightarrow{\tilde{\delta}} \ldots \xrightarrow{\tilde{\delta}} \mathcal{M}_n \longrightarrow 0$$

where $\tilde{\delta}$ is the adjoint of $\tilde{\partial}$.

II. The Cup Product

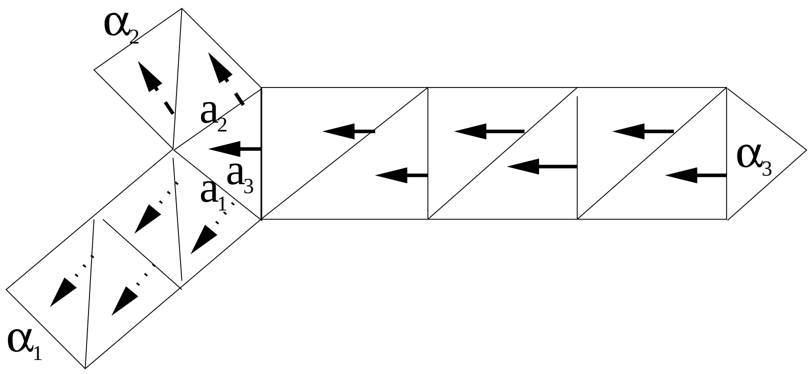
Now that we have a Morse representation of the cohomology of K, it is natural to ask how one can see the cup product, and other cohomological operations, from the point of view of Morse theory. In the smooth setting this question was answered by Betz-Cohen ([5]) and independently by Fukaya ([24],[26]), who show how one can express cohomology operations in terms of a counting of gradient paths. One can apply the same procedure here. In this section, we indicate only how one can construct the cup product as a map of the Morse cochains. Let V_1, V_2, V_3 be any 3 gradient vector fields on a finite simplicial complex K. The question we answer is: Suppose we have a cohomology class c_1 represented in terms of the Morse cocomplex induced by V_1 (i.e. given as a closed cochain in the corresponding Morse cocomplex), and a cohomology class c_2 represented in terms of the Morse cocomplex induced by V_2. How can we express the cup product $c_1 \cup c_2$ in terms of the Morse cocomplex induced by V_3?

In this case, rather than just counting single gradient paths, we will have to count "Morse embeddings" of the configuration shown in Figure 7.3, i.e. triples of gradient paths. One a V_1-gradient path ending at a V_1 critical α_1, and another a V_2 gradient path ending at a V_2-critical simplex α_2, beginning at any simplices a_1 and a_2, resp., and the third a V_3 path from a V_3 critical simplex α_3 to any simplex a_3. We require that a_1^*, a_2^*, and a_3^*, the cosimplices dual to a_1, a_2, and a_3 resp., satisfy $a_1^* \cup a_2^* = \pm a_3^*$ (where $\cup$ represents any map on cosimplices which induces the cup product on cohomology). We draw such a configuration in Figure 7.4. In this figure, the solid arrows represent the discrete gradient vector field V_3, and the dashed and dotted arrows represent V_2 and V_1, resp.



Counting Morse embeddings of this figure results in a map which induces the cup product.

Figure 7.3.



A more precise picture of the configuration shown in Figure 7.3.

Figure 7.4.

We now proceed more precisely. Define

$$\kappa : \mathcal{M}^*(V_1) \otimes \mathcal{M}^*(V_2) \longrightarrow \mathcal{M}^*(V_3).$$

By

$$\kappa(\alpha_1, \alpha_2) = \sum_{\alpha_3} \kappa_{\alpha_1,\alpha_2}^{\alpha_3} \alpha_3 \tag{7.4}$$

where α_i is critical for V_i and

$$\kappa_{\alpha_1,\alpha_2}^{\alpha_3} = \sum_{\alpha_1 \cup \alpha_2 = \alpha_3} \sum_{\gamma_1,\gamma_2,\gamma_3} \pm 1 \tag{7.5}$$

where

γ_1 is a "V_1-gradient path" from α_1 to a_1

γ_2 is a V_2-gradient path from α_2 to a_2

and

γ_3 is a V_3-gradient path from a_3 to α_3

As usual, the $\pm$ is determined by a careful consideration of orientations.

In [21] we prove the following theorem.

