BOCHNER'S METHOD FOR CELL COMPLEXES AND COMBINATORIAL RICCI CURVATURE

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§0. Introduction.

In this paper we will present a new notion of curvature for cell complexes. For each p, we define a p^{th} combinatorial curvature function, which assigns a number to each p-cell of the complex. The curvature of a p-cell depends only on the relationships between the cell and its neighbors. In the case that $p = 1$, the curvature function appears to play the role for cell complexes that Ricci curvature plays for Reimannian manifolds. Much of this paper is devoted to comparing the properties of the combinatorial Ricci curvature with those of its Riemannian avatar. The definitions of these curvature functions come from an analysis of the combinatorial Laplace operator acting on p-chains. In the Riemannian setting, Bochner and Weitzenböck showed that the Riemannian Laplace operator can be decomposed into a sum of a Laplace-type operator built from the covariant derivative operator, and a term built from the curvature of the manifold. In the combinatorial setting, we simply mimic the Bochner- Weitzenböck decomposition, and define the combinatorial curvature to be the corresponding term in the combinatorial Bochner- Weitzenböck decomposition.

The roots of this work can be found in many sources. The combinatorial Laplace operator, as an analogue of the Riemannian Laplace operator was introduced by Eckmann [Ec]. There has been much work on the ways in which the combinatorial Laplace operator mimics the properties of the Riemannian operator. See, for example, [Do1,2] and [DK] for some of the earlier contributions to this study. This work has much in common with two earlier works. In [Ga1-3], Garland presented a combinatorial version of the slight modification of Bochner's approach used by Matsushima in [M]. Garland applied his method to prove some quite striking vanishing theorems for the cohomology of p-adic symmetric spaces. From another point of view, Stone [St1,2] presented a combinatorial Ricci curvature and deduced a version of Myers' Theorem for combinatorial manifolds. The curvature functions we define are different from the curvatures defined in these papers. However, the derivation of our curvature function has much in common with that of [Ga1-3], since both

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are based on a combinatorial interpretation of Bochner's method [Bo1-3]. The proof of Myers' Theorem in Section 7 of this paper has much in common with the proofs in [St1,2], since both are based upon a combinatorial interpretation of Myers' original proof in [My].

Let us begin the presentation of this work with some examples . Although the theory can be applied to very general cell complexes, in this introduction we will restrict attention to CW complexes satisfying a combinatorial condition modelled on the notion of convexity. The reader may think only of simplicial complexes, but it is insightful to allow more general convex cells.

Definition 0.1. *A regular CW complex M is quasiconvex if for each pair of p-cells α_1, α_2, if $\overline{\alpha}_1 \cap \overline{\alpha}_2$ contains a $(p-1)$-cell β, then $\overline{\alpha}_1 \cap \overline{\alpha}_2 = \overline{\beta}$. For example, all simplicial complexes and all polyhedral complexes are quasiconvex. From now on, by a quasiconvex complex we will always mean a compact regular CW complex which is quasiconvex.*

Let M be a cell complex. If α and β are cells of M, we will write $\alpha < \beta$ or $\beta > \alpha$ if α is contained in the boundary of β. Throughout this paper, a primary role will be played by the notion of a *neighbor*.

Definition 0.2. *If α_1 and α_2 are p-cells of M, say α_1 and α_2 are neighbors if*

(1) *α_1 and α_2 share a $(p+1)$-cell, that is, there is a $(p+1)$-cell β with $\beta > \alpha_1$ and $\beta > \alpha_2$,*
or
(2) *α_1 and α_2 share a $(p-1)$-cell, that is, there is a $(p-1)$-cell γ with $\gamma < \alpha_1$ and $\gamma < \alpha_2$.*

We say that α_1 and α_2 are parallel neighbors if either (1) or (2) is true but not both. If both (1) and (2) are true, we say α_1 and α_2 are transverse neighbors.

The example shown in Figure 1 illustrates the origin of these terms. This is an illustration of the case $p = 1$. The edge e_0 has 8 neighbors (e_1 through e_8). Of these, 4 are parallel neighbors (e_2, e_4, e_6 and e_8). The edges e_2 and e_6 share a vertex with e_0 but not a face, and the edges e_4 and e_8 share a face with e_0 but not a vertex.

We are now ready to state the main new definition of this paper.

Definition 0.3. *For each p, define the p^{th} curvature function*

$$\mathcal{F}_p : \{p\text{-}cells\} \to \mathbb{R}$$

as follows. For any p-cell α, set

$$\mathcal{F}_p(\alpha) = \#\{(p+1)\text{-}cells\ \beta > a\} + \#\{(p-1)\text{-}cells\ \gamma < a\} - \#\{parallel\ neighbors\ of\ \alpha\}.$$

This is actually a special case of our main definition. As explained later, one is permitted to assign a positive weight to each cell. The formula displayed in Definition 0.3 is the curvature in the case that every cell has been assigned a weight of 1. If each cell α has been assigned a weight $\omega_\alpha > 0$, then for any p-cell α,

$$\mathcal{F}_p(\alpha) = w_\alpha \Bigg\{ \Bigg[\sum_{\beta^{(p+1)} > \alpha} \frac{w_\alpha}{w_\beta} + \sum_{\gamma^{(p-1)} < \alpha} \frac{w_\gamma}{w_\alpha} \Bigg] - \sum_{\tilde{\alpha}^{(p)} \neq \alpha} \Bigg| \sum_{\substack{\beta^{(p+1)} > \alpha \\ \beta > \tilde{\alpha}}} \frac{\sqrt{\omega_\alpha \omega_{\tilde{\alpha}}}}{\omega_\beta} - \sum_{\substack{\gamma^{(p-1)} < \alpha \\ \gamma < \tilde{\alpha}}} \frac{\omega_\gamma}{\sqrt{\omega_\alpha \omega_{\tilde{\alpha}}}} \Bigg| \Bigg\}.$$

where $\sum_{\beta^{(p+1)} > \alpha}$ denotes the sum over all $(p+1)$-cells β which have α as a face, and all other summations are defined similarly (see Theorem 3.10). See Theorem 2.1 for the formula in the case of a general CW complex.

The first point of this definition is the following combinatorial analogue of "Bochner's Theorem" ([Bo1,2,3]). Note that we will label all theorems with the label with which they appear in the text.

Theorem 2.8. *Let M be a quasiconvex complex. Suppose, for some p, that $\mathcal{F}_p(\alpha) > 0$ for each p-cell α. Then*

$$H_p(M, \mathbb{R}) = 0.$$

Of particular interest is the case $p = 1$. In this case we give the curvature function a special name. For any edge (i.e. 1-cell) e of M, define the *Ricci curvature* of e by

$$\mathrm{Ric}(e) = \mathcal{F}_1(e).$$

More explicitly,

$$\mathrm{Ric}(e) = \#\{\text{2-cells } f > e\} + 2 - \#\{\text{parallel neighbors of } e\}.$$

For example, in Figure 1

$$\mathrm{Ric}(e_0) = 2 + 2 - 4 = 0.$$

Thus, Figure 0.1 is a "Ricci flat" cell complex. In Figure 0.2(i) we show a 2-dimensional sphere with "cubic" cell decomposition. In this case the edge e_0 has 2 parallel neighbors (the edges e_1 and e_2) so that

$$\mathrm{Ric}(e_0) = 2 + 2 - 2 = 2.$$

One purpose of Figure 0.2(ii) is to demonstrate that the definitions and theorems discussed in this paper can, with a few exceptions, be applied to very general complexes. The complex need not be a manifold, or have any other special structure. In Figure 0.2(ii) we have indicated the Ricci curvature of each edge.

In the Riemannian case Bochner showed that one does not need the p^{th} curvature function to be strictly positive, as we required in Theorem 2.8, in order to draw some

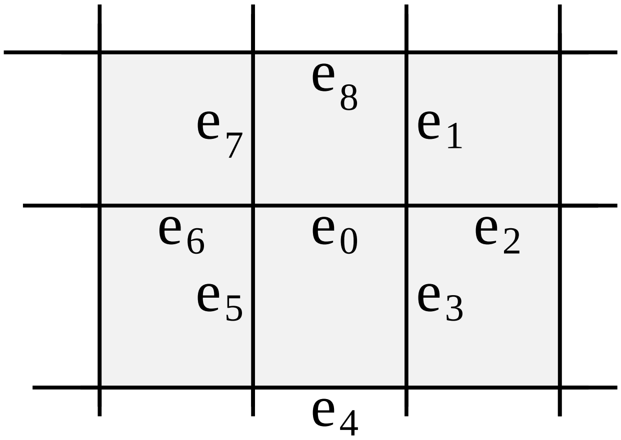
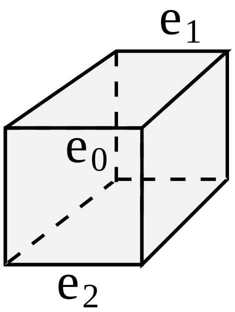
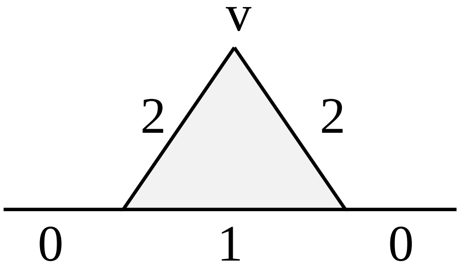


Figure 0.1



(i)



(ii)

Figure 0.2

topological conclusions. In fact, non-negativity also has strong implications. In the combinatorial setting, we can prove a corresponding result in the case p=1.

Theorem 4.4. *Let M be a connected quasiconvex complex.*
(i) Suppose that $\mathrm{Ric}(e) \geq 0$ *for all edges e of M. If there is a vertex v such that* $\mathrm{Ric}(e) > 0$ *for all $e > v$ then*

$$H_1(M, \mathbb{R}) = 0.$$

(ii) Suppose that M is a combinatorial n-manifold, $n \leq 3$, and $\mathrm{Ric}(e) \geq 0$ *for all edges of M, then*

$$\dim H_1(M, \mathbb{R}) \leq n.$$

This theorem is true for all $n \geq 0$ with an additional technical hypothesis. In particular, we have the following theorem.

Theorem 4.6. *Suppose that M is a combinatorial n-manifold and* $\mathrm{Ric}(e) \geq 0$ *for all edges of M. If M^*, the dual complex to M, contains an n-simplex or an n-cube, then*

$$\dim H_1(M, \mathbb{R}) \leq n.$$

Applying this theorem to M^*, and using Poincaré duality, one can restate this theorem as follows.

Theorem 4.7. *Suppose that M is a combinatorial n-manifold and $\mathcal{F}_{n-1}(\alpha) \geq 0$ for all $(n-1)$-cells α. If M contains an n-simplex or an n-cube, then*

$$\dim H_1(M, \mathbb{R}) \leq n.$$

We will later indicate the proof of these results. We conjecture that, just as in the Riemannian setting, Theorem 4.6 holds for general p (with the right hand side of the inequality in (ii) replaced by $\binom{n}{p}$). However, we have run into interesting obstacles in trying to prove Theorem 5 in the desired generality. We will later give an idea as to how $p = 1$ differs from the general case. For a more complete discussion of this point, see Section 5.

It is interesting to see how these theorems apply to the very simple complexes shown in Figures 0.1 and 0.2. In the case of the 2-sphere S^2 given the cell decomposition illustrated in Figure 0.2(i), each edge has Ricci curvature 2. Thus we may apply Theorem 2.8 to conclude

$$H_1(S^2, \mathbb{R}) = 0.$$

Each edge of the complex M in Figure 0.2(ii), has Ricci curvature ≥ 0. Moreover, every edge $e > v$ has positive Ricci curvature. In this case, Theorem 4.6(i) implies

$$H_1(M, \mathbb{R}) = 0.$$

The 2-dimensional torus T^2 can be given a cell decomposition which looks everywhere like the cell complex shown in Figure 0.1. With this cell structure T^2 is a connected combinatorial 2-manifold. Since every edge has Ricci curvature $= 0$, we may apply Theorem 5(ii) to learn

$$\dim H_1(T^2, \mathbb{R}) \leq 2.$$

More generally, for every n, the n-dimensional torus T^2 can be given such a cubic cell decomposition decomposition. In this case, T^n will be a connected combinatorial n-manifold, with every edge having Ricci curvature 0. From Theorem 4.6 (or Theorem 4.7) we may conclude

$$\dim H_1(T^n, \mathbb{R}) \leq n.$$

Since, in fact, $\dim H_1(T^n, \mathbb{R}) = n$, we see that the conclusion of this combinatorial Bochner theorem, just as the conclusion of the classical Bochner theorem, is sharp.

In [My], a precursor to Bochner's work, Myers proves a theorem (in the Riemannian setting) which is a strengthening of Theorem 2.8 in the case $p = 1$. The analogous result holds in the combinatorial setting.

Theorem 6.1. *Let M be a quasiconvex complex. Suppose that for every edge e,* $\mathrm{Ric}(e) > 0$. *Then $\pi_1(M)$ is finite.*

In [Sto1,2] a different notion of curvature is defined, in a somewhat more restricted setting, from which a version of Myers' theorem is deduced. The proof of Theorem 7 is very similar to the proof in [Sto1,2] (and the proof in [My]), but the relationship between the two notions of combinatorial curvature is not clear to this author at present.

These theorems show that there are strong topological restrictions for a topological space to have a quasiconvex cell decomposition with $\mathrm{Ric}(e) > 0$ (or ≥ 0) for each edge e. It is natural to consider the other extreme, and to investigate the possibility of decompositions with $\mathrm{Ric}(e) < 0$ for every edge e. In [Gao],[GY] and [Lo1,2] it was shown that every smooth manifold of dimension ≥ 3 has a Riemannian metric with everywhere negative Ricci curvature. The same, and more, is true in the combinatorial setting.

Theorem 7.2. *Let M be a (not necessarily compact) combinatorial n-manifold with $n \geq 2$. Then there is a finite subdivision M^* of M such that* $\mathrm{Ric}(e) < 0$ *for all edges e of M^*.*

In dimension $n = 2$, the Gauss-Bonnet Theorem provides an obstruction to the existence of a Riemannian metric with negative Ricci (= Gauss if $n = 2$) curvature. Theorem 7.2 implies that there is no simple Gauss-Bonnet theorem for combinatorial Ricci curvature.

In Section 8, we focus attention on the important special case of quasicomplex complexes in which every cell is a combinatorial cube. In this case, the formulas simplify somewhat, and we have the following result.

Theorem 8.1. *Let M be a quasiconvex complex of cubes, and endow every cell with the weight 1, then for any p-cell α,*

$$\mathcal{F}_p(\alpha) = 2p(2 + \deg(\alpha)) - \sum_{\gamma^{(p-1)} < \alpha} \deg(\gamma),$$

where $\deg(\alpha^{(p)}) = \#\{\beta^{(p+1)} > \alpha\}$.

In the case that the quasiconvex complex of cubes is a topological manifold, there is a more classical notion of curvature, defined by comparing $\deg(\alpha^{(p)})$ to the corresponding degree in the case of the standard cubic decomposition of $\mathbb{R}^n$ (see, e.g. [AR]). In particular, say M has positive (resp. nonnegative, nonpositive, negative) $(n-2)$-curvature if $\deg(\alpha) < 4$ (resp. $\leq 4, \geq 4, > 4$) for each $(n-2)$-cell α. From Theorem 9 we easily see that

Coroallary 8.2. *Let M be a quasiconvex complex of cubes which is homeomorphic to a topological n-manifold. Suppose that M has positive (resp. nonnegative, nonpositive, negative) $(n-2)$-curvature. Then $\mathcal{F}_{n-1} > 0$ (resp. $\geq 0, \leq 0, < 0$).*

Corollary 8.3. *Let M be a quasiconvex complex of cubes which is homeomorphic to a topological n-manifold. Suppose that M has positive (resp. nonnegative) $(n-2)$-curvature. Then* $\dim H_1(M, \mathbb{R}) = 0$ *(resp. $\leq n$).*

We will now give an idea of the proofs of Theorems 2.8 and 4.4. The readers familar with Bochner's original proof will notice that we are simply following his proof (now known as

"Bochner's method") in a combinatorial setting. Let M be a finite CW complex. Consider the cellular chain complex

$$0 \longrightarrow C_n(M,\mathbb{R}) \xrightarrow{\partial} C_{n-1}(M,\mathbb{R}) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(M,\mathbb{R}) \longrightarrow 0$$

Endow each $C_p(M,\mathbb{R})$ with a positive definite inner product such that the cells of M are orthogonal. This involves choosing a positive weight ω_α for each cell α and setting

$$\langle \alpha, \alpha \rangle = \omega_\alpha.$$

Let $\partial^* : C_p(M,\mathbb{R}) \longrightarrow C_{p+1}(M,\mathbb{R})$ denote the adjoint of ∂ and

$$\Box_p = \partial\partial^* + \partial^*\partial : C_p(M,\mathbb{R}) \longrightarrow C_p(M,\mathbb{R})$$

the corresponding Laplacian. The combinatorial Hodge theorem

$$\operatorname{Ker} \Box_p \cong H_p(M,\mathbb{R})$$

follows from basic linear algebra.

Following Bochner's proof, our next step is to develop a Bochner-Weitzenböck formula for $\Box_p$, the combinatorial Laplace operator. The Bochner-Weitzenböck formula for the Riemannian Laplace operator states

$$\Box_p = (\nabla_p)^* \nabla_p + F_p$$

where now $\Box_p$ is the Riemannian Laplace operator on p-forms on a compact Riemannian manifold, ∇_p is the (Levi-Civita) covariant derivative operator on $\Omega^p(M)$, and F_p is a 0^{th} order operator whose value at $m \in M$ depends only on derivatives of the Riemannian metric at m. In teh combinatorial setting, this is rather mysterious, since we begin by knowing neither a combinatorial analogue for $(\nabla_p^*)\nabla_p$ nor for F_p. However, we show that there is, in fact, a canonical decomposition

$$\Box_p = B_p + F_p$$

where B_p is a nonnegative operator and, when expressed as a matrix with respect to a basis of $C_p(M,\mathbb{R})$ consisting of the p-cells of M, F_p is diagonal. To describe this decomposition, we begin by defining a decomposition of any $n \times n$ symmetric matrix. Rather than writing a general definition, this is done in Section 1, we will illustrate the decomposition in the case of a symmetric 3×3 matrix.

$$\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} = \begin{pmatrix} |b|+|c| & b & c \\ b & |b|+|e| & e \\ c & e & |c|+|e| \end{pmatrix} + \begin{pmatrix} a-(|b|+|c|) & 0 & 0 \\ 0 & d-(|b|+|e|) & 0 \\ 0 & 0 & f-(|c|+|e|) \end{pmatrix} \tag{0.1}$$

$$= B + F.$$

The linear map defined by the matrix B is nonnegative (with respect to the standard inner product on R^3). This is a special case of the fact that any symmetric matrix $(a_{i,j})$ such that $a_{i,i} \geq \sum_{j \neq i} |a_{i,j}|$ for each i is nonnegative. This is one of those very useful facts in matrix theory which is constantly being rediscovered. For the history of this result (not stated in quite the same way) and extensions see [Ta].

The nonnegativity of B implies that in this simple example we can already make the Bochner-like statement that if each of the 3 numbers $a-(|b|+|c|)$, $d-(|b|+|e|)$, $f-(|c|+|e|)$ is positive then the original 3×3 matrix has a trivial kernel.

To apply this decomposition to the combinatorial Laplace operator, we must first represent the Laplace operator as a symmetric matrix, and this requires choosing an ordered orthonormal basis for the space of chains. We can define such a basis by choosing an orientation for each cell of M, and then choosing an ordering of the cells. Making a different choice for the orientation of a cell has the effect of multiplying the corresponding row and column of the symmetric matrix by -1. Permuting the ordering of the cells has the effect of applying the same permutation to both the rows and the columns of the matrix. Therefore, for a decomposition of symmetric matrices to induce a well-defined decomposition of the combinatorial Laplace operator, the decomposition must be equivariant under multiplying a row and column by -1, and applying a single permutation to both the rows and the columns. It is easy to see that the decomposition shown in (0.1) has this property, and hence induces a decomposition $\Box_p = B_p + F_p$. In analogy with the Bochner-Weitzenböck formula, we call B_p the combinatorial Bochner Laplacian, and F_p the p^{th} combinatorial curvature function.

It follows immediately that if $F_p > 0$ then $\Box_p > 0$ so that $H_p(M, \mathbb{R}) = 0$. For any p-cell α, we define

$$\mathcal{F}_p(\alpha) = < F_p(\alpha), \alpha > .$$

In the Riemmannian setting, for $p > 1$ the curvature functions $\mathcal{F}_p$ are rather mysterious. On the other hand, $\mathcal{F}_1$ is equal to the Ricci curvature (i.e. for every $\omega \in T^*M, \mathcal{F}_1(\omega) = Ric(\omega)$). By analogy we define combinatorial Ricci curvature by setting, for each edge e of M

$$\text{Ric}(e) = \mathcal{F}_1(e). \tag{0.2}$$

We note that (0.2) is a theorem in the Riemannian setting, but a definition in the combinatorial case. Proving Theorem 2.8 is now reduced to explicitly calculating $\mathcal{F}_p$ as defined above, and showing that this is equivalent to the formula given in Definition 3.

In the Riemannian case Theorem 4.4 follows from the Bochner-Weitzenböck formula with just a bit more work. Namely, if F_p is a nonnegative operator, then since $(\nabla_p)^* \nabla_p$ is also a nonnegative operator, we have

$$\text{Ker}(\Box_p) = \text{Ker}((\nabla_p)^* \nabla_p) \cap \text{Ker}(F_p).$$

We note that

$$\text{Ker}((\nabla_p)^* \nabla_p) = \text{Ker}(\nabla_p).$$

Any $\omega \in \mathrm{Ker}(\nabla_p)$ is said to be *parallel*, and is completely determined by its value at any one point. This is essentially all that is required to complete the proof.

The proofs of Theorem 5(i) and (ii) in the combinatorial case, just as in the Riemannian case, would follow if we knew that a 1-chain $c \in \mathrm{Ker}\, B_1$ is completely determined by its values in some neighborhood in M, that is, if there were a unique continuation theorem for 1-chains in $\mathrm{Ker}\, B_1$. This is not quite true. However, if $c \in \mathrm{Ker}\,\Box_1 \cap \mathrm{Ker}\, B_1$, then c is, in fact, determined by its value in a neighborhood of any vertex. This is sufficient for the proof of Theorem 5(i) and (ii) . It can also be shown that for $p > 1$, p-chains $c \in \mathrm{Ker}\,\Box_p \cap \mathrm{Ker}\, B_p$, in drastic contradiction to the Riemannian case, are not determined by local information. This explains the difficulty in extending Theorem 5 to the case of general p (see Section 5).

We briefly indicate the proof of the unique continuation theorem for $c \in \mathrm{Ker}\,\Box_1 \cap \mathrm{Ker}\, B_1$. Consider, for example, a vector $x = (x_1, x_2, x_3)$ in the kernel of the matrix B in (0.1). It is not too hard to see that if $b \neq 0$ then x_1 determines x_2. Namely,

$$x_2 = -(sign(b))x_1.$$

More generally, if $x \in \mathrm{Ker}\, B$, and the $(i,j)^{th}$ entry of B is nonzero, then x_i determines x_j (so that x_i is zero if and only if x_j is zero). See Section 1 for proofs of these facts.

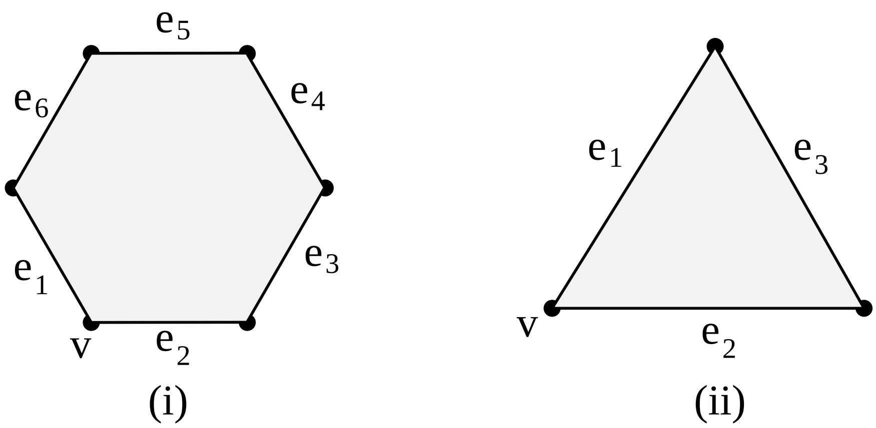


Figure 0.3

In the case of the Laplace operator, if α_1 and α_2 are oriented p-cells, then the entry of B_p corresponding to α_1 and α_2 is nonzero if α_1 and α_2 are parallel neighbors. Therefore, if $c = \sum_{\alpha^{(p)}} c_\alpha \alpha \in \mathrm{Ker}\, B_p$ (where we have arbitrarily chosen an orientation for each p-cell), and α_1 and α_2 are parallel neighbors, then c_{α_1} determines c_{α_2}. Consider the polygon shown in Figure 3(i). Every side of the polygon is a parallel neighbor of either e_1 of e_2. This is the case whenever the polygon has at least 4 sides. Therefore, if $c = \sum_e c_e e \in \mathrm{Ker}\, B_1$, then the value of c on the boundary of the polygon is completely determined by c_{e_1} and c_{e_2}. This argument doesn't quite work if the polygon has 3 sides, as in Figure 0.3 (ii).

Here, e_3 is a transverse neighbor of both e_1 and e_2. However, if we knew that $c \in \operatorname{Ker} \partial^*$, then c_{e_3} would be determined by c_{e_1} and c_{e_2}. Putting these ideas together, it is not too hard to get to the general result that if M is connected, and $c \in \operatorname{Ker} \partial^* \cap \operatorname{Ker} B_1$, then for any vertex v, c is completely determined by c_e for $e > v$. Since

$$\operatorname{Ker} \square_p = \operatorname{Ker} \partial \cap \operatorname{Ker} \partial^* \subset \operatorname{Ker} \partial^*$$

the desired theorem follows.

To see how Theorem 5(i) follows, suppose $\operatorname{Ric}(e) \geq 0$ for each edge e. Then, since B_1 is also a nonnegative operator,

$$\operatorname{Ker}(\square_1) = \operatorname{Ker}(B_1) \cap \operatorname{Ker}(\operatorname{Ric}) = (\operatorname{Ker}(B_1) \cap \operatorname{Ker}(\square_1)) \cap \operatorname{Ker}(\operatorname{Ric})$$

$$= (\operatorname{Ker}(B_1) \cap \operatorname{Ker}(\partial^*)) \cap \operatorname{Ker}(\operatorname{Ric}).$$

Now suppose $c = \sum_e c_e e \in \operatorname{Ker}(\square_p)$. Then $c \in \operatorname{Ker}(\operatorname{Ric})$. Since $\operatorname{Ric}(e) > 0$ for each edge $e > v$, we must have $c_e = 0$ for each edge $e > v$. Now we observe that since $c \in \operatorname{Ker}(B_1) \cap \operatorname{Ker}(\partial^*)$, c is completely determined by c_e for $e > v$, and hence we must have $c = 0$. This demonstrates that $H_1(M, \mathbb{R}) = 0$.

This paper is organized as follows. In §1 we present the combinatorial Weitzenböck formula. In §2 we apply the formula to the combinatorial Laplace operator and deduce Theorem 2.8 for general cell complexes. In §3 we specialize to polyhedral complexes and derive the formula stated in Definition 3. In §4 we proof a unique continuation theorem for harmonic 1-chains and deduce the more refined versions of Bochner's Theorem, i.e. Theorems 4.4 and 4.5. In §5 we demonstrate that there is no such unique continuation result for harmonic p-chains for general p. In §6 we prove the combinatorial Myers' Theorem. In §7 we investigate the existence of cell decompositions with everywhere negative Ricci curvature. Lastly, in §8, we focus attention on the case of quasiconvex complexes of cubes.

§1. The Combinatorial Weitzenböck Formula.

Our goal in this section is to present a simple Weitzenböck-type decomposition of a symmetric matrix. In the next section we will apply this idea to the combinatorial Laplace operator.

Let $A = (A_{ij})$ be a symmetric $n \times n$ matrix.

Definition 1.1. *Let $B(A) = (B_{ij})$ denote the symmetric $n \times n$ matrix whose entries are given by*

$$B_{ij} = \begin{cases} A_{ij} & \text{if } i \neq j \\ \sum_{j \neq i} |A_{ij}| & \text{if } i = j \end{cases}$$

and $F(A) = (F_{ij})$ the $n \times n$ diagonal matrix whose entries are given by

$$F_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ A_{ij} - \sum_{j \neq i} |A_{ij}| & \text{if } i = j \end{cases}.$$

We will refer to $B(A)$ as the Bochner matrix associated to A and $F(A)$ as the curvature matrix associated to A. Moreover, we refer to the identity

$$A = B(A) + F(A)$$

as the Weitzenböck decomposition of A.

The goal of this section is to study the properties of this decomposition. The fundamental property is that $B(A)$ is a nonnegative definite matrix. This sort of result is standard in matrix theory (see, e.g. [Ta]), but we include a proof here, as the technique used in the proof will appear again later.

Definition 1.2. *Say a symmetric $n \times n$ matrix B is strongly nonnegative if, for each $1 \leq i \leq n$*

$$B_{ii} \geq \sum_{j \neq i} |B_{ij}|.$$

Note that for every A, the associated Bochner matrix $B(A)$, is strongly nonnegative.

Theorem 1.3. *Suppose B is strongly nonnegative. Then B is nonnegative definite, i.e., $\forall v \in \mathbb{R}^n$ $\langle Bv, v \rangle \geq 0$.*

Proof: Suppose B is not nonnegative definite. Since B is symmetric it is diagonalizable. Thus, there must be a $\lambda < 0$ and a nonzero vector $v \in \mathbb{R}^n$ such that $Bv = \lambda v$. The contradiction will be reached via a maximum principle (which we appear again in the proof of Theorem 1.6).

Let

$$|v_i| = \max_{1 \leq j \leq n} |v_j|.$$

Since $v \neq 0$, $|v_i| > 0$. Multiplying v by -1 if necessary, we may assusme that $v_i > 0$. Then

$$(B - \lambda)v = 0$$

so

$$\begin{aligned}
0 = [(B-\lambda)v]_i &= \sum_j (B-\lambda)_{ij} v_j \\
&= (B_{ii} - \lambda)v_i + \sum_{j \neq i} B_{ij} v_j \\
&> B_{ii} v_i + \sum_{j \neq i} B_{ij} v_j \\
&\geq \left(\sum_{j \neq i} |B_{ij}| \right) v_i - \sum_{j \neq i} |B_{ij}| \, |v_j| \\
&\geq \left(\sum_{j \neq i} |B_{ij}| \right) v_i - \sum_{j \neq i} |B_{ij}| \, |v_i| \\
&= 0.
\end{aligned}$$

This is a contradiction. $\square$

As an immediate corollary, we have a Bochner-type theorem.

Corollary 1.4. (i) *Suppose $F(A) > 0$ (i.e., $F_{ii} > 0$ for each i) then*

$$\operatorname{Ker} A = 0.$$

(ii) *Suppose $F(A) \geq 0$ (i.e., $F_{ii} \geq 0$ for each i) then*

$$\operatorname{Ker} A = \operatorname{Ker} B(A) \cap \operatorname{Ker} F(A).$$

Part (i) of the above corollary will be sufficient for the proof of Theorem 4. However, for the more refined Theorem 5, we will require a more detailed understanding of the kernel of $B(A)$.

Definition 1.5. *Let B be a symmetric nxn matrix. For $1 \leq i \neq j \leq n$ say i and j are B-neighbors if $B_{ij} \neq 0$. Let $\sim$ denote the equivalence relation generated by the B-neighbor relation. That is, $i \sim i$, and $i \sim j$ if and only if there exist $i = k_0, k_1, \ldots, k_\ell = j$ such that for each $1 \leq r \leq \ell - 1$ k_r and k_{r+1} are neighbors. Let $\mathcal{C}(B)$ denote the set of equivalence classes of $\{1, 2, \ldots, n\}/\sim$ and let $\mathcal{N}(B) = \#\mathcal{C}(B)$. Note that if $B = B(A)$ for some A, then the A-neighbor relation is equivalent to the B-neighbor relation.*

Theorem 1.6. *Let B be a strongly nonnegative symmetric matrix. Let $\sim$ denote the B-neighbor equivalence relation on $\{1, 2, \ldots, n\}$. Then, with the above notation*

(i) $\operatorname{Dim} \operatorname{Ker} B \leq \mathcal{N}(B)$.