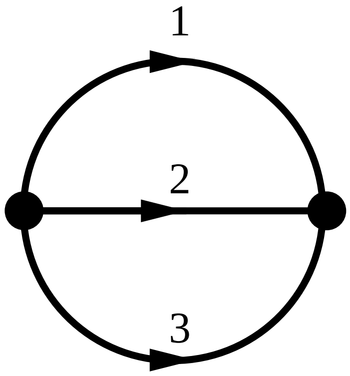
Theorem 7.5. *The map κ defined in (7.4) and (7.5) induces the cup product on cohomology.*

The "same" result holds in the smooth case (as demonstarted by Betz-Cohen and Fukaya) with an additional complication. Namely, in the smooth case these is again a genericity condition which requires that each of the gradient vector fields be Morse-Smale and, in addition, be in general position with respect to each other. There are no such conditions in the combinatorial setting. In particular, one may choose each of V_1,V_2 and V_3 to be the trivial gradient vector field, that is the vector field consisting of no pairs, so that the space of Morse cochains is equal to the space of all cochains. Then the map introduced above is just the original cup product on cochains.

III. Finite-Type Invariants

In this section we discuss some work in progress. As a consequence, we will not even state a precise theorem. (In fact, it would take quite a bit of work to state the precise theorem.)

In [25], Fukaya showed how to define a finite type invariant of a (smooth) rational homology S^3 by choosing a generic triple of Morse functions, f_1, f_2, f_3, counting the Morse embeddings of the diagram shown in Figure 7.6, and adding correction terms corresponding to related diagrams. Much work goes into proving that the resulting invariant does not depend on the choice of the Morse functions.



Counting Morse embeddings of this "theta graph" leads to a finite-type invariant.

Figure 7.6.

The "same" construction using discrete Morse theory leads to a combinatorial invariant of combinatorial $\mathbb{Q}$-homology S^3's. At this point the reader may wonder why we must restrict attention to combinatorial manifolds. After all, up to this point everything was true for a general simplical complex, and even more generally. Let us take another look at Figure 7.6. Rather than viewing the diagram as three arcs beginning together and ending together, it is more accurate to view the figure as a set of loops. One loop is formed by going in the positive direction along the arc labelled 1, followed by going in the negative direction along the arc labelled 2. Another loop is formed by going in the positive direction along the arc labelled 2 and in the negative direction along the arc labelled 3, etc.

In the smooth category, there is no difficulty in flowing in the direction of a gradient vector field V or in the opposite direction, since flowing in the opposite direction is equivalent to flowing in the direction of the gradient vector field $-V$. In the combinatorial category, at least for general cell complexes, the negative of a vector field does not exist. For a combinatorial manifold, however, the negative of a discrete vector field does exist as a discete vector field on the combinatorial manifold with the dual cell decomposition. It is precisely for this reason that we restrict attention to combinatorial manifolds.

The upshot is that following Fukaya's work, one can show that the analogous construction in the combinatorial category yields a combinatorial invariant. At this point, however, our proof rests on converting all of the combinatorial ingredients into continuous ingredients, and then applying Fukaya's proof. This is not completely satisfying. The main reason is that Fukaya's construction requires the three smooth gradient vector fields be in general position with respect to each other, and a similar condition remains in the combinatorial setting. However, just as no such condition was required to define a cup product map in Morse theory in Part II of this section, it seems clear that no such condition is really necessary for the construction of the combinatorial invariant. In order to provide a proof that holds for any triple of gradient vector fields, one should work entirely in the combinatorial category. Thus, what remains in this project is to use the ideas of Part II of this section to define all necessary intersections combinatorially, and then to give a direct combinatorial proof of the combinatorial invariance of the resulting invariant.

Finding a construction which does not require the three vector fields to be mutually transverse has more than just aesthetic value. In [1], Axelrod and Singer define a finite type invariant of a rational homology 3-sphere. It would seem that in Fukaya's approach, if one formally chooses all three Morse functions to be constants (which are not Morse functions at all, let alone mutually transverse) then the result is the invariant of Axelrod and Singer. Even though Fukaya's construction does not permit this choice of functions, it is natural to conjecture that his invariant is equal to that of Axelrod and Singer. (Fukaya presents some ideas - along the line of those presented in the next section of this paper- for how one might be able to go about proving the equality of Fukaya's invariant and that of Axelrod and Singer). One advantage of working in the combinatorial category in which all triples of gradient vector fields are premitted is that the combinatorial version of Fukaya's invariant and the combinatorial analogue of the invariant of Axelrod and Singer are all on equal footing, different special cases of the same construction, and thus is would follow imediately that they are equal. Of course the difficulty would then remain of showing that the combinatorial versions of these invariants are equal to the smooth versions.