(ii) *Suppose $v = (v_1, \ldots, v_n) \in \operatorname{Ker} B$. Then if $B_{ij} \neq 0$*

$$v_i = -(\mathit{sign}\ B_{ij}) v_j.$$

In particular, let $c \in \mathcal{C}(B)$ be an equivalence class

(iii) *If $i \in c$ then, for all $j \in c$, v_j is completely determined by v_i.*

(iv) *If $v_i = 0$ for some $i \in c$, then $v_j = 0$ for all $j \in c$.*

Proof: Let $0 \neq v \in \operatorname{Ker} B$. Choose an equivalence class c, and let

$$|v_i| = \max_{j \in c} |v_j|.$$

Multiplying v by -1 if necessary, we may assume $v_i \geq 0$. Then $Bv = 0$ implies

$$\begin{aligned}
0 = (Bv)_i &= \sum_j B_{ij} v_j \\
&= B_{ii} v_i + \sum_{j \neq i} B_{ij} v_j \\
&\geq \sum_{j \neq i} |B_{ij}| v_i + \sum_{j \neq i} B_{ij} v_j \\
&\geq \sum_{j \neq i} |B_{ij}| v_i - \sum_{j \neq i} |B_{ij}|\ |v_j| \qquad (1.1) \\
&\geq \sum_{j \neq i} |B_{ij}| v_i - \sum_{j \neq i} |B_{ij}| v_i \qquad (1.2) \\
&= 0.
\end{aligned}$$

For this to be true, we must have equality at (1.1) and (1.2). Equality at (1.1) implies

$$\text{For each } \ j \neq i \quad B_{ij} v_j \leq 0. \tag{1.3}$$

Equality at (1.2) implies

$$\text{For each } i \neq j, \text{ if } B_{ij} \neq 0 \text{ then } |v_j| = v_i. \tag{1.4}$$

Together, (1.3) and (1.4) imply (ii), (iii) and (iv) and hence (i). $\square$

We may now combine this result with Corollary 1.3 to learn

Corollary 1.7. *Suppose $F(A) \geq 0$*

(i) *If, in addition, for each equivalence class c there is an $i \in c$ with $F_{ii} > 0$, then*

$$\operatorname{Ker} A = 0.$$

(ii) *Let $\mathcal{N}^0(A)$ denote the number of A-equivalence classes $c \in \mathcal{C}(A)$ such that $F_{ii} = 0$ for all $i \in c$. Then*

$$\dim \operatorname{Ker} A \leq \mathcal{N}^0(A).$$

There are many decompositions $A = B'(A) + F'(A)$ which satisfy the properties we have used so far. However, the Weitzenböck decomposition we have defined has an additional crucial property, which implies that it induces a decomposition of the combinatorial Laplace operator. Namely, let E_{ij} denote the $n \times n$ matrix which exchanges the i^{th} and j^{th} entries of a vector, i.e.,

$$(E_{ij})_{k\ell} = \begin{cases} \delta_{k\ell} & \text{if } \{k,\ell\} \neq \{i,j\} \\ 1 & \text{if } \{k,\ell\} = \{i,j\} \end{cases}.$$

Say a matrix T is simple if one of the following is true.

(i) $T = E_{ij}$ for some i and j.

(ii) T is a diagonal matrix with all diagonal entries equal to ± 1.

Say a matrix $\mathcal{U}$ is a *change of choices* if $\mathcal{U}$ can be expressed as a product of simple matrices. Then, for all $n \times n$ symmetric matrices A and all change of choices $\mathcal{U}$, it is easy to see that

$$\begin{aligned} B(\mathcal{U}A\mathcal{U}^{-1}) &= \mathcal{U}(B(A))\mathcal{U}^{-1} \\ F(\mathcal{U}A\mathcal{U}^{-1}) &= \mathcal{U}(F(A))\mathcal{U}^{-1}. \end{aligned} \tag{1.5}$$

This has the following significance. Let V be a vector space of dimension n with an inner product, and $L : V \to V$ a symmetric linear map. Suppose that we are given an (unordered) set $\{\pm\omega_1, \pm\omega_2, \ldots, \pm\omega_n\} \subset V$ such that $\omega_1, \ldots, \omega_n$ are orthogonal. Then we can define a Weitzenböck decomposition

$$L = B(L) + F(L).$$

Namely, choose an orthonormal basis $\{v_1, \ldots, v_n\}$ of V, with each $v_i = \pm\omega_j$ for some j. Let A be the matrix of L with respect to this basis. Then A is a symmetric matrix. Let $B(L)$ be the linear map represented by the matrix $B(A)$, and $F(L)$ the linear map represented by the matrix $F(A)$. Choosing a different such basis, i.e., reordering the ω_j's and multiplying some by -1, is equivalent to acting on the original basis by a change of choices matrix $\mathcal{U}$. This has the effect of conjugating A by $\mathcal{U}$. From (1.5) the result is to conjugate $B(A)$ and $F(A)$ by the same change of basis matrix, and hence the linear maps $B(L)$ and $F(L)$ are well-defined.

One may think of this phenomenon as the combinatorial equivalent of the fact that in Riemannian geometry the curvature is often defined with respect to a choice of local coordinates, but is independent of the choice.

§2. The Combinatorial Bochner Method: General Cell Complexes.

In this section we let M be a general finite CW complex. Beginning in §3, we will make our results more precise in the case that M has some additional structure. Denote by $C_p = C_p(M, \mathbb{R})$ the oriented real chains of M. The (real-) homology of M is the homology of the chain complex

$$\mathcal{C}_* : 0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

where $n = \dim M$, and $\partial_p : C_p \to C_{p-1}$ denotes the standard boundary operator.

Choose a positive weight $w_\alpha > 0$ for each cell α, and endow each C_p with an inner product, which we denote by $\langle\ ,\ \rangle_p$, by declaring the p-cells to be orthogonal, and setting for each p-cell α

$$\langle \alpha, \alpha \rangle_p = \omega_\alpha.$$

We can now define the adjoint operator

$$\partial_p^* : C_p \longrightarrow C_{p+1}$$

by

$$\langle \partial_{p+1} c_{p+1}, c_p \rangle_p = \langle c_{p+1}, \partial_p^* c_p \rangle_{p+1}$$

for each $c_p \in C_p$ and $c_{p+1} \in C_{p+1}$. This leads us to the combinatorial Laplace operator

$$\Box_p = \partial_{p+1}\partial_p^* + \partial_{p-1}^*\partial_p : C_p \longrightarrow C_p. \tag{2.1}$$

Standard linear algebra implies

$$\operatorname{Ker} \Box_p \cong \frac{\operatorname{Ker} \partial_p}{\operatorname{Im} \partial_{p+1}} \cong H_p(M, \mathbb{R}).$$

Our next goal is to find a more explicit representation of $\Box_p$. For any cell α of M, we write $\alpha^{(p)}$ if $\dim(\alpha) = p$. For cells α and β we write $\alpha < \beta$ (or $\beta > \alpha$) if $\alpha \subseteq \dot{\beta}$ $(= \overline{\beta} - \beta)$ and say α is a face of β.

Choose an orientation for each cell. For any $(p+1)$-cell β we can write

$$\partial \beta = \sum_{\alpha^{(p)}} \epsilon_{\alpha\beta}\, \alpha$$

(we will leave the subscript off of ∂ whenever it will not cause confusion) where $\epsilon_{\alpha\beta}$ is the incidence number of β relative to α. Note that $e_{\alpha\beta} = 0$ unless $\alpha < \beta$. For each oriented p-cell α and $(p+1)$-cell β

$$\langle \partial^* \alpha, \beta \rangle_{p+1} = \langle \alpha, \partial \beta \rangle_p = \left\langle \alpha, \sum_{\tilde{\alpha}^{(p)}} \epsilon_{\tilde{\alpha}\beta} \tilde{\alpha} \right\rangle = \sum_{\tilde{\alpha}^{(p)}} \epsilon_{\tilde{\alpha}\beta} \langle \alpha, \tilde{\alpha} \rangle = \epsilon_{\alpha\beta} w_\alpha.$$

Thus

$$\partial^* \alpha^{(p)} = \sum_{\beta^{(p+1)}} \epsilon_{\alpha\beta} \frac{w_\alpha}{w_\beta} \beta.$$

Plugging this expression into (2.1) we see that for any p-cell α_1

$$\Box_p \alpha_1 = \sum_{\alpha_2^{(p)}} \left[\sum_{\beta^{(p+1)}} \epsilon_{\alpha_1\beta}\epsilon_{\alpha_2\beta} \frac{w_{\alpha_1}}{w_\beta} + \sum_{\gamma^{(p-1)}} \epsilon_{\gamma\alpha_1}\epsilon_{\gamma\alpha_2} \frac{w_\gamma}{w_{\alpha_2}} \right] \alpha_2.$$

We observe that $\Box_p$ is a self-adjoint operator with respect to the inner product $\langle\ ,\ \rangle_p$. It is natural to express $\Box_p$ as a matrix with respect to a basis of C_p that is orthonormal with respect to the inner product. With this in mind, for each p-cell α, with its chosen orientation, define

$$\alpha^* = \frac{1}{\sqrt{w_\alpha}} \alpha.$$

Then the α^*'s form an orthonormal basis. Let N_p^* denote the set of normalized p-cells. For $\alpha_1^* \in N_p^*$ we have

$$\Box_p \alpha_1^* = \sum_{\alpha_2^* \in N_p^*} \left[\sum_{\beta^{(p+1)}} \epsilon_{\alpha_1\beta}\epsilon_{\alpha_2\beta} \frac{\sqrt{w_{\alpha_1} w_{\alpha_2}}}{w_\beta} + \sum_{\gamma^{(p-1)}} \epsilon_{\gamma\alpha_1}\epsilon_{\gamma\alpha_2} \frac{w_\gamma}{\sqrt{w_{\alpha_1} w_{\alpha_2}}} \right] \alpha_2^*.$$

We can think of the Laplace operator $\Box_p$ as a symmetric matrix, with the rows and columns indexed by the elements of N_p^*. For α_1 and α_2 in N_p^*

$$\begin{aligned} \Box_p(\alpha_1, \alpha_2) &= \Box_p(\alpha_2, \alpha_1) \\ &= \sum_{\beta^{(p+1)}} \epsilon_{\alpha_1\beta}\epsilon_{\alpha_2\beta} \frac{\sqrt{w_{\alpha_1} w_{\alpha_2}}}{w_\beta} + \sum_{\gamma^{(p-1)}} \epsilon_{\gamma\alpha_1}\epsilon_{\gamma\alpha_2} \frac{w_\gamma}{\sqrt{w_{\alpha_1} w_{\alpha_2}}}. \end{aligned} \tag{2.2}$$

Following the recipe described in §1, we now define 2 new operators, $B_p = B(\Box_p)$, the combinatorial Bochner Laplacian, and $F_p = F(\Box_p)$, the p^{th} combinatorial curvature operator, in matrix form, by

$$\begin{aligned} B_p(\alpha_1^*, \alpha_2^*) &= \begin{cases} \Box_p(\alpha_1^*, \alpha_2^*) & \text{if } \alpha_1^* \neq \alpha_2^* \\ \sum_{\tilde{\alpha}^* \neq \alpha_1^*} \mid \Box_p(\alpha_1^*, \tilde{\alpha}^*) \mid & \text{if } \alpha_1^* = \alpha_2^* \end{cases} \\ F_p(\alpha_1^*, \alpha_2^*) &= \Box_p(\alpha_1^*, \alpha_2^*) - B_p(\alpha_1^*, \alpha_2^*) \\ &= \begin{cases} 0 & \text{if } \alpha_1^* \neq \alpha_2^* \\ \Box_p(\alpha_1^*, \alpha_1^*) - \sum_{\tilde{\alpha}^* \neq \alpha_1^*} \mid \Box_p(\alpha_1^*, \tilde{\alpha}^*) \mid & \text{if } \alpha_1^* = \alpha_2^* \end{cases}. \end{aligned}$$

These operators have the following desirable properties.

(1) (The combinatorial Bochner-Weitzenböck formula) $\Box_p = B_p + F_p$.
(2) B_p is a strongly nonnegative matrix (see Definition 1.2).
(3) F_p is a diagonal matrix.

We now take a closer look at the curvature operator F_p. For any p-chain $c \in C_p$, define the p^{th} curvature of c by

$$\mathcal{F}_p(c) = \langle F_p(c), c \rangle$$

(here we are thinking of F_p as a linear may from C_p to C_p). In particular, for any p-cell α

$$\mathcal{F}_p(\alpha) = \langle F_p(\alpha), \alpha \rangle .$$

Note that $\mathcal{F}_p(\alpha)$ is independent of the orientation of α. More explicitly, we have

Theorem 2.1. *For any p-cell α, the p^{th} curvature function applied to α, $\mathcal{F}_p(\alpha)$, is given by*

$$\begin{aligned} \mathcal{F}_p(\alpha) &= \langle F_p(\alpha), \alpha \rangle = w_\alpha \langle F_p(\alpha^*), \alpha^* \rangle \\ &= w_\alpha \Bigg\{ \Bigg[\sum_{\beta^{(p+1)}} \epsilon^2_{\alpha\beta} \frac{w_\alpha}{w_\beta} + \sum_{\gamma^{(p-1)}} \epsilon^2_{\gamma\alpha} \frac{w_\gamma}{w_\alpha} \Bigg] \\ &\qquad - \sum_{\tilde{\alpha}^{(p)} \neq \alpha} \Bigg| \sum_{\beta^{(p+1)}} \epsilon_{\alpha\beta}\epsilon_{\tilde{\alpha}\beta} \frac{\sqrt{\omega_\alpha \omega_{\tilde{\alpha}}}}{\omega_\beta} + \sum_{\gamma^{(p-1)}} \epsilon_{\gamma\alpha}\epsilon_{\gamma\tilde{\alpha}} \frac{\omega_\gamma}{\sqrt{\omega_\alpha \omega_{\tilde{\alpha}}}} \Bigg| \Bigg\}. \end{aligned}$$

In later sections we will often restrict attention to cell complexes with additional structure, and to special sets of weights. In such cases this formula will often greatly simplify.

We say that the operator F_p is ≥ 0 (> 0) if any of the following three equivalent conditions hold

(i) $\mathcal{F}_p(\alpha) \geq 0$ (> 0) for every p-cell α.
(ii) $\mathcal{F}_p(\alpha^*) \geq 0$ (> 0) for all $\alpha^* \in N_p^*$.
(iii) $\mathcal{F}_p(c) \geq 0$ (> 0) for all $0 \neq c \in C_p$.

In analogy with the Riemannian case, we make the following definition.

Definition 2.2. *For any 1-cell α, we define the Ricci curvature of α by*

$$\text{Ric}(\alpha) = \mathcal{F}_1(\alpha).$$

Definition 1.5 suggests that we now consider the neighbor relation among p-cells defined by the combinatorial Laplace operator.

Definition 2.3. *For p-cells $\alpha_1 \neq \alpha_2$, say that α_1 and α_2 are metric neighbors if $\Box_p(\alpha_1, \alpha_2) = B_p(\alpha_1, \alpha_2) \neq 0$ (equivalently, thinking of $\Box_p$ and B_p as matrices, if α_1 and α_2 are $\Box_p$-neighbors, or B_p-neighbors, in the sense of Definition 1.5).*

The main point is that if $\alpha_1^{(p)} \neq \alpha_2^{(p)}$ are metric neighbors, and $c = \sum_{\alpha^{(p)}} c_\alpha \alpha$ is a p-chain in the kernel of B_p, then (by Theorem 1.5(i)) c_{α_1} completely determines c_{α_2}. In particular, if $c_{\alpha_1} = 0$ then $c_{\alpha_2} = 0$.

It is useful to relate the concept of metric neighbors to more directly topological notions.

Definition 2.4. *For p-cells $\alpha_1 \neq \alpha_2$, say α_1 and α_2 are topological neighbors if either*

(1) $\exists \beta^{(p+1)}$ *with* $\beta > \alpha_1$ *and* $\beta > \alpha_2$.

or

(2) $\exists \gamma^{(p-1)}$ *with* $\gamma < \alpha_1$ *and* $\gamma < \alpha_2$.

If both (1) and (2) are true, we say α_1 and α_2 are transverse neighbors. If either (1) or (2) is true, but not both, we say α_1 and α_2 are parallel neighbors.

Much of Section 3 will be devoted to studying the relationships between these various notions. For now, we state a simple lemma which follows immediately from the definitions.

Theorem 2.5. (i) *If p-cells $\alpha_1 \neq \alpha_2$ are metric neighbors, then they are topological neighbors.*

(ii) *Let $\mathcal{W} = \{\text{Maps}\, W : \{\text{cells of}\, M\} \to \mathbb{R}_+\}$ denote the set of all possible weight assignments. Then, for an open dense set of weight assignments (with $\mathcal{W}$ given the obvious topology) we have that 2 cells in M are metric neighbors if and only if they are topological neighbors.*

Denote by $\approx$ the equivalence relation generated by the metric neighbor relation. That is, we set $\alpha_1 \approx \alpha_2$ if there are p-cells $\alpha_1 = \beta_1, \beta_2, \beta_3, \ldots, \beta_k = \alpha_2$ so that for each $i = 1, \ldots, k-1$, β_i and β_{i+1} are metric neighbors. Let $\mathcal{N}(p)$ denote the number of equivalence classes of p-cells.

It follows from Theorem 1.5(i) that

$$\dim \operatorname{Ker} B_p \leq \mathcal{N}(p). \tag{2.3}$$

We note that if $F_p \geq 0$ then (since $B_p \geq 0$)

$$\operatorname{Ker} \Box_p = \operatorname{Ker}(B_p + F_p) = \operatorname{Ker} B_p \cap \operatorname{Ker} F_p. \tag{2.4}$$

Combining (2.3) and (2.4) we learn

Theorem 2.6. *If $F_p \geq 0$ then*

$$\dim H_p(M, \mathbb{R}) \leq \mathcal{N}(p).$$

This theorem can be improved. We now restrict attention to the case $F_p \geq 0$. Say an equivalence class Λ of p-cells is *positive* if there is an $\alpha^{(p)} \in \Lambda$ with $\mathcal{F}_p(\alpha) > 0$. Otherwise, say Λ is *flat*. Let $\mathcal{N}^0(p)$ denote the number of flat equivalence classes. Theorem 1.5(iv) implies

Theorem 2.7. *If $F_p \geq 0$ then*

$$\dim H_p(M, \mathbb{R}) = \mathcal{N}^0(p).$$

Theorem 2.8. *If $F_p > 0$ then*

$$H_p(M, \mathbb{R}) = 0.$$

In particular, if $\operatorname{Ric}(e) > 0$ for each edge e, then

$$H_1(M, \mathbb{R}) = 0.$$

§3. Quasiconvex Complexes.

In this section we restrict our attention to cell complexes with some additional structure. This structure will enable us to simplify the formulas of the preceeding section. Although we will be primarily interested in polyhedral complexes, all of the ideas of this section will apply equally well to a more general class of complexes which we call quasiconvex. We begin this section with a review of some basic types of cell complexes.

Let M be a CW complex. For each p-cell α of M there is a characteristic map

$$h_\alpha : e^p \longrightarrow M$$

where e^p is the closed ball of dimension p, which maps the interior of e^p homeomorphically onto α.

Definition 3.1. *We say M is a regular cell complex if, for each cell α of M the characteristic map $h_\alpha : e^{(\dim \alpha)} \to M$ maps $e^{(\dim \alpha)}$ homeomorphically onto its image.*

We are primarily interested in the following properties of regular cell complexes.

Theorem 3.2. *Let M be a regular cell complex*

(i) *For any $(p+1)$-cell β and p-cell α*

$$\epsilon_{\alpha\beta} = \begin{cases} \pm 1 & \text{if } \alpha < \beta \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *For any $(p+1)$-cell β and $(p-1)$-cell γ. If $\gamma < \beta$ then*

$$\#\{\alpha^{(p)} \mid \gamma < \alpha < \beta\} = 2.$$

(iii) *If $\beta^{(p+1)} > \gamma^{(p-1)}$ and α_1^p, α_2^p are the unique p-cells satisfying $\beta^{(p+1)} > \alpha_i > \gamma^{(p-1)}$, then*

$$\epsilon_{\gamma\alpha_1}\epsilon_{\alpha_1\beta} + \epsilon_{\gamma\alpha_2}\epsilon_{\alpha_2\beta} = 0.$$

Proof: See [LW] Lemma 4.1 and Theorem 4.2. □

The results of this paper simplify in the case of cell complexes which exhibit a property that one often associates with convexity. For this reason, we call the following property *quasiconvexity* and note that every simplicial complex, and more generally every polyhedral complex, satisfies this property.

Definition 3.3. *Let M be a regular CW complex. We say M is quasiconvex if for every two $(p+1)$-cells, $\beta_1 \neq \beta_2$, if $\overline{\beta}_1 \cap \overline{\beta}_2$ contain a p-cell α, then $\overline{\beta}_1 \cap \overline{\beta}_2 = \overline{\alpha}$. This implies, in particular, that $\overline{\beta}_1 \cap \overline{\beta}_2$ contains at most 1 p-cell.*

From now on, by a quasiconvex complex, we will mean a compact, regualr CW complex which is quasiconvex.

Theorem 3.4. *Let M be a quasiconvex complex. Let $\alpha_1 \neq \alpha_2$ be p-cells. If α_1 and α_2 are parallel neighbors* (Definition 2.2)*, then α_1 and α_2 are metric neighbors* (Definition 2.1)*.*

Proof: Suppose α_1 and α_2 are parallel neighbors, so that either

(i) $\exists \beta^{(p+1)}$ such that $\beta > \alpha_1$ and $\beta > \alpha_2$

or

(ii) $\exists \gamma^{(p-1)}$ such that $\gamma < \alpha_1$ and $\gamma < \alpha_2$

but not both. We must check that $\Box_p(\alpha_1, \alpha_2) \neq 0$. From (2.2)

$$\Box_p(\alpha_1, \alpha_2) = \sum_{\substack{\beta^{(p+1)} > \alpha_1 \\ \beta^{(p+1)} > \alpha_2}} \epsilon_{\alpha_1 \beta} \epsilon_{\alpha_2 \beta} \frac{\sqrt{w_{\alpha_1} w_{\alpha_2}}}{w_\beta} + \sum_{\substack{\gamma^{(p-1)} < \alpha_1 \\ \gamma^{(p-1)} < \alpha_2}} \epsilon_{\gamma \alpha_1} \epsilon_{\gamma \alpha_2} \frac{w_\gamma}{\sqrt{w_{\alpha_1} w_{\alpha_2}}}.$$

Suppose (i) is true. From the definition of quasiconvexity, there can be at most one $(p+1)$-cell which satisfies (i). In addition, from the definition of a parallel nerighbor, there is no $\gamma^{(p-1)}$ which is a face of both α_1 and α_2. Therefore

$$\Box_p(\alpha_1, \alpha_2) = \epsilon_{\alpha_1 \beta} \epsilon_{\alpha_2 \beta} \frac{\sqrt{w_{\alpha_1} w_{\alpha_2}}}{w_\beta}$$

where β is the unique $(p+1)$-cell satisfying (i). From Theorem 3.2 this in nonzero, so α_1 and α_2 are metric neighbors.

The same proof works in case (ii). $\Box$

Definition 3.5. *Let $\sim$ denote the equivalence relation generated by the parallel neighbor relation, and $\mathcal{P}(p)$ the number of $\sim$ equivalence classes of p-cells. Note that the parallel neighbor relation is independent of the choice of weights, and hence so is $\mathcal{P}(p)$.*

Corollary 3.6. *Let M be a quasiconvex complex. Then*

(i) *for any $\alpha_1^{(p)}$ and $\alpha_2^{(p)}$*

$$\alpha_1 \sim \alpha_2 \Longrightarrow \alpha_1 \approx \alpha_2$$

(ii) $\mathcal{N}(p) \leq \mathcal{P}(p)$.

Corollary 3.7. *Let M be a quasiconvex complex and suppose $F_p \geq 0$.*

(i) *Then* $\dim \operatorname{Ker} B_p \leq \mathcal{P}(p)$.

(ii) *Let $\mathcal{P}^0(p)$ denote the set of flat $\sim$ equivalence classes of p-cells, i.e., those classes c such that for all $\alpha^{(p)} \in c$, $F_p(\alpha) = 0$. Then*

$$\dim H_p(M, \mathbb{R}) \leq \mathcal{P}^0(p).$$

Proof: Part (i) follows from Theorem 2.2 and Corollary 3.6(ii). Part (ii) follows from Theorem 2.3 and the observation that $\mathcal{N}^0(p) \leq \mathcal{P}^0(p)$. $\Box$

We now observe that in some important special cases, the notions of parallel neighbor and metric neighbor are equivalent. All of the examples shown in the introduction to this paper are of this form.

Definition 3.8. *Suppose there are positive constants ω_1 and ω_2 such that for all cells α, $\omega_\alpha = \omega_1 \cdot \omega_2^{(\dim \alpha)}$. Then we say the $\{\omega_\alpha\}$ form a standard set of weights. Note that this includes the case in which $\omega_\alpha = 1$ for each cell α.*

Theorem 3.9. *Let M be a quasiconvex complex, and suppose the $\{\omega_\alpha\}$ form a standard set of weights. Then for any p and any p-cells $\alpha_1 \neq \alpha_2$*

$$\begin{aligned} &\alpha_1 \textit{ and } \alpha_2 \textit{ are metric neighbors } \Longleftrightarrow \\ &\alpha_1 \textit{ and } \alpha_2 \textit{ are parallel neighbors.} \end{aligned}$$

Proof: We know that

(i) If α_1 and α_2 are metric neighbors then they are topological neighbors (Lemma 2.3).
(ii) If α_1 and α_2 are parallel neighbors then they are metric neighbors (Theorem 3.4).

We must only see that if α_1 and α_2 are transverse neighbors (Definition 2.2) then they are not metric neighbors.

Suppose α_1 and α_2 are transverse neighbors. Then

(i) $\exists \beta^{(p+1)}$ such that $\beta > \alpha_1$ and $\beta > \alpha_2$.
and
(ii) $\exists \gamma^{(p-1)}$ such that $\gamma < \alpha_1$ and $\gamma < \alpha_2$.

From the quasiconvexity of M, the above β and γ must be unique. Thus, from (2.2)

$$\Box_p(\alpha_1, \alpha_2) = \epsilon_{\alpha_1\beta}\epsilon_{\alpha_2\beta} + \epsilon_{\gamma\alpha_1}\epsilon_{\gamma\alpha_2}.$$

From Theorem 3.2,

$$\epsilon_{\gamma\alpha_1}\epsilon_{\alpha_1\beta} + \epsilon_{\gamma\alpha_2}\epsilon_{\alpha_2\beta} = 0.$$

Since each term in this expression is ± 1, this is equivalent to $\Box_p(\alpha_1, \alpha_2) = 0$, so α_1 and α_2 are not metric neighbors. $\Box$

For future reference, we end this section with an explicit presentation of the curvature of a quasiconvex complex. The proof follows by comparing the general formula for curvature found in Theorem 2.1 with the above-mentioned properties of a quasiconvex complex.

Theorem 3.10. *Let M be a quaisiconvex complex, and α a p-cell of M.*

$$\begin{aligned} \mathcal{F}_p(\alpha) = w_\alpha \Bigg\{ &\Bigg[\sum_{\beta^{(p+1)} > \alpha} \frac{w_\alpha}{w_\beta} + \sum_{\gamma^{(p-1)} < \alpha} \frac{w_\gamma}{w_\alpha} \Bigg] \\ &- \sum_{\tilde{\alpha}^{(p)} \neq \alpha} \Bigg| \sum_{\substack{\beta^{(p+1)} > \alpha \\ \beta > \tilde{\alpha}}} \frac{\sqrt{\omega_\alpha \omega_{\tilde{\alpha}}}}{\omega_\beta} - \sum_{\substack{\gamma^{(p-1)} < \alpha \\ \gamma < \tilde{\alpha}}} \frac{\omega_\gamma}{\sqrt{\omega_\alpha \omega_{\tilde{\alpha}}}} \Bigg| \Bigg\}. \end{aligned}$$

§4. Bochner's Theorems for 1-chains.

In Corollary 2.4 we saw that Ric > 0 implies $H_1(M, \mathbb{R}) = 0$. In this section we consider Ric ≥ 0. In this case, Bochner's theorems for smooth manifolds follow from the fact that any 1-form in the kernel of the Bochner Laplacian is completely determined by its value at any one point. Our main idea will be to prove a unique continuation theorem for 1-chains in $\operatorname{Ker} \Box_1 \cap \operatorname{Ker} B_1$ (in Section 5 we will see that there is no unique continuation for all 1-chains in $\operatorname{Ker} \Box_1$ or all 1-chains in $\operatorname{Ker} B_1$). In what follows, we will assume that a positive weight has been assigned to each cell of M, and that $C_*(M, \mathbb{R})$ has been endowed with the corresponding inner product with respect to which the cells are orthogonal.

Lemma 4.1. *Suppose M is a finite regular CW complex. Let $c = \sum c_e e$, $c_e \in \mathbb{R}$ (where the sum is over all 1-cells e, and we have arbitrarily chosen an orientation for each 1-cell), denote a 1-chain satisfying*

$$c \in \operatorname{Ker} \partial^* \cap \operatorname{Ker} B_1 \, [\supseteq \operatorname{Ker} \Box_1 \cap \operatorname{Ker} B_1] \, .$$

Let f be a 2-cell of M. Suppose there are two 1-cells e_1, e_2 such that

(i) $e_1 < f$, $e_2 < f$.

(ii) $\overline{e}_1 \cap \overline{e}_2 \neq \emptyset$

(so that e_1 and e_2 are transverse neighbors).

(iii) $c_{e_1} = c_{e_2} = 0$.

Then $c_e = 0$ for all $e < f$.

Proof: Let $e \neq e_1$, e_2 be another codimension-one face of f. Then e is a topological neighbor to both e_1 and e_2. If e is a parallel neighbor to either e_1 or e_2 then Theorem 3.9 (see also Definition 2.1) implies that, since $c \in \operatorname{Ker} B_1$, $c_e = 0$.

Now suppose e is a parallel neighbor of neither e_1 nor e_2. Then we must have $\overline{e} \cap \overline{e}_1 \neq \emptyset$ and $\overline{e} \cap \overline{e}_2 \neq \emptyset$. This implies that e, e_1, e_2 are the only 1-dimensional faces of f (since M is a regular complex). Thus

$$\partial f = \pm e \pm e_1 \pm e_2$$

so that

$$\langle \partial f, c \rangle = \pm w_e c_e.$$

However, since $c \in \operatorname{Ker} \partial^*$

$$\langle \partial \beta, c \rangle = \langle \beta, \partial^* c \rangle = 0.$$

which implies that

$$c_e = 0. \quad \Box$$

Theorem 4.2. (The Unique Continuation Theorem). *Let M be a connected regular CW complex. Suppose $c = \sum_e c_e e$, $c_e \in \mathbb{R}$, is a 1-chain satisfying*

$$c \in \operatorname{Ker} \partial^* \cap \operatorname{Ker} B_1.$$

Suppose, in addition, that there is a vertex v such that $c_e = 0$ for all $e > v$. Then $c = 0$.

Proof: Define a map

$$D : \{\text{1-cells of } M\} \longrightarrow \mathbb{Z}_{\geq 0},$$

the distance from v, as follows. If $e > v$ set $D(e) = 0$. If $D(e) \neq 0$, and there is a 1-cell e_1 so that $\overline{e} \cap \overline{e}_1 \neq \emptyset$ and $D(e_1) = 0$, then we set $D(e) = 1$. We continue inductively. If $D(e)$ is not $\leq k$, and there is a 1-cell e_1 so that $\overline{e} \cap \overline{e}_1 \neq \emptyset$ and $D(e_1) = k$ then we set $D(e) = k + 1$. Since M is connected, every 1-cell is assigned a finite integer.

We will prove $c_e = 0$ for all e by working inductively on $D(k)$. From the hypotheses, $c_e = 0$ if $D(e) = 0$. Suppose $c_{e_1} = 0$ for all 1-cells e_1 with $D(e_1) \leq k$. Let e be a 1-cell with $D(\alpha) = k + 1$. Then there is a 1-cell e_1 such that $\overline{e} \cap \overline{e}_1 \neq \emptyset$ and $D(e_1) = k$. In particular, e and e_1 are topological neighbors. If e and e_1 are parallel neighbors, then since $c \in \operatorname{Ker} B_1$, and $c_{e_1} = 0$ we must have $c_e = 0$.

Now suppose e and e_1 are not parallel neighbors. Then there is a 2-cell f with $e < f$ and $e_1 < f$. Label the endpoints of e_1 $\{v_1, v_2\}$ with $v_1 < e$. Let $e_2 \neq e_1$ denote the other 1-cell satisfying

$$v_2 < e_2 < f$$

(by Theorem 3.2, such a 1-cell exists). See Figure 4.1. If $c_{e_2} = 0$ then $c_e = 0$ by Lemma 4.1, so it is sufficient to prove that $c_{e_2} = 0$.

Suppose $k = 0$, i.e., $D(e_1) = 0$. Then either $v_1 = v$ or $v_2 = v$. If $v_1 = v$ then $D(e) = 0$ which is a contradiction. Thus $v_2 = v$, which implies $D(e_2) = 0$, so, by hypothesis, $c_{e_2} = 0$.