8. Supersymmetry and Discrete Morse Theory

In the paper "Supersymmetry and Morse theory", Witten showed how Morse theory may be derived from supersymmetric considerations in quantum field theory. The same approach may be taken with discrete Morse theory, with, as usual, some simplifications. Perhaps most significantly, Witten derives the Morse complex from a mathematically ill-defined path integral. Following the same ideas in the combinatorial setting, we are lead instead to a well-defined discrete sum. Unfortunately, this last point is a bit too far afield to go into here (see section 8 of [16]). We will have to content ourselves here with a "supersymmetric" derivation of the Morse inequalities.

First, we must take another look at the idea of a discrete Morse function. Previously in these notes, such a function was defined to be a Lyapunnov function associated to a gradient vector field. It is useful to give a direct definition here.

Theorem 8.1. *A function is a discrete Morse function if and only if for each simplex α,*

$$\#\{\beta \succ \alpha \mid f(\beta) < f(\alpha)\} \leq 1$$

and

$$\#\{\gamma \prec \alpha \mid f(\gamma) > f(\alpha)\} \leq 1.$$

In fact, in [15], Theorem 8.1 is used as the definition of a discrete Morse function, and the equivalence with the definition used here - a Lyapunov function corresponding to a gradient vector field with no closed orbits - is proved in Theorem 9.3 of [15]. It is quite useful, when doing the sort of analysis which follows, to restrict attention to a special class of discrete Morse functions. However, this restriction does not affect the generality ofthis approach, as we will now explain.

Definition 8.2. *Say a discrete Morse function is flat if* $\beta \succ \alpha$ *implies that* $f(\beta) \geq f(\alpha)$.

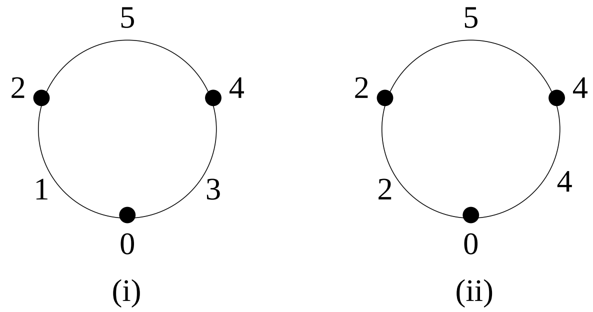
Definition 8.3. *Say that two discrete Morse functions* f *and* g *are equivalent if for every pair* $\beta \succ \sigma$

$$f(\beta) > f(\alpha) \iff g(\beta) > g(\alpha).$$

In particular, equivalent discrete Morse functions have the same critical simplices and gradient vector fields.

Figure 8.5(i) displays a discrete Morse function which is not flat. In Figure 8.5 (ii) we show a flat discrete Morse function which is equivalent to the function in 8.5(i). In fact, we have the following result.

Theorem 8.4. *Every discrete Morse function is equivalent to a flat discrete Morse function.*



(i). A nonflat discrete Morse function. (ii). A flat discrete Morse fuction

Figure 8.5.

It follows from Theorem 8.4 that in order to prove the Morse inequalities for a general discrete Morse function, it is sufficient to restrict attention to flat Morse functions. With that in mind, let f denote a flat Morse function. Beginning with the real chain complex

$$0 \longrightarrow C_n(M,\mathbb{R}) \xrightarrow{\partial} C_{n-1}(M,\mathbb{R}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(M,\mathbb{R}) \longrightarrow 0$$

we replace the boundary operator ∂ by

$$\partial_t \;=\; e^{tf}\partial e^{-tf}$$

where e^{tf} multiplies each simplex α by $e^{tf(\alpha)}$ and is extended linearly to all chains. The resulting differential complex has the same homology for all $t \in \mathbb{R}$. We will now investigate this homology using ideas from Hodge theory. Let ∂_t^* denote the adjoint of ∂_t with respect to the canonical inner product $<,>$ on chains, and $\Delta_p(t)$ the induced Laplacian on p-chains

$$\Delta_p(t) \;=\; \partial_t\partial_t^* + \partial_t^*\partial_t : C_p(M,\mathbb{R}) \longrightarrow C_p(M,\mathbb{R}).$$