If $k > 0$, then by definition there is a 1-cell e_3 so that $D(e_3) = k - 1$ and $\overline{e}_3 \cap \overline{e}_1 \neq \emptyset$. Thus, either $v_1 < e_3$ or $v_2 < e_3$. If $v_1 < e_3$ then $\overline{e} \cap \overline{e}_3 \neq \emptyset$ which implies $D(e) \leq k$. This is a contradiction. Thus, $v_2 < e_3$. This implies that $\overline{e}_3 \cap \overline{e}_2 \neq \emptyset$ so that $D(e_2) \leq k$, so, by the inductive hypothesis $c_{e_2} = 0$. $\square$

Corollary 4.3. *Suppose M is a finite connected regular CW complex and for all 1-cells e, $\operatorname{Ric}(e) \geq 0$. Suppose, in addition, there is a vertex v such that for all 1-cells $e > v$, $\operatorname{Ric}(e) > 0$. Then $H_1(M, \mathbb{R}) = 0$.*

Proof: Suppose $c = \sum_{e^{(1)}} c_e e$, $c_e \in \mathbb{R}$, is a 1-chain satisfying

$$c \in \operatorname{Ker} \Box_1 = \operatorname{Ker}(B_1 + \operatorname{Ric}).$$

Since B_1 and Ric are nonnegative symmetric operators

$$\operatorname{Ker}(B_1 + \operatorname{Ric}) = \operatorname{Ker} B_1 \cap \operatorname{Ker} \operatorname{Ric}.$$

For $c \in \operatorname{Ker} \operatorname{Ric}$ we must have

$$c_e = 0 \qquad \forall\ e > v. \tag{4.1}$$

Moreover, if $c \in \operatorname{Ker} \Box_1$ we learn

$$c \in \operatorname{Ker} \Box_1 \cap \operatorname{Ker} B_1 \subseteq \operatorname{Ker} \partial^* \cap \operatorname{Ker} B_1. \tag{4.2}$$

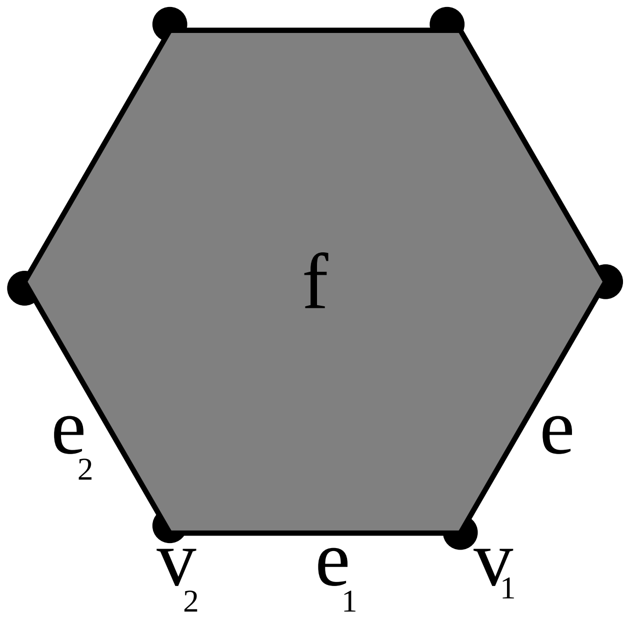


Figure 4.1

From (4.1), (4.2) and Theorem 4.2 we learn $c = 0$. Since

$$\mathrm{Ker}\,\Box_1 \cong H_1(M, \mathbb{R})$$

the corollary follows. $\Box$

We now turn our efforts to proving the remaining part of Bochner's theorem, which in the standard setting of Riemannian manifolds states that if M is a compact connected Riemannian manifold with $\mathrm{Ric} \geq 0$, then

$$\dim H_1(M, \mathbb{R}) \leq \dim(M).$$

In the combinatorial setting, it is easy to see that such an inequality cannot be true for general cell complexes. For example, let T^n denote the n-dimensional torus with a standard cubic cell decomposition and all cells assigned a weight of 1. Then $\mathrm{Ric}(e) = 0$ for all edges e. Let M_n denote the 2-skeleton of T^n. Then M_n is a 2-dimensional complex with $\mathrm{Ric} \equiv 0$, and

$$\dim H_1(M_n, \mathbb{R}) = n.$$

To recover such an inequality in the combinatorial setting, we must restrict attention to cell complexes with extra structure. For the remainder of this section, we will restrict attention to combinatorial manifolds. Rather than reviewing the definition (for which the reader may consult [St]) we simply recall that a combinatorial manifold is, in particular, a polyhedral complex which is a topological manifold. Conversely, any polyhedral complex which is a topological manifold of dimension 1, 2 or 3 is a combinatorial manifold.

Theorem 4.4. *Let M be a compact connected combinatorial n-manifold, $n \leq 3$, with* $\mathrm{Ric} \geq 0$. *Then*

$$\dim H_1(M, \mathbb{R}) \leq n.$$

Proof: We will prove this independently for each value of n.

$\mathbf{n = 0}$: A compact connected 0-manifold must be a single point, so we have

$$\dim H_1(M, \mathbb{R}) = 0.$$

$\mathbf{n = 1}$: A compact connected 1-manifold must be a circle, so we have

$$\dim H_1(M, \mathbb{R}) = 1.$$

$\mathbf{n = 2}$: Let M be a compact connected 2-manifold satisfying $\mathrm{Ric} \geq 0$. Let v be a vertex of M and label the edges at v $e_1, e_2, \ldots, e_k$. They may be labeled in any fashion as long as e_1 and e_2 are consecutive edges (i.e., share a 2-cell).

Suppose $c = \sum_{e^{(i)}} c_e e$ is a harmonic 1-chain (where we have arbitrarily chosen an orientation for each edge). We will prove that if $c_{e_1} = c_{e_2} = 0$ then $c = 0$. This is sufficient to imply

$$\dim H_1(M, \mathbb{R}) = \dim \mathrm{Ker}\, \Box_1 \leq 2.$$

Assume that $c_{e_1} = c_{e_2} = 0$. Since $\mathrm{Ric} \geq 0$, if c is harmonic, then $c \in (\mathrm{Ker}\, B_1) \cap (\mathrm{Ker}\, \partial^*)$. By Corollary 4.3 to prove $c = 0$ it is sufficient to prove that $c_{e_3} = \cdots = c_{e_k} = 0$. If $k \geq 4$ then each e_i, $i \geq 3$, is a parallel neighbor of either e_1 or e_2 and hence we must have $c_{e_i} = 0$. Suppose $k = 3$. Since c is harmonic, $\partial c = 0$. If $c_{e_1} = c_{e_2} = 0$ then

$$\langle \partial c, v \rangle = \pm \omega_v c_{e_3}$$

so we must have $c_{e_3} = 0$.

$\mathbf{n = 3}$: Let v be a vertex of M. There must be at least 4 edges incident to v. Suppose there are exactly 4 edges e_1, e_2, e_3, e_4 at v. Let $c = \sum_{e^{(i)}} c_e e$ be a harmonic 1-chain. We will see that if $c_{e_1} = c_{e_2} = c_{e_3} = 0$ then $c = 0$. This is sufficient to prove

$$\dim H_1(M, \mathbb{R}) = \dim \mathrm{Ker}\, \Box_1 \leq 3.$$

Following the argument presented in the case $n = 2$, it is sufficient to prove $c_{e_4} = 0$. Since c is harmonic, $\partial c = 0$. Therefore, since $c_{e_1} = c_{e_2} = c_{e_3} = 0$

$$0 = \langle \partial c, v \rangle = \pm \omega_v c_{e_4}$$

so $c_{e_4} = 0$.

From now on we assume that there are ≥ 5 edges at v. The following argument is perhaps best presented by first reducing the dimension of the problem. We do this by considering the link of v (see [St]). To construct the link, embed M in some Euclidean space so that the image of each cell is convex. Choose an ϵ small enough so that the closed sphere in M centered at v with radius ϵ (in the Euclidean metric) intersects only cells α with $\alpha > v$. Then the intersection of this spehere with M is a topological 2-sphere S which inherits a cell structure from M. In particular, the vertices in S are in $1-1$ correspondence with the edges in M which are incident to v. Vertices w_1 and w_2 in S are connected by an edge if and only if the corresponding edges e_1 and e_2 in M share a 2-cell (i.e., if and only if the neighbors e_1 and e_2 are transverse neighbors).

The 1-skeleton S_1 of S forms a connected graph embedded in S. Let K be a maximal complete subgraph of S_1 (a graph is complete if each vertex is connected to every other vertex). It is clear that K has at least two vertices. On the other hand, it is well known that the complete graph with five vertices does not embed in the 2-sphere. Therefore, K has ≤ 4 vertices.

Suppose K has ≤ 3 vertices, denoted by $\{w_i\}_{i\in I}$ ($|I| = 2$ or 3). Let $\{e_i\}_{i\in I}$ be the corresponding edges of M. Let $c = \sum_{e^{(i)}} c_e e$ be a harmonic 1-chain in M. We will see that if $c_{e_i} = 0$, $\forall i \in I$, then $c = 0$. This is sufficient to imply

$$\dim H_1(M, \mathbb{R}) = \dim \operatorname{Ker} \square_1 \leq 3.$$

Suppose $c_{e_i} = 0$, $\forall i \in I$. As in the previous cases to show $c = 0$ it is sufficient to show that $c_e = 0$ $\forall e > v$. Let $e \notin \{e_i\}_{i\in I}$ be an edge at v and let w be the corresponding vertex in S. Then, since K was chosen to be maximal, $\{w\} \cup \{w_i\}_{i\in I}$ does not form a complete graph. Hence there is some w_i, $i \in I$, so that w does not share an edge with w_i. This implies that e is a parallel neighbor of e_i, so that $c_{e_i} = 0 \Rightarrow c_e = 0$. Since this is true for every $e > v$ we must have $c = 0$.

Now suppose there are exactly four vertices w_1, w_2, w_3, w_4 in K, with corresponding edges e_1, e_2, e_3, e_4. We have assumed that there are at least five edges at v, so let e_5 denote any other edge at v, and ω_5 the corresponding vertex in S.

The graph K partitions S into four "triangles," each bounded by a triple w_i, w_j, w_k, $i \neq j \neq k \neq i$, and the edges connecting them. Suppose, without loss of generality, that w_5 lies in the interior of the w_1, w_2, w_3 triangle. Then w_5 does not share an edge with w_4 (see Figure 4.2). Moreover, if e_6 is any additional edge at v, with corresponding vertex ω_6 in S, then w_6 cannot share an edge with both w_4 and w_5. Hence e_6 is a parallel neighbor of either e_4 or e_5.

Let $c = \sum_e c_e e$ be a harmonic 1-chain on M. We will prove that if $c_{e_2} = c_{e_3} = c_{e_4} = 0$, then $c = 0$. This is sufficient to imply

$$\dim H_1(M, \mathbb{R}) = \dim \operatorname{Ker} \square_1 \leq 3.$$

As in the previous cases, it is sufficient to prove that $c_e = 0$ for all $e > v$. Since e_4 and e_5 are parallel neighbors $c_{e_4} = 0 \Rightarrow c_{e_5} = 0$. If $e_0 \notin \{e_1, \ldots, e_5\}$ is any other edge at v, then e_6 is a parallel neighbor of either e_4 or e_5. Hence $c_{e_4} = c_{e_5} = 0$ implies $c_{e_6} = 0$.

Summarizing, $c_{e_2} = c_{e_3} = c_{e_4} = 0$ implies that $c_e = 0$ for all $e > v$ except possibly e_1. Now we observe that since c is harmonic, $c \in \operatorname{Ker} \partial$. Therefore,

$$0 = \langle \partial c, v \rangle = \pm \omega_v c_{e_1}$$

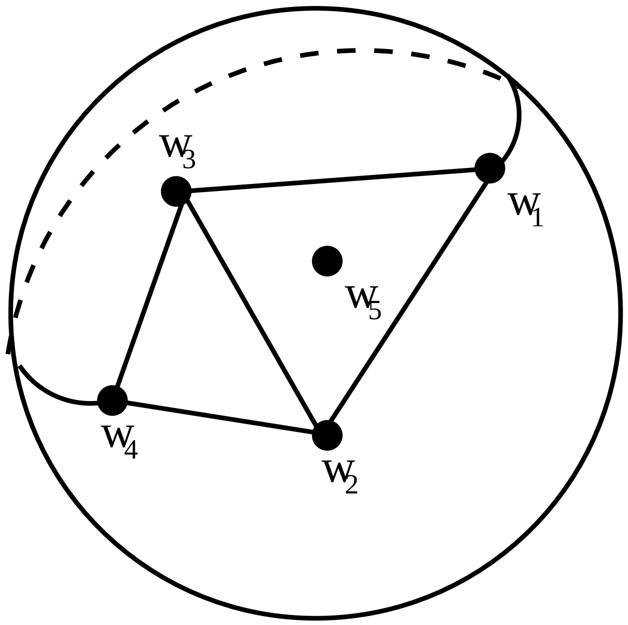


Figure 4.2

so $c_{e_1} = 0$ as well. $\square$

There are numerous difficulties involved in extending this result to dimensions > 3. Nevertheless it is possible to make some general statements. Let M be any cell complex. For each vertex v of M let $e(v)$ denote the set of 1-cells e of M with $e > v$. Define a relation on $e(v)$ by setting $e \sim e'$ if there is a sequence of edges in $e(v)$

$$e = e_0, e_1, e_2, \ldots, e_k = e'$$

such that for each $i = 0, 1, \ldots, k-1$ e_i and e_{i+1} do not share a 2-cell (i.e., e_i and e_{i+1} are parallel neighbors).

Let $N(v)$ denote the number of equivalence classes.

Define the *local homological dimension of M at v, $D(v)$, by*

$$D(v) = N(v)$$

if each equivalence class has at least two elements. If there exists an equivalence class with exactly one element (i.e., there is an edge with no parallel neighbors) then set

$$D(v) = N(v) - 1.$$

Define the homological dimension $D(M)$ of M by

$$D(M) = \inf_v D(v).$$

We then have the following general statement.

Theorem 4.5. *Suppose M is a compact connected quasiconvex cell complex with* $\mathrm{Ric} \geq 0$. *Then*

$$\dim H_1(M, \mathbb{R}) \leq D(M).$$

Proof: It is sufficient to prove that for each vertex v

$$\dim \operatorname{Ker} \square_1 \leq D(v).$$

It is a simple matter to prove that

$$\dim \operatorname{Ker} \square_1 \leq N(v).$$

Namely, choose one representative $e_1,\ e_2, \ldots, e_{\mathcal{N}(v)}$ from each equivalence class in $e(v)$. Let $c = \sum_e c_e e$ be a harmonic 1-chain. We will see that if $c_{e_1} = c_{e_2} = \cdots = c_{e_{\mathcal{N}(v)}} = 0$ then $c = 0$. However, if $e > v$ then $e \sim e_i$ for some i, so $c_{e_i} = 0 \Rightarrow c_e = 0$. This implies, by Theorem 4.2 , that $c = 0$.

Now suppose that there is an equivalence class with exactly one element. Let e^* denote an edge which has no parallel neighbors. Choose representatives $e_1,\ e_2, \ldots, e_{\mathcal{N}(v)-1}$ of each of the other equivalence classes. Let $c = \sum_e c_e e$ be a harmonic 1-chain. We will see that if $c_{e_1} = c_{e_2} = \cdots = c_{e_{\mathcal{N}(v)-1}} = 0$ then $c = 0$. By the previous argument, if $c_{e_i} = 0$ for all $1 \leq i \leq \mathcal{N}(v) - 1$, then $c_e = 0$ for all $e > v$ except possibly e^*. Now, since c is harmonic, $c \in \operatorname{Ker} \partial$ so

$$0 = \langle 2c, v\rangle = \pm\omega_v c_{e^*}$$

so $c_{e^*} = 0$. This implies, by Theorem 4.2 , that $c = 0$, so in this case

$$\dim H_1(M, \mathbb{R}) = \dim \operatorname{Ker} \square_1 \leq N(v) - 1 = D(v). \quad \square$$

In general, it may be difficult to determine $D(v)$. However, if M is a combinatorial manifold then one may be able to be more explicit. Let $n = \dim(M)$. Let M^* denote the dual complex to M. That is M^* is a polyhedral complex, homeomorphic to M, such that there is an identification of the p-cells of M with the $(n-p)$-cells of M^*, satisfying

$$\alpha < \beta \Longleftrightarrow \beta^* < \alpha^*$$

(where α and β are cells in M, and α^* and β^* are the corresponding cells in M^*).

For any n-cell α^* in M^*, let $\mathcal{F}(\alpha^*)$ denote the set of codimension 1 faces of α^*. Define an equivalence relation on $\mathcal{F}(\alpha^*)$ by declaring $f \sim f'$, where $f,\ f' \in \mathcal{F}(\alpha^*)$, if there exists a sequence

$$f = f_0,\ f_1, \ldots, f_k = f'$$

of elements of $\mathcal{F}(\alpha^*)$ such that for each $i = 0,\ 1, \ldots, k-1,\ f_i$ and f_{i+1} do not share an $(n-2)$-cell (i.e., f_i and f_{i+1} are parallel neighbors). Let $N^*(\alpha^*)$ denote the number of equivalence classes. Define

$$D^*(\alpha^*) = N^*(\alpha^*)$$

if each equivalence class has at least two members, and

$$D^*(\alpha^*) = N^*(\alpha^*) - 1$$

otherwise (i.e., if $\exists f \in \mathcal{F}(\alpha^*)$ with no parallel neighbors).

Let us do two examples. (i) Suppose α^* is an n-simplex (with its standard cell decomposition). Then $\mathcal{F}(\alpha^*)$ consists of the $n+1$ $(n-1)$-dimensional faces of α^*. Every two such faces share an $(n-2)$-cell, so no element is equivalent to any other. Therefore $N^*(\alpha^*) = n+1$. Since each equivalence class consists of a single element

$$D^*(\alpha^*) = N^*(\alpha^*) - 1 = n.$$

(ii) Suppose α^* is an n-cube (with its standard cell decomposition). Then $\mathcal{F}(\alpha^*)$ consists of the $2n$ $(n-1)$-dimensional faces. Each such face is a parallel neighbor of the opposite face, and has no other parallel neighbors. Thus, there are n equivalence classes, each consisting of two elements, so

$$D^*(\alpha^*) = N^*(\alpha^*) = n.$$

If v is a vertex of M, and v^* is the dual n-cell in M^*, then it follows directly from the definitions that

$$D(v) = D^*(v^*).$$

From the above, we can conclude the following.

Theorem 4.6. *Let M be a compact connected n-manifold satisfying $\mathrm{Ric} \geq 0$. Let M^* denote the dual complex to M. If M^* contains at least one n-simplex or at least one n-cube, then*

$$\dim H_1(M, \mathbb{R}) \leq n.$$

We end this section by observing that every statement about M corresponds to a dual statement about M^*.

More precisely, let us attach weights to the cells of M^* as follows. To each $(n-p)$-cell α^* of M^*, dual to the p-cell α of M, set

$$\omega_{\alpha^*} = (\omega_\alpha)^{-1}.$$

With these weights it is simple to see that

$$\mathcal{F}_p(\alpha) = \mathcal{F}_{n-p}(\alpha^*).$$

In particular, each statement about $\mathrm{Ric}(= \mathcal{F}_1)$ and $H_1(M, \mathbb{R})$ is equivalent to a statement about $\mathcal{F}_{n-1}$ and $H_{n-1}(M^*, \mathbb{R})$. We summarize these statements as follows.

Theorem 4.7. (i) *Let M be a finite cell complex. If $\mathcal{F}_{n-1} > 0$ then*

$$H_{n-1}(M, \mathbb{R}) = 0.$$

(ii) *Let M be a compact connected combinatorial n-manifold. Suppose $\mathcal{F}_{n-1} \geq 0$. If there is an n-cell α such that $\mathcal{F}_{n-1}(\beta) > 0$ for all $(n-1)$-cells $\beta < \alpha$, then*

$$H_{n-1}(M, \mathbb{R}) = 0.$$

(iii) *Let M be a compact connected n-manifold. Suppose $\mathcal{F}_{n-1} \geq 0$. Then, if either $n = 1, 2, 3$, or M contains an n-simplex or an n-cube*

$$\dim H_{n-1}(M, \mathbb{R}) \leq n.$$

§5. Counterexamples to Unique Continuation Theorems.

In this section we demonstrate that the Laplacian, and the Bochner Laplacian, behave differently, in a fundamental way, on smooth manifolds and cell complexes. In particular, on smooth, compact, connected manifolds, solutions of these operators are completely determined by local data. We pause to make this precise.

Theorem 5.1 ([AKS],[Ka]). *Let M be a smooth ($= C^\infty$) connected complete Riemannian manifold. Let ω be a harmonic p-form on M. For any $m \in M$, ω is completely determined by its infinite Taylor series at m. In particular, if ω is 0 on any open set in M, then $\omega \equiv 0$ everywhere on M.*

Theorem 5.2. Theorem 4.1 *is also true for p-forms ω which satisfy $\nabla_p^* \nabla_p \omega = 0$ (*[AKS], [Ka]*). Moreover, if M is compact, then for any $m \in M$, $\omega \in \mathrm{Ker}(\nabla_p^* \nabla_p)$ is completely determined by $\omega(m)$.*

We will show, by presenting examples, that these theorems fail to hold in the combinatorial setting. In the first example we show that harmonic p-chains are not determined by local data. More precisely, we show that for any CW structure on a ball B, there is a manifold M which contains B as a subcomplex, and a nonzero harmonic chain c which vanishes on B. Hence c is not determined by its values on B.

Example 5.3: Given $n \geq 2$, let $B \subseteq \mathbb{R}^n$ be the closed unit ball endowed with *any* finite CW decomposition. For any p, $1 \leq p \leq n-1$, let $C(p)$ be the number of p-cells in B. Let M^n be an n-manifold with

$$\dim H_p(M, \mathbb{R}) > N(p).$$

Embed B in M and extend the CW decomposition of B to a CW decomposition of M. Choose any inner products on $C_*(M, \mathbb{R})$ and let $\Box_p$ denote the induced Laplacian on $C_p(M, \mathbb{R})$. Then

$$\dim \mathrm{Ker}\, \Box_p = \dim H_p(M, \mathbb{R}) > C(p).$$

Hence there is a nonzero p-chain $c = \sum_{\alpha^{(p)} \subseteq M} c_\alpha \alpha$, $c_\alpha \in \mathbb{R}$ such that $\Box_p c = 0$ and $c_\alpha = 0$ for all $\alpha \subseteq B$.

We now present some very simple examples to show that elements in the kernel of the Bochner Laplacian are not necessarily determined by their values at a point.

Example 5.4: Let S_{n+1} be the closed $(n+1)$-simplex with its standard cell structure. For $1 \leq p \leq n$, let M be any subcomplex of S_{n+1} containing the $(p+1)$-skeleton. Choose a weight $\omega > 0$ and endow each $C_k(M, \mathbb{R})$ with an inner product by declaring the cells to be orthogonal and for each cell α

$$\langle \alpha, \alpha \rangle = w^{\dim \alpha}.$$

We now apply Theorem 3.13, and observe that for each $p \leq n-1$, each p-cell of M has no parallel neighbors. Hence, with respect to the basis consisting of p-cells, $\Box_p = I$ where

I is the $\binom{n+2}{p+1} \times \binom{n+2}{p+1}$ identity matrix $\left(\binom{n+2}{p+1}\right.$ is the number of p-cells in $\left.M\right)$. In particular,

$$B_p = 0, \quad F_p = \Box_p.$$

Therefore, there is no proper subset P of the p-cells so that if $c = \sum_{\alpha^{(p)}} c_\alpha \alpha$, $c_\alpha \in \mathbb{R}$, satisfies $c \in \operatorname{Ker} B_p$ then c is completely determined by $\{c_a | \alpha \in P\}$.

In Section 4 we proved the 2^{nd} Bochner Theorem for 1-chains by proving that on any finite CW complex an element in $\operatorname{Ker}(\Box_1) \cap \operatorname{Ker}(B_1)$ is completely determined by local data. However, this need not be true for p-chains with $p \geq 2$. We now demonstrate how to construct some examples.

Example 5.5: Let M_1 be any finite CW complex, and assign to each cell α of M_1 a weight $\omega_\alpha > 0$. Let $\Box_p$, B_p denote the induced Laplacian and Bochner Laplacian on p-chains. Let $c = \sum_{\alpha^{(p)}} c_\alpha \alpha$ be any nonzero p-chain satisfying

$$c \in (\operatorname{Ker} \Box_p) \cap (\operatorname{Ker} B_p).$$

Suppose there is a $(p-1)$-cell $\gamma \subseteq M_1$ such that for any $\alpha^{(p)} > \gamma$, $c_\alpha = 0$ (we will show later that such M_1, c and γ exist).

Let M_2 be any CW complex and $\tilde{\gamma}$ any $(p-1)$-cell of M_2. Endow each cell α of M_2 with a weight $w_\alpha > 0$ so that $w_{\tilde{\gamma}} = w_\gamma$.

Construct a new complex M by identifying γ and $\tilde{\gamma}$. Extend c to a p-chain $c^* = \sum_{\alpha^{(p)}} c^*_\alpha \alpha$ on M by setting

$$c^*_\alpha = \begin{cases} c_\alpha & \text{if } \alpha \subseteq M_1 \\ 0 & \text{if } \alpha \subseteq M_2 \end{cases}.$$

Then $0 \neq c^* \in (\operatorname{Ker} \Box^M_p) \cap (\operatorname{Ker} B^M_p)$ (where $\Box^M_p$ and B^M_p are defined with respect to the inner products on $C_*(M, \mathbb{R})$ induced by those on $C_*(M_1, \mathbb{R})$ and $C_*(M_2, \mathbb{R})$). Moreover, c^* is 0 on all of M_2.

We will now show that, for $p = 2$, such M, c and γ exist. Analogous constructions yield examples for any $p \geq 2$. Let M be the 3-torus T^3. We endow M with a standard cubic cell decomposition as follows. Begin with $\mathbb{R}^3$ given the integer cubic cell decomposition. Let M be $\mathbb{R}^3$ modulo translations in the x, y, z directions of distance ≥ 2. Then M inherits a cell structure from $\mathbb{R}^3$. Endow $c_*(T^3, \mathbb{R})$ with an inner product by declaring the cells to be orthonormal. Then $\dim \operatorname{Ker} \Box_2 = 3$, with canonical generators c^{xy}, c^{xz}, c^{yz}, with $c^{xy}_\alpha = 1$ if α is a 2-cell which is parallel to the xy-plane, and 0 otherwise. Let γ be any edge parallel to the z-axis. Then $c^{xy}_\alpha = 0$ for all $\alpha > \gamma$.

§6. Myer's Theorem.

In this section we prove a combinatorial version of Myer's theorem [My]. A theorem along these lines was presented in [Sto1,2]. The results, and the proofs, in this section are closely related to those in [Sto1,2], as well as those in [My]. In this section, we will work only with special choices of weights. More precisely, the main result of this section is

Theorem 6.1. *Let M be a finite quasiconvex complex (see Definition 3.3). Suppose that the cells have been assigned a standard set of weights (see Definition 3.8). If* $\mathrm{Ric}(e) > 0$ *for every edge e then $\pi_1(M)$ is finite.*

We will first describe the strategy of the proof. Note that it is sufficient to assume that M is connected, as otherwise one could simply analyze each component of M independently. It is convenient to introduce a distance function on the vertices of M. We follow the standard sort of definition. Suppose $\gamma = v_0^{(0)}, e_1^{(1)}, v_1^{(0)}, e_2^{(1)}, \ldots, v_{k-1}^{(0)}, e_k^{(1)}, v_k^{(0)}$ describes a connected path in M (i.e., for each i, v_i and v_{i+1} are the endpoints of e_i). We say γ is a path from v_0 to v_k and define the length of γ to be $\mathrm{length}(\gamma) = k$. For vertices v_1 and v_2 define

$$\text{distance } (v_1, v_2) = \inf_{\gamma} \text{ length } (\gamma)$$

where the infimum is taken over all paths γ from v_1 to v_2. We define

$$\mathrm{diam}(M) = \sup_{v_1, v_2} \text{distance } (v_1, v_2)$$

where the supremum is taken over all pairs of vertices in M.

Before going further, we remark that to more accurately reflect the geometry, we should define the length of $\gamma = v_0^{(0)}, e_1^{(1)}, v_1^{(0)}, e_2^{(1)}, \ldots, v_{k-1}^{(0)}, e_k^{(1)}, v_k^{(0)}$ to be $k(\omega_e)^{\frac{1}{2}}$, where ω_e is the weight (= size squared) that has been assigned to each edge. Other similar changes should be made along the way. Although these changes are natural from the geometric pont of view, they seem (surprisingly, at least to the author) only to complicate the proof, so we will make due with the definitions as stated.

We now note the following simple observation.

Lemma 6.2. *Let M be a (not necessarily finite) connected CW complex. Suppose that there is a V so that valence$(v) \le V$ for all vertices v. Then M has finitely many vertices if and only if* $\mathrm{diam}(M)$ *is finite.*

Proof: The number of vertices is bounded above by $V^{\mathrm{diam}(M)}$. This implies the lemma. □

With this preliminary, our main result is a corollary of the following statement.

Theorem 6.3. *Let M be a (not necessarily finite) quasiconvex regular CW complex, with the cells assigned a standard set of weights $\omega_a = \omega_1 \cdot \omega_2^{(\dim \alpha)}$. Define*

$$F(M) = \left[\sup_e \#\{f^{(2)} > e\}\right] + 1$$

where the supremum is taken over all edges e. *Suppose* $\mathrm{Ric}(e) \geq c > 0$ *for all edges* e. *Then*

$$\mathrm{Diam}(M) \leq \frac{2\omega_1}{c} F(M).$$

Proof of Theorem 6.1 (assuming Theorem 6.3): Let M be a finite CW complex in which the cells have been assigned a standard set of weights. Suppose $\mathrm{Ric} > 0$. Since M is finite, we can find a $c > 0$ so that $\mathrm{Ric}(e) \geq c$ for all edges e. The finiteness of M also implies $D(M) < \infty$ and the existence of a $V < \infty$ so that $\mathrm{valence}(v) \leq V$ for all vertices v of M.

Let $\pi : \widetilde{M} \to M$ denote the universal cover of M, with $\widetilde{M}$ endowed with the cell structure induced from M (so that if $\tilde{\alpha}$ is a p-cell of $\widetilde{M}$ then π maps $\tilde{\alpha}$ homeomorphically onto a p-cell of M), and the corresponding weights. It is easy to see that for any edge $\tilde{e}$ of $\widetilde{M}$

$$\mathrm{Ric}_{\widetilde{M}}(\tilde{e}) = \mathrm{Ric}_M(\pi(\tilde{e})).$$

In particular, $\mathrm{Ric}(\tilde{e}) \geq c$ for all edges $\tilde{e}$ of $\widetilde{M}$. Moreover, for every edge $\tilde{e}$ of $\tilde{M}$

$$\#\{\tilde{f}^{(2)} > \tilde{e}\}_{\widetilde{M}} = \#\{f^{(2)} > \pi(e)\}_M$$

so that

$$F(\widetilde{M}) = F(M) < \infty.$$

For each vertex $\tilde{v}$ of $\widetilde{M}$

$$\mathrm{valence}(\tilde{v}) = \mathrm{valence}(\pi(\tilde{v}))$$

so the valences of the vertices of $\widetilde{M}$ are bounded above by V.

Theorem 6.3 now implies that $Diam(\widetilde{M}) < \infty$. We can apply Lemma 6.2 to conclude that $\widetilde{M}$ has finitely many vertices. Since

$$\#\pi_1(M) = \#\{\text{vertices of } \widetilde{M}\}/\#\{\text{vertices of } M\}$$

this implies that $\pi_1(M)$ is finite. $\square$

To proceed further, we must introduce some new definitions. Let $\gamma = v_0, e_1, \ldots, e_k, v_k$ describe a connected path in M.

Definition 6.4. *A (combinatorial) Jacobi field along* γ *is a map*

$$J : \{e_i\}_{1 \leq i \leq k} \longrightarrow \{2\text{-}cells\}$$

such that

(i) $J(e_i) > e_i$ *for all* i

and

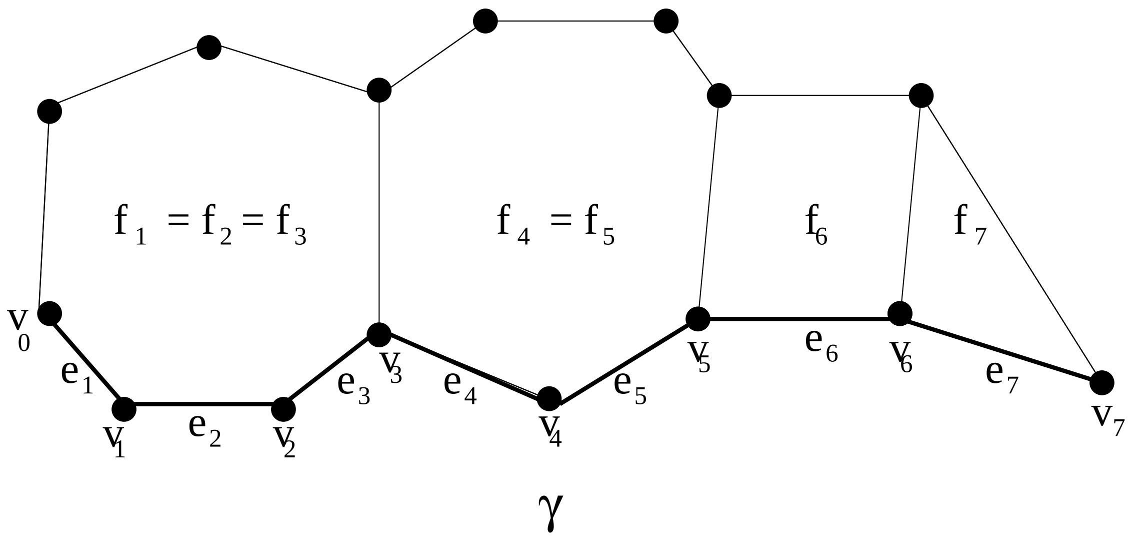


Figure 6.1

(ii) For every $1 \leq i \leq k-1$, $J(e_i)$ *and* $J(e_{i+1})$ *share an edge other than* e_i *and* e_{i+1} *(*$J(e_i) = J(e_{i+1})$ *is allowed).*

A Jacobi field is illustrated in Figure 6.1 (we have written f_i for $J(e_i)$). We have chosen the term combinatorial Jacobi field because these objects will play the role in our combinatorial proof that classical Jacobi fields play in the Riemannian proof.