It follows from basic linear algebra that for each $t \in \mathbb{R}$,

$$\operatorname{Ker}\Delta_p(t) \cong H_p(M,\mathbb{R}).$$

The next step is to derive a more explicit representation for $\Delta_p(t)$. Choose an orientation for each simplex. Then

$$\partial_t\sigma^{(p)} \;=\; \sum_{\gamma^{(p-1)}<\sigma} \pm e^{t(f(\gamma)-f(\sigma))}u$$

(where the $\pm$ sign is determined by whether the orientations on γ and σ agree or not)

$$\partial_t^*\sigma^{(p)} \;=\; \sum_{\beta^{(p+1)}>\sigma} \pm e^{t(f(\sigma)-f(\beta))}\beta,$$

$$\Delta_p(t)\sigma^{(p)} \;= \sum_{\tilde\sigma^{(p)}}\left[\sum_{\gamma^{(p-1)}<\sigma\gamma<\tilde\sigma} \pm e^{t(2f(\gamma)-f(\sigma)-f(\tilde\sigma))} + \sum_{\beta^{(p+1)}>\sigma\beta>\tilde\sigma} \pm e^{t(f(\sigma)+f(\tilde\sigma)-2f(\beta))}\right]\tilde\sigma.$$

It follows from our restrictions on f that if $\tilde\sigma \neq \sigma$, then the coefficient of $\tilde\sigma$ in the above expression is $O(e^{-tc})$ as $t \to \infty$, for some $c > 0$. Thus, up to exponentially small errors as $t \to \infty$, the Laplacian $\Delta_p(t)$ is diagonalized by the p-simplices. A closer look reveals

$$\left\langle \Delta_p(t)\sigma^{(p)}, \sigma^{(p)} \right\rangle \;= \begin{cases} O(e^{-tc}) & \text{if } \sigma \text{ is critical} \\ 1 + O(e^{-tc}) & \text{if } \sigma \text{ is not critical.} \end{cases} \tag{8.1}$$

In particular, we learn

Theorem 8.6. *As $t \to \infty$,*

$$\#\{\textit{eigenvalues of } \Delta_p(t) \textit{ which are } O(e^{-tc})\} = \#\{\textit{critical } p-\textit{simplices}\}.$$

All other eigenvalues of $\Delta_p(t)$ *are* $1 + O(e^{-tc})$.

That is, the dimension of the kernel of $\Delta_p(t)$ is equal to $b_p = \dim H_p(K, \mathbb{R})$ for each $t \in \mathbb{R}$, and the dimension of the kernel of $\Delta_p(\infty)$ is m_p, the number of critical p-simplices. Since the dimension of the kernel is an upper-semicontinuous function of t we learn the weak Morse inequalities

$$m_p \geq b_p.$$

In fact, Theorem 8.6 is sufficient to imply the strong Morse inequalities (over $\mathbb{R}$). See, for example, [7] (where the proof is discussed in the smooth category) or [16] for the completion of the argument.

We now take a closer look at the small eigenspaces. Let $W_p(t) \subseteq C_p(M, \mathbb{R})$ denote the span of the eigenvectors of Δ_t^p corresponding to the eigenvalues which are $O(e^{-tc})$. Then

$$\partial_t(W_p(t)) \subseteq W_{p-1}(t)$$

so for all t we have a differential complex (the Witten complex) with the same homology as M

$$\mathcal{W}(t) : 0 \longrightarrow W_n(t) \xrightarrow{\partial_t} W_{n-1}(t) \xrightarrow{\partial_t} \cdots \xrightarrow{\partial_t} W_0(t) \longrightarrow 0.$$