In the Riemannian category, Jacobi fields are solutions of a 2^{nd} order ordinary differential equation, and hence are completely determined by their value and derivative at any point. We can state an important combinatorial version of this uniqueness property.

Lemma 6.5. *Let* $\gamma = v_0, e_1, \ldots, e_k, v_k$ *describe a connected path. Suppose* J_1 *and* J_2 *are combinatorial Jacobi fields along* γ*, and there is an* i *with*

$$J_1(e_i) = J_2(e_i).$$

Then for each $i = 1, 2, \ldots, k$

$$J_1(e_i) = J_2(e_i).$$

Proof: It is sufficient to show $J_1(e_{i+1}) = J_2(e_{i+1})$ if $i < k$, and $J_1(e_{i-1}) = J_2(e_{i-1})$ if $i > 1$, as the lemma then follows by induction.

Suppose $i < k$. We will show that $J_1(e_{i+1}) = J_2(e_{i+1})$. First suppose that $J_1(e_i) = J_1(e_{i+1})$. In particular e_i and e_{i+1} are both faces of $J(e_i)$. Then, by definition, $J_2(e_{i+1})$ shares an edge with $J_2(e_i) = J_1(e_i)$ other than e_i and e_{i+1}. Let e' be this edge. Then e_{i+1} and e' are faces of both $J_2(e_{i+1})$ and $J_2(e_i)$ so by quasiconvexity we must have

$$J_2(e_{i+1}) = J_2(e_i) = J_1(e_i) = J_1(e_{i+1}).$$

Now suppose $J_1(e_i) \neq J_1(e_{i+1})$. Then, by definition, $\overline{J_1(e_i)}$ and $\overline{J_1(e_{i+1})}$ share an edge other than e_i and e_{i+1}. Let e' denote this edge. Since $\overline{J_1(e_i)} \cap \overline{J_1(e_{i+1})} \supset e'$, and $J_1(e_i) \neq J_1(e_{i+1})$, by quasiconvexity we must have

$$\overline{J_1(e_i)} \cap \overline{J_1(e_{i+1})} = \overline{e'}.$$

Moreover, $J_1(e_i) > e_i > v_i$ and $J_1(e_{i+1}) > e_{i+1} > v_i$, so

$$v_i \in \overline{J_1(e_i)} \cap \overline{J_1(e_{i+1})}$$

which implies

$$e' > v_i.$$

Now consider J_2. We must have $J_2(e_i) \neq J_2(e_{i+1})$, since if $J_2(e_i) = J_2(e_{i+1})$ the earlier argument would imply $J_1(e_i) = J_1(e_{i+1})$. Just as in the case of J_1, we find that there is an edge e'' with $J_2(e_i)$, $J_2(e_{i+1}) > e''$ and e_i, $e_{i+1} \neq e'' > v_i$.

We note that $J_1(e_i) = J_2(e_i)$ is a 2-cell, and so has exactly two 1-dimensional faces which have v_i as an endpoint. The edge e_i is one such edge. On the other hand, the edges e' and e'' both share these properties (and are distinct from e_i). Hence we must have

$$e' = e''.$$

From this we can conclude that $J_1(e_{i+1})$ and $J_2(e_{i+1})$ both have e_{i+1} and $e' = e''$ as faces. Quasiconvexity now implies

$$J_1(e_{i+1}) = J_2(e_{i+1}).$$

□

There is one significant difference between Jacobi fields in the combinatorial Riemannian settings. Namely, a Riemannian Jacobi field which is defined on a segment γ' of a geodesic γ can always be extended to all of γ. This is not true in the combinatorial setting. In Figure 6.2 the faces f_2 and f_3 form a Jacobi field over the segment $\gamma' = v_1,\ e_2,\ v_2,\ e_3,\ v_3$ of the path $\gamma = v_0,\ e_1,\ v_1, \ldots, e_4, v_4$, but it cannot be extended in either direction.

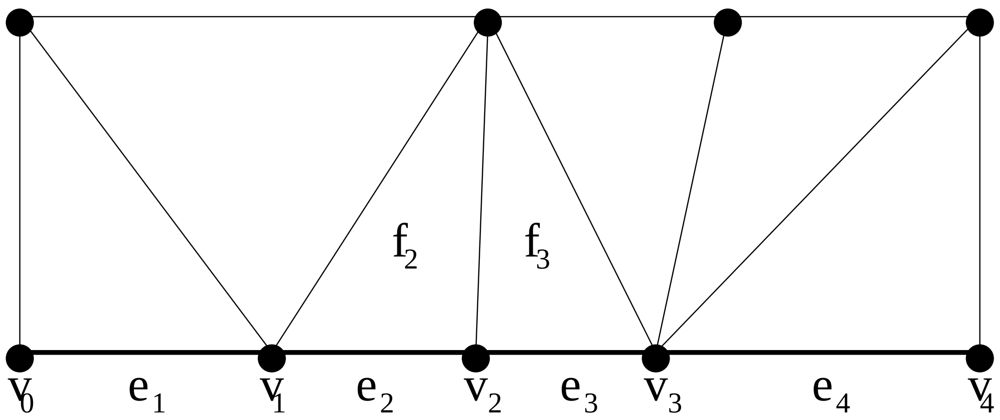


Figure 6.2

Let f_i be any 2-cell with $f_i > e_i$. We say f_i *can be continued to* e_{i+1} if there is a 2-cell $f_{i+1} > e_{i+1}$ such that $J(e_i) = f_i$, $J(e_{i+1}) = f_{i+1}$ forms a Jacobi field along the path $\gamma' = v_{i-1}$, e_i, v_i, e_{i+1}, v_{i+1}. Equivalently, f_i can be continued to e_{i+1} if there is a 2-cell $f_{i+1} > e_{i+1}$ such that f_i and f_{i+1} share an edge other than e_i and e_{i+1} (again, $f_i = f_{i+1}$ is allowed). Let

$$C(e_i, e_{i+1}) = \{f^{(2)} > e_i \text{ such that } f \text{ can be continued to } e_{i+1}\}$$

and

$$NC(e_i, e_{i+1}) = \{f^{(2)} > e_i \text{ such that } f \text{ can not be continued to } e_{i+1}\}$$

We define $C(e_i, e_{i-1})$ and $NC(e_i, e_{i-1})$ analogously.

The rest of this section is devoted to proving Theorem 6.3. Changing the weight of each cell to 1 has the effect of dividing the Ricci curvature by ω_1. Therefore, it is sufficient to prove the theorem in the case that all cells have been assigned the weight 1. Let v and v' be two vertices of M, and

$$\gamma : v = v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k = v'$$

a minimal path from v to v' (so that distance$(v, v') = k$). For each $1 \le i \le k$

$$c \le \mathrm{Ric}(e_i) \;=\; \#\{f^{(2)} > e_i\} + 2 - \#\{\text{parallel neighbors of } e_i\}.$$

Let us take a closer look at this formula. Parallel neighbors of e_i may occur in two ways, either by sharing a 2-cell with e_i or sharing a vertex with e_i (but not both). Say a parallel neighbor e' of e_i is a 2-neighbor of e_i if there exists a 2-cell f with $f > e_i$ and $f > e'$. Say a parallel neighbor e' is a 0-neighbor if there exist a vertex v with $v < e_i$ and $v > e'$. In particular

$$\#\{\text{parallel neighbors of } e_i\} \;=\; \#\{\text{2-neighbors of } e_i\} \;+\; \#\{\text{0-neighbors of } e_i\}.$$

Each 2-cell $f > e_i$ contributes $(\mathrm{sides}(f) - 3)$ 2-neighbors of e_i ,where sides(f) is the number of 1-cells in the boundary of f . Namely, every 1-face of f is a 2-neighbor of e_i, except for e_i and the two edges which share an endpoint with e. Every 2-neighbor of e_i arises in this way. Therefore,

$$\#\{\text{2-neighbors of } e_i\} \;=\; \sum_{f^{(2)} > e} (\mathrm{sides}(f) - 3)$$

so that

$$\#\{f^{(2)} > e_i\} \;-\; \#\{\text{2-neighbors of } e_i\} \;=\; \sum_{f^{(2)} > e_i} (4 - \mathrm{sides}(f)).$$

We now examine the number of 0-neighbors of e_i. Suppose that $i > 1$. If $f > e_{i-1}$ is a 2-cell, then there is exactly one edge e' other than e_{i-1} such that $f > e' > v_{i-1}$. The edge

e' is a 0-neighbor of e_i if and only if there is no 2-cell f' with $f' > e'$ and $f' > e_i$. That is, if and only if $f \in NC(e_{i-1}, e_i)$.

Similarly, if $i < k$, each element of $NC(e_{i+1}, e_i)$ gives rise to a 0-neighbor of e_i. In addition, the edge e_{i-1} is a 0-neighbor of e_i if and only if e_{i-1} and e_i do not share a 2-cell, and e_{i+1} is a 0-neighbor of e_i unless e_i and e_{i+1} share a 2-cell. Let $\epsilon_i \in \{0, 1, 2\}$ denote the number of 1-cells from among $\{e_{i-1}, e_{i+1}\}$ which share a face with e_i. Summarizing, for $2 \leq i \leq k-1$

$$\#\{\text{0-neighbors of } e_i\} \geq 2 + NC(e_{i-1}, e_i) + NC(e_{i+1}, e_i) - \epsilon_i,$$

so that

$$\mathrm{Ric}(e_i) \leq \sum_{f^{(2)} > e_i} (4 - \mathrm{sides}(f)) - NC(e_{i-1}, e_i) - NC(e_i, e_{i+1}) + \epsilon_i.$$

Following the same argument for $i = 1$ we find

$$\mathrm{Ric}(e_1) \leq \sum_{f^{(2)} > e_1} (4 - \mathrm{sides}(f)) - NC(e_2, e_1) + 1 + \epsilon_1$$

where $\epsilon_1 = 1$ if e_1 shares a 2-cell with e_0, and $\epsilon_1 = 0$ otherwise. Similarly, for $i = k$ we find

$$\mathrm{Ric}(e_k) \leq \sum_{f^{(2)} > e_k} (4 - \mathrm{sides}(f)) - NC(e_{k-1}, e_k) + 1 + \epsilon_k$$

where $\epsilon_k = 1$ if ϵ_{k-1} shares a 2-cell with e_k, and $\epsilon_k = 0$ otherwise. Summing over i, and using the fact that $\mathrm{Ric}(e_i) \geq c$ for all i, yields

$$ck \leq 2 + \sum_{i=1}^{k} \sum_{f^{(2)} > e_i} (4 - \mathrm{sides}(f)) - \sum_{i=1}^{k-1} NC(e_i, e_{i+1}) - \sum_{i=2}^{k} NC(e_i, e_{i-1}) + 2\epsilon(\gamma) \tag{6.1}$$

where $\epsilon(\gamma)$ is the number of pairs (e_i, e_{i+1}) which share a 2-cell.

We will now make use of the fact that γ is a minimal path. Let $\gamma' = v_r$, $e_{r+1}, \ldots, v_{s-1}$, e_s, v_s be a sub-path of γ. It follows that γ' is a minimal path from v_r to v_s. Let J be a Jacobi field along γ'. Then J provides an alternative path $\tilde{\gamma}$ from v_r to v_s (see Figure 6.3)

Namely, suppose $J(e_r) = J(e_{r+1}) = \cdots = J(e_{r+t-1}) \neq J(e_{r+t})$. Then, in particular, the edges e_r, $e_{r+1}, \ldots, e_{r+t-1}$ all share the 2-cell $J(e_r)$. Since γ' is minimal, these edges are all distinct. The 2-cells $J(e_r)$ and $J(e_{r+t})$ have an edge e' in common not equal to e_{r+t-1} or e_r. The vertex v_{r+t-1} is one endpoint of e'. Let v' denote the other endpoint. Then, starting at v_r and heading away from e_r, we can follow the boundary of $J(e_r)$ until we reach v'. Note that this path has $(\mathrm{sides}(J(r)) - (t+1))$ edges. That is, this path includes all sides of $J(r)$ except $e_r, \ldots, e_{r+t-1}$ and e'. Now suppose that $J(e_{r+t}) = J(e_{r+t-1}) = \cdots = J(e_{r+t+u-1}) \neq J(e_{r+t+u})$. Then $J(e_{r+t})$ and $J(e_{r+t+u})$ share an edge e'' with endpoints $v_{r+t+u-1}$ and v''. Starting at v' and heading away from e', we can follow the boundary of $J(e_{r+t})$ from v' to v''. This path has $(\mathrm{sides}(J(e_{r+t})) - (u+2))$

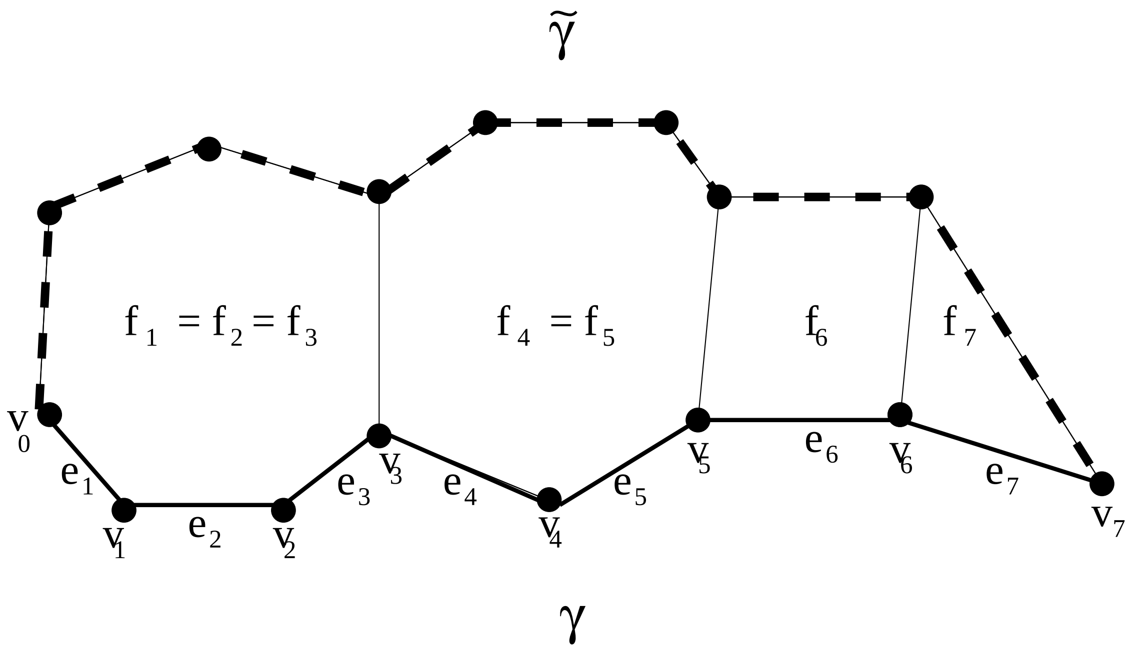


Figure 6.3

edges (i.e., all sides of $J(e_{r+t})$ except $e_{r+t}, \ldots, e_{r+t+u-1}$, e' and e'', which are all distinct). Continuing in this fashion, we construct a path $\tilde{\gamma}$ from v_r to v_s with length

$$\text{length}(\tilde{\gamma}) = 2 + \sum_U [\text{sides}(J(U)) - (\#U + 2)]$$

where the sum is over the maximal connected sets $U \subset \{e_{r+1}, \ldots, e_s\}$ along which J is constant, $J(U)$ is the common value of $J(e_i)$ for $e_i \in U$, and $\#U$ is the number of elements in U. Since every term in the summation is nonnegative, we have

$$\begin{aligned}
\text{length}(\tilde{\gamma}) &\leq 2 + \sum_U \#U\,[\text{sides}(J(U)) - (\#U + 2)] \\
&= 2 + \sum_U \#U\,[\text{sides}(J(U)) - 3] - \sum_U \#U(\#U - 1) \\
&= 2 + \sum_{i=r+1}^{s} [\text{sides}\,(J(e_i)) - 3] - \sum_U \#U(\#U - 1) \\
&\leq 2 + \sum_{i=r+1}^{s} [\text{sides}\,(J(e_i)) - 3] - 2\sum_U (\#U - 1).
\end{aligned}$$

Let

$$\delta(J) = \#\{(i, i+1) \text{ such that } J(e_i) = J(e_{i+1})\}.$$

It is easy to see that

$$\delta(J) = \sum_U (\#U - 1)$$

so that

$$\text{length}(\tilde{\gamma}) \leq 2 + \sum_{i+r+1}^{s} [\text{sides}\,(J(e_i)) - 3] - 2\delta(J).$$

Since γ' is minimal, we must have

$$\text{length}(\tilde{\gamma}) \geq \text{length}(\gamma') = s - r = \sum_{i=r+1}^{s} 1$$

which implies

$$0 \leq 2 + \sum_{i=r+1}^{s} [\text{sides}\,(J(e_i)) - 4] - 2\delta(J). \tag{6.2}$$

Say a Jacobi field J along γ' is maximal if there is no connected path γ'' with $\gamma' \subsetneqq \gamma'' \subset \gamma$ such that J can be extended to γ''. It follows from Lemma 6.4 that for every edge e_i of γ and every 2-cell $f > e_i$, $J(e_i) = f$ in at most one maximal Jacobi field. Moreover, $J(e_i) = f$ trivially forms a Jacobi field over the sub-path $\gamma' = v_{i-1},\, e_i,\, v_i$, so that $J(e_i) = f$ in some maximal Jacobi field. Therefore, for every e_i and every $f^{(2)} > e_i$, $J(e_i) = f$ in exactly 1 maximal Jacobi field. Thus, summing (6.2) over the maximal Jacobi fields yields

$$0 \leq 2\#\{\text{maximal Jacobi fields}\} + \sum_{i=r+1}^{s} \sum_{f^{(2)} > e_i} [\text{sides}(f) - 4] - 2 \sum_{\substack{\text{maximal} \\ \text{Jacobi fields} \\ J}} \delta(J).$$

The same reasoning implies that

$$\sum_{\substack{\text{maximal} \\ \text{Jacobi fields} \\ J}} \delta(J) = \epsilon(\gamma).$$

Substitution into (6.1) yields

$$ck \leq 2 + 2\#\{\text{maximal Jacobi fields}\} - \sum_{i=1}^{k-1} NC(e_i, e_{i+1}) - \sum_{i=2}^{k} NC(e_i, e_{i-1}). \tag{6.3}$$

A Jacobi field on a subinterval $\gamma' = v_r, \ldots, v_s$ is maximal if and only if
(i) $r = 0$ or $J(v_r) \in NC(e_r, e_{r-1})$
and

(ii) $s = k$ or $J(v_s) \in NC(e_s, e_{s+1})$.