It follows from (8.1) that

$$\lim_{t \to \infty} W_p(t) \;=\; \mathcal{M}_p,$$

where, as in the previous section, $\mathcal{M}_p$ denotes the Morse p-chains, i.e. the p-chains which are supported on the critical p-simplices. We can say more. In fact, after a simple change of coordinates the differential ∂_t on $\mathcal{W}(t)$ converges to the differential $\widetilde{\partial}$ on $\mathcal{M}$. To describe this change of coordinates, let π denote the orthogonal projection onto $\mathcal{M}_p$. For large t

$$\pi : W_p(t) \longrightarrow \mathcal{M}_p$$

is an isomorphism. Define a function θ on $W_p(t)$ by setting, for any critical simplex $\sigma^{(p)}$

$$\theta(\pi^{-1}(\sigma)) \;=\; e^{tf(\sigma)}$$

and extending linearly. Let $\widetilde{\partial}_t = \theta^{-1}\partial_t\theta$.

Theorem 8.7. $\lim_{t\to\infty}(\mathcal{W}(t), \widetilde{\partial}_t) = (\mathcal{M}, \widetilde{\partial})$.

In [15] we use this theorem to give a second proof, along the lines suggested by Witten in [56] that the discrete Morse complex $\mathcal{M}$ has the same homology as the underlying manifold.

We now return to the observation that all eigenvalues of $\Delta_p(t)$ which do not converge to 0, converge to 1. This may seem like just a curiosity, but it obviously has interesting implications when one is considering the determinant of the Laplace operator. Given a representation

$$\rho : \pi_1(M) \longrightarrow O(n, \mathbb{R})$$

we can twist the chain complex by ρ to define a complex

$$\mathcal{C}(\rho) : 0 \longrightarrow C_n(M,\rho) \xrightarrow{\partial} C_{n-1}(M,\rho) \xrightarrow{\partial} \cdots \longrightarrow C_0(M,\rho) \longrightarrow 0.$$

There is a canonical inner product on each $C_p(M,\rho)$ and hence we can define an adjoint operator ∂^* and a Laplacian

$$\Delta_p(\rho) \;=\; \partial\partial^* + \partial^*\partial : C_p(M,\rho) \longrightarrow C_p(M,\rho).$$

If each Laplacian $\Delta_p(\rho)$ is invertible, we can define the Reidemeister torsion of M with respect to the representation ρ by

$$T(M,\rho) \;=\; \prod_{p=0}^{n} (\mathrm{Det}\,\Delta_p(\rho))^{(-1)^p \frac{p}{2}}$$

(this formula is due to Ray and Singer [46]). Now replace the differential ∂ by ∂_t as before, form the induced Laplacians, and consider

$$T(M,\rho,t) \;=\; \prod_{p=0}^{n} (\mathrm{Det}\,\Delta_p(\rho,t))^{(-1)^p \frac{p}{2}}.$$

As with the "untwisted" Laplacian, all eigenvalues of $\Delta_p(\rho,t)$ which do not converge to 0, converge to 1. Thus, as $t \to \infty$, the right hand side "collapses" onto the complex $\mathcal{W}(\rho,t)$ of small eigenspaces. As $t \to \infty$, the complex of small eigenspaces converges, after a change of variables, to the Morse complex (suitably twited by ρ.. Keeping track of how $T(M,\rho,t)$ varies with t, and the effect of conjugating ∂_t by θ we prove

Theorem 8.8. $T(M,\rho) = \mathit{Torsion}(\mathcal{M}(\rho), \widetilde{\partial})$

where $\mathcal{M}(\rho)$ denotes the twisted Morse complex, and the torsion of $\mathcal{M}(\rho)$ is

$$\prod_{p=0}^{n} (\operatorname{Det} \widetilde{\Delta}_p(\rho))^{(-1)^p \frac{p}{2}}$$

where

$$\widetilde{\Delta}_p(\rho) \;=\; \widetilde{\partial}\widetilde{\partial}^* + \widetilde{\partial}^*\widetilde{\partial} : \mathcal{M}_p(\rho) \longrightarrow \mathcal{M}_p(\rho).$$

Note that Theorem 8.8 is precisely Theorem 3.2, in the special case that V is a gradient vector field. The general case of Theorem 3.2 can be proved by an analogous technique. Given a general discrete vector field V, one can deform the Laplace operator via a "flat Lyapunov function." In this case it os no longer true that the nonsmall eigenvalues converge to 1. However, they do converge, and taking a closer look at the limiting behavior of the Laplace operator as $t \to \infty$ enables one to equate the product of the nonsmall eigenvalues with the special value of the zeta function.

In section 3 we discussed the relevance of the smooth version of Theorem 3.2. The smooth version of the special case of Theorem 8.8 is also of great interest, as it plays a key role in recent proofs of the equality of the Reidemeister torsion and Ray and Singer's analytic torsion ([6], [8]).

9. Discrete Morse Theory and the Distance Function

In this section we present a recent application of discrete Morse theory to geometry. Just as in the smooth setting, Morse theory meets geometry when one tries to apply Morse theory to the study of the distance function on a metric space. Some of the fundamental theorems in Riemannian geometry which relate curvature to topology are proved in this fashion. In this section we present one such result in the combinatorial setting which is due to my graduate student Katherine Crowley. Let K be a contractible triangulated 3-manifold with boundary. Our goal is to find some geometric conditions which imply that K collapses.

The main theorem is the following

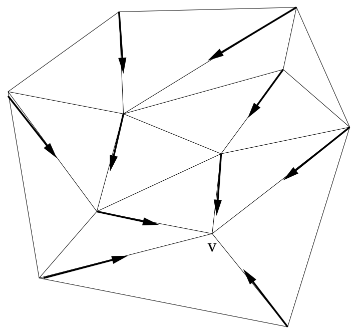
Theorem 9.1 (Crowley [11]). *Let K be a combinatorial 3-manifold with boundary. Endow K with the piecewise Euclidean metric resulting from declaring all edges to have length 1. If K is $CAT(0)$ then K collapses.*

The rest of this section will be devoted to giving an outline of the proof, but we begin with some general remarks. The notion of CAT(0) is a global condition meaning (roughly) "simply-connected and with non-positive curvature". With this in mind, Crowley's theorem should be compared to Hadamard's theorem: A complete simply-connected nonpositively curved Riemannian manifold is contractible (in fact, is diffeomorphic to $\mathbb{R}^n$).

I find Crowley's result to be fascinating. Two different distance functions are being examined here. The hypothesis that K is Cat(0) is a restriction on the distance function induced by the piecewise Euclidean metric. The conclusion that K collapses is more a statement about the combinatorial distance function on the vertex set, in which the distance between two vertices is defined to be the minimal number of edges needed to connect one vertex to the other. The proof of this theorem demonstrates some rather intriguing relationships between these two distance functions. The two distant functions seem to be most closely related when all of the edges have the same length, and that is the reason for hypothesis that all edges have length 1. However, this is not a rigig phenomenon. The theorem remains true, without any changes in the proof, if the ratio of the lengths of edges is sufficiently close to 1.

Recall that Hadamard's Theorem is proved by picking a point p in the manifold, and showing that the function "distance to p" has no critical points other than p. Crowley's theorem is proved in an analogous way. We pick a vertex v, and use the function "distance to v" to construct a gradient vector field on K with only a single critical simplex. By Theorem 4.8 it is sufficient to construct a gradient vector field with no critical edges.

The first step is to have all vertices flow to v along "geodesics". That is, for every vertex $w \neq v$, choose a "geodesic" from w to v (that is, a minimal set of edges connecting w to v). Begin the construction of our gradient vector field by drawing an arrow from w into the first edge of the geodesic.

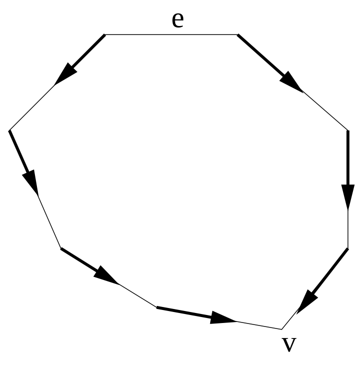


The first step in the construction of the gradient vector field.

Figure 9.2.

It is rather easy to check that this vector field has no closed orbits, and hence is a gradient vector field. The only critical vertex is v. However this gradient vector field will have lots of critical 1-simplices.

Let e be an edge that is is still critical (i.e. is not paired up with any vertex). We must now choose which 2-simplex to pair with e. The edge e, along with the geodesics from its endpoints to v, forms a loop (actually, the 2 geodesics may meet before reaching v)



A look at a critical edge.