Associate to each maximal Jacobi field its initial endpoint $J(v_r)$. This maps {*maximal Jacobi fields*} onto $\{f^{(2)} > e_1\} + \sum_{i=2}^{k} NC(e_i, e_{i-1})$.

Since every Jacobi field is uniquely determined by its value at any edge, this map is $1-1$. Therefore,

$$\#\{\text{maximal Jacobi fields}\} = \#\{f^{(2)} > e_1\} + \sum_{i=2}^{k} NC(e_i, e_{i-1}).$$

Considering the final endpoint of the Jacobi field yields

$$\#\{\text{maximal Jacobi fields}\} = \#\{f^{(2)} > e_k\} + \sum_{i=1}^{k-1} NC(e_i, e_{i+1}).$$

Substituting these values into (6.3) yields

$$ck \leq 2 + \#\{f^{(2)} > e_1\} + \#\{f^{(2)} > e_k\} \leq 2F(M)$$

so that

$$\text{distance}(w_1, w_2) = k \leq \frac{2}{c} F(M).$$

Since v and v' were arbitrary, this is true for all pairs of points, so that

$$\text{diam}(M) \leq \frac{2}{c} F(M).$$

$\square$

§7. Negative Ricci Curvature.

As we have seen, Bochner's and Myers' theorems, in both their smooth and combinatorial forms, provide some strong topological restrictions for a manifold to have either a Riemannian metric or a combinatorial cell structure with positive or nonnegative Ricci curvature.

The situation is very different for negative Ricci curvature. The Gauss-Bonnet theorem implies that a 2-dimensional manifold M cannot have a metric with negative Ricci curvature (= Gauss curvature) unless M has negative Euler characteristic. However, this is the only restriction.

Theorem 7.1 ([Gao],[GY] [Lo1,2]). *Let M be a manifold of dimension $n \geq 3$. Then M has a Riemannian metric with everywhere negative Ricci curvature.*

In this section we show that the situation is even more extreme for combinatorial manifolds. Namely, there is no Gauss-Bonnet theorem for general combinatorial structures with general inner products so, in fact, every manifold of dimension ≥ 2 has a triangulation with everywhere negative Ricci curvature.

Given a triangulation of a manifold M, before defining Ricci curvature, one must choose positive weights for each of the simplices. We will restrict attention to standard sets of weights (see Definition 3.8).

We can now state the main theorem of this section.

Theorem 7.2. *Let M be a simplicial complex which is a combinatorial n-manifold, with $n \geq 2$. Then there is a simplicial complex M^* which is a subdivision of M, so that, with respect to any standard inner product, every edge has negative Ricci curvature.*

Note that every subdivision of a combinatorial manifold is itself a combinatorial manifold. Our proof of Theorem 7.2 is completely independent of Theorem 7.1. Before proving 7.2 we recall that in [Ca] and [Wh] it is shown that every smooth manifold is homeomorphic to a simplicial complex which is a combinatorial manifold. Hence we have the following corollary.

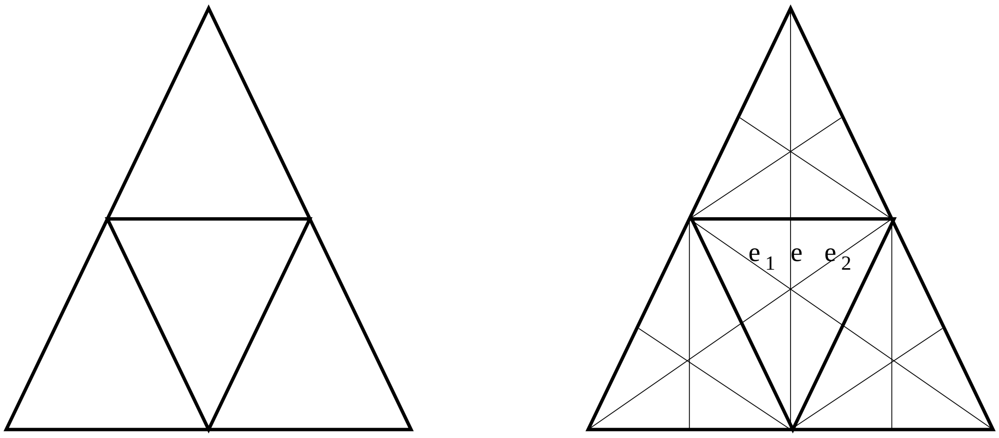
Corollary 7.3. *Let M be a smooth manifold of dimension $n \geq 2$. Then M can be triangulated so that, with respect to any standard inner product, every edge has negative Ricci curvature.*

The remainder of this section is devoted to proving Theorem 7.2. We first note that the sign of the Ricci curvature is independent of the standard set of weights. Thus, we can assume that each cell has been assigned the weight 1, so that for any edge e

$$\text{Ric}(e) = \#\{f^{(2)} > e\} + 2 - \#\{\text{parallel neighbors } \tilde{e} \text{ of } e\}. \tag{7.1}$$

The proof of Theorem 7.2 will be in 2 steps. The first step is to let M' denote the barycentric subdivision of M (see [GL] p. 7). We will show that for every edge e of M', $\text{Ric}(e) \leq 0$.

The next step is to subdivide each n-simplex of M' in a stellar way ([GL] p. 8). We then show that by inserting edges in this manner, the Ricci curvature of each edge is strictly



(i) (ii)

Figure 7.1

decreased, and the new edges have negative Ricci curvature. These statements rely on the fact that M' is, in fact, the barycentric subdivision of some complex.

We briefly illustrate main ideas in the case that $\dim M = 2$. Suppose that Figure 7.1(i) occurs somewhere in M. After barycentric subdivision it is transformed into Figure 7.1(ii). We now observe that $\mathrm{Ric}(e) = 0$. The same is true for all edges which connect a vertex in a 2-simplex of M to a vertex in a 1-simplex of M. All other edges have negative Ricci curvature. We now subdivide each 2-simplex in a stellar way (Figure 7.2).
This strictly decreases the Ricci curvature of each edge. For example, the edges e_1 and e_2 are parallel neighbors of e in Figure 7.2 but not in Figure 7.1(ii). The last thing to check is that each new edge, e.g. edge e_3 in Figure 7.2, has negative Ricci curvature. This is true as long as the vertex v has degree at least 4 in M'. Since M' is the barycentric subdivision of another complex, each vertex has degree at least 4.

Although the numerology is more complicated, the general case follows the same path. Let M be a simplicial complex which is a combinatorial n-manifold, with $n \geq 2$. Let M' denote the barycentric subdivision of M. Recall that p-simplices of M' can be identified with nested sequences $\alpha_p > \alpha_{p-1} > \cdots > a_1 > \alpha_0$ of simplices in M. We will denote such a simplex in M' by $\alpha' = [\alpha_p > \alpha_{p-1} > \cdots > a_1 > \alpha_0]$. A simplex α' in M' is a face of a simplex β' of M', if and only if the sequence of simplices of M corresponding to α' is a subsequence of that corresponding to β'. In particular, an edge of M' can be identified with a pair $\beta^{(b)} > \alpha^{(a)}$ of simplices in M. Of course, we must have $b > a$. Let $e' = [\beta^{(b)} > \alpha^{(a)}]$.

To apply formula (7.1) we must calculate

$$\#\{f'^{(2)} > e'\}$$

and

$$\#\{\tilde{e}'^{(1)} \text{ s.t. } \exists\, v' \text{ with } \tilde{e}' > v',\, e' > v' \text{ but } \nexists\, f'^{(2)} \text{ with } f' > \tilde{e}',\, f' > e'\}.$$

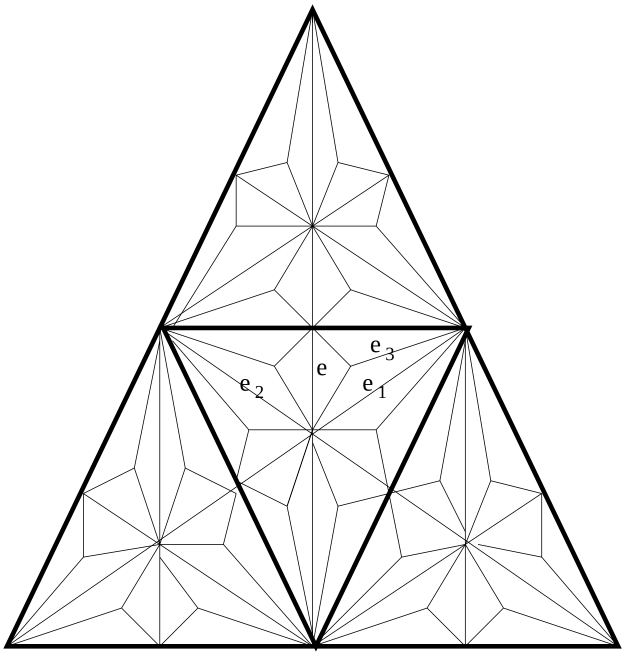


Figure 7.2

A face $f'^{(2)} > e'$ corresponds to a simplex γ of M such that either

$$\gamma > \beta > \alpha \quad \text{or} \quad \beta > \gamma > \alpha \quad \text{or} \quad \beta > \alpha > \gamma.$$

Let

$$\begin{aligned} S_1 &= \{\gamma \quad \text{s.t. } \gamma > \beta\} \\ S_2 &= \{\gamma \quad \text{s.t. } \beta > \gamma > \alpha\} \\ S_3 &= \{\gamma \quad \text{s.t. } \alpha > \gamma\}. \end{aligned}$$

An edge $\tilde{e}' \neq e'$ such that $\exists\ v'$ with $\tilde{e}' > v'$ and $e' > v'$ corresponds to a simplex γ of M with $\alpha \neq \gamma \neq \beta$ and

$$\gamma > \beta \quad \text{or} \quad \beta > \gamma \quad \text{or} \quad \gamma > \alpha \quad \text{or} \quad \alpha > \gamma.$$

There is a 2-simplex containing $\tilde{e}'$ and e' if and only if the simplices α, β, γ form a nested sequence, i.e., if

$$\gamma > \beta > \alpha \quad \text{or} \quad \beta > \gamma > \alpha \quad \text{or} \quad \beta > \alpha > \gamma.$$

In particular, if $\tilde{e}' = [\gamma > \beta]$ or $\tilde{e}' = [\alpha > \gamma]$ then $\tilde{e}'$ and e' are faces of a single 2-simplex. Thus

$$\{\tilde{e}' \neq e' \text{ such that } \exists\ v' \text{ with } \tilde{e}' > v',\ e' > v' \text{ and } \nexists\ f'^{(2)} \text{ with } f' > \tilde{e}',\ f' > e'\}$$

can be identified with

$$\{\gamma \text{ s.t. } \beta > \gamma \neq \alpha,\ \gamma \not< \alpha,\ \gamma \not> \alpha\} \cup \{\gamma \text{ s.t. } \beta \neq \gamma > \alpha,\ \gamma \not> \beta,\ \gamma \not< \beta\}.$$

It will be useful to decompose the first set into 2 sets; those γ's which share a common face with α, and those that don't. Similarly, we decompose the second set into 2 sets; those γ's such that γ and β are faces of the same simplex, and those γ's such that γ and β are not faces of the same simplex. Let

$$\begin{aligned}
S_4 &= \{\gamma \text{ s.t. } \beta > \gamma \neq \alpha,\ \gamma \not< \alpha,\ \gamma \not> \alpha \text{ and } \exists\ \delta \text{ with } \delta < \alpha,\ \delta < \gamma\}\\
S_5 &= \{\gamma \text{ s.t. } \beta > \gamma \neq \alpha,\ \gamma \not< \alpha,\ \gamma \not> \alpha \text{ and } \nexists\ \delta \text{ with } \delta < \alpha,\ \delta < \gamma\}\\
S_6 &= \{\gamma \text{ s.t. } \beta \neq \gamma > \alpha,\ \delta \not> \beta,\ \delta \not< \beta \text{ and } \exists\ \delta \text{ with } \delta > \beta,\ \delta > \gamma\}\\
S_7 &= \{\gamma \text{ s.t. } \beta \neq \gamma > \alpha,\ \delta \not> \beta, \delta \not< \beta \text{ and } \nexists\ \delta \text{ with } \delta > \beta,\ \delta > \gamma\}.
\end{aligned}$$

We can now write

$$\operatorname{Ric}\left(\left[\beta^{(b)} > \alpha^{(a)}\right]\right) = \#S_1 + \#S_2 + \#S_3 + 2 - \#S_4 - \#S_5 - \#S_6 - \#S_7.$$

Suppose $\gamma \in S_1$, i.e., $\gamma > \beta$. Let $\dim \gamma = c > b$. We can label the vertices of γ $\{v_0, v_1, \ldots, v_c\}$ such that

$$\alpha = \text{ span } \{v_0, \ldots, v_a\},\ \beta = \text{ span } \{v_0, \ldots, v_b\}.$$

We write

$$\alpha = [v_0, \ldots, v_a],\ \beta = [v_0, \ldots, v_a, \ldots, v_b],\ \gamma = [v_0, \ldots, v_a, \ldots, v_b, \ldots, v_c].$$

Consider the simplex

$$\tilde{\gamma} = [v_0, \ldots, v_a, (v_{a+1}, \ldots, v_b), v_{b+1}, \ldots, v_c]$$

where the notation $(v_{a+1}, \ldots, v_b)$ means replace the set of vertices $\{v_{a+1}, \ldots, v_b\}$ by any (possibly empty) proper subset. Then $\tilde{\gamma} \in S_6$. Moreover, there are $2^{b-a} - 1$ such $\tilde{\gamma}$'s, and the sets of such $\tilde{\gamma}$'s arising from different γ's are disjoint. Hence

$$\#S_6 \geq \left(2^{b-a} - 1\right) \#S_1.$$

(In fact, $\#S_6 = (2^{b-a} - 1)\#S_1$ but we will not need this fact).

Now suppose $\gamma \in S_3$, i.e., $\gamma < \alpha$. Let $\dim \gamma = c < a$. Label the vertices of β so that

$$\gamma = [v_0, \ldots, v_c] \quad \alpha = [v_0, \ldots, v_c, \ldots, v_a] \quad \beta = [v_0, \ldots, v_c, \ldots, v_a, \ldots, v_b].$$

Consider

$$\tilde{\gamma} = [v_{c+1}, \ldots, v_a, (v_{a+1}, \ldots, v_b)]$$

where $(v_{a+1}, \ldots, v_b)$ denotes any nonempty possibly nonproper subset