Figure 9.3.

Now we reach the main new idea of the proof. Fill in this loop with a minimal simplicial disc in K (i.e. a simplicial map from a triangulated disc into K), where "minimal" means fewest number of triangles

Now draw an arrow from e pointing into this minimal disc. Doing this for every critical edge results in a **gradient vector field** with no critical 1-simplices.

It is non-trivial to show that the resulting vector field is a gradient vector field. In fact, it is not trivial that the resulting pairing is a discrete vector field at all (after all, what is to stop this procedure from pairing two edges with the same 2-simplex?) It is at this point that we must use the hypotheses of the theorem. The key point is the following.

Theorem 9.4 (Crowley)**.** *A minimal simplicial disc spanning a loop in a 3-dim $CAT(0)$ complex is $CAT(0)$.*

In this context, a triangulated idsc is CAT(0) if and only if every vertex in its interior has degree ¡6. This theorem is a combinatorial analogue of the fact that a minimal surface in R^3 always has nonpositive Gauss curvature. I believe that Crowley's work hints at the existence of a more complete theory of "combinatorial differential geometry."

10. Main Ingredient

In this final section, we introduce what is the main ingredient in the proof of many of the results we have discussed. When given a smooth vector field on a smooth manifold, there are a couple different ways to extract information about the dynamics of induced flow. For example, one may fix a point in the manifold and let time vary. That is, fixing a point on the manifold, one may consider the flow line through that point. Another possibility is to fix a time and to consider all initial points. That is, given a real number t, to consider the diffeomorphism of the manifold resulting from letting every point flow along the vector fielld for time t. These two different points of view bring with them their own insights. We have already presented a combinatorial analog of the notion of a flow line, we have not yet seen a combinatorial analog of the induced diffeomorphism.

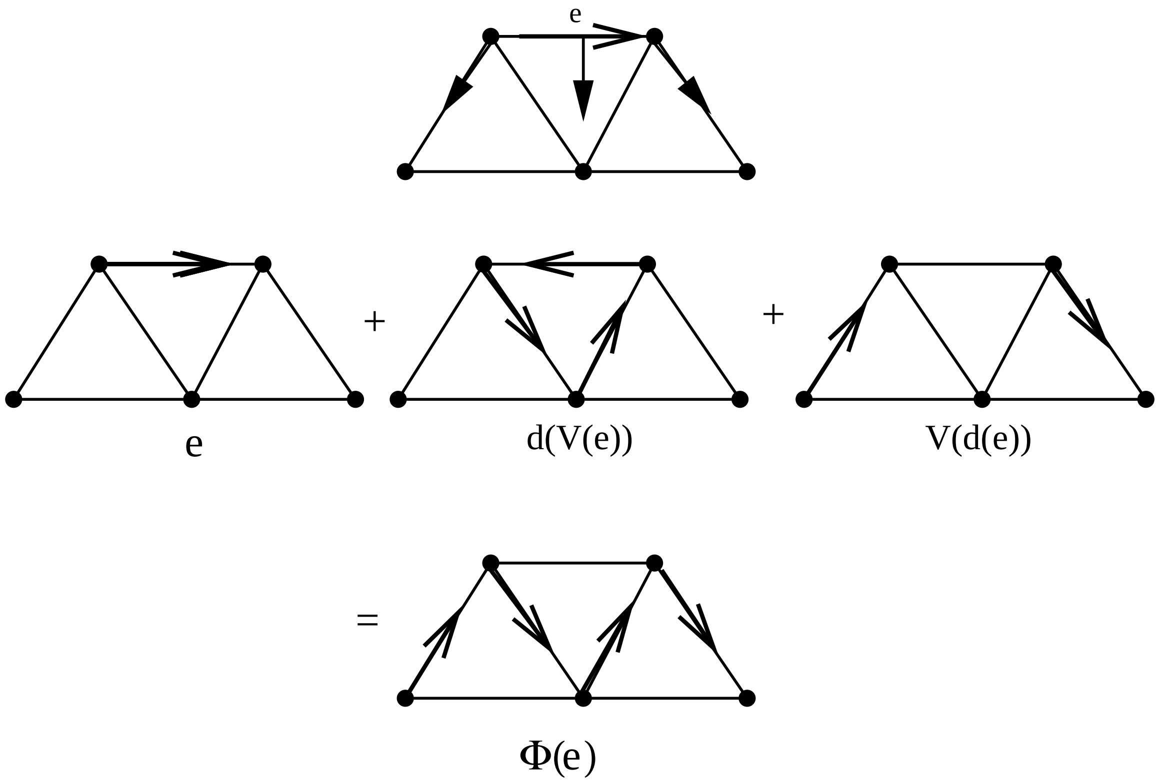
Before defining the flow, we must slightly change our notion of a discrete vector field. So far, we have treated a discrete vector field as a collection of pairs of simplices. Now we think of V as a map of oriented chains which has degree $+1$. Namely, if $\alpha^{(p)} < \beta^{(p+1)}$ are paired in V, then we define the map of oriented chains, which we also call V, by setting $V(\alpha) = \pm\beta$, with the sign chosen so that $< \alpha, \partial V(\beta) >= -1$. Given V, define the corresponding (discrete time t=1) flow

$$\Phi : C_p(K) \longrightarrow C_p(K); \qquad p = 0, 1, 2, \ldots$$

by the formula

$$\Phi := 1 + \partial V + V\partial. \tag{10.1}$$

For example, in Figure 10.1 we illustrate the flow of an oriented edge.



The flow of an oriented edge.

Figure 10.1.

We can now desrcibe briefly how this map enters into the proofs of results presented earlier. Consider first Theorem 3.1 concerning the analytic continuation of the zeta function.

To give a hint of the proof it is perhaps best to consider the simple example of a shift of finite type. Let A be a finite set. Suppose that for each $a \in A$, we are given a subset $F(a) \subset A$, the set of letters which are permitted to follow a. Let Σ denote the set of all allowable double infinite sequences. That is, the set of all

$$x = \ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots,$$

where for each i, $x_i \in A$ and $x_{i+1} \in F(x_i)$. Say x is periodic with period k if $x_i = x_{i+k}$ for all I. Consider the corresponding zeta function

$$\zeta(z) = \exp[\sum_{k=0}^{\infty} \frac{z^k}{k} \mathcal{P}_k].$$

where $\mathcal{P}_k$ is the number of elements in Σ which are periodic with period k. It is one of the foundational results of symbolic dynamics that $\zeta(z)$ has an analytic continuation to all of $\mathbb{C}$. The proof is rather simple. Let $\mathcal{A}$ denote the square matrix whose rows and columns are identified with the elements of a, such that for i, j in A, the $(i,j)^{th}$ element of $\mathcal{A}$ is 1 if $j \in F(i)$, and 0 otherwise. It is then easy to check that $\mathcal{P}_k = trace(\mathcal{A}^k)$, so that

$$\zeta(z) = \det(I - z\mathcal{A})$$

from which the desired result follows.

Without going into details, suffice to say that the proof of Theorem 3.1 is along the same lines as that in the previous paragraph, with only minor complications. The key point is that the role of $\mathcal{A}$ played by the flow map Φ. (See section 6 of [17].)

It also remains to give some idea of the proofs of the results of section 7, in which we show how one can view the homology of a simplicial complex from the point of view of discrete Morse theory. The crucial point here is that it follows immediately from the formula (10.1) that the flow map commutes with the coboundary map, and induces the identity map on homology. Similarly, one can consider the coflow

$$\Phi^* = 1 + \delta V^* + V^* \delta : C^*(K) \to C^*(K),$$

which commutes with the coboundary operator δ and induces the identity map on cohomology. In the case that V is the gradient vector field associated to a Morse function f, then Φ has many of the properties that one would expect from flowing along the (negative of) the gradient of a function. For example, if f is a discrete Morse function corresponding to the gradient vector field V and α is any simplex, then $\Phi(\alpha)$ is supported on simplices β with $f(\alpha) < f(\alpha)$. One major simplification that occurs in the combinatorial setting is that Φ stabilizes in finite time. That is, there is an N such that

$$\Phi^N = \Phi^{N+1} = \cdots = \Phi^\infty.$$

Theorem 7.2 is proved by showing that the map Φ^∞ maps the Morse complex to the standard cellular chain complex inducing the identity map on homology. Theorem 7.5, and the generalizations which appear in [21], is proved by examining this idea a bit closer.

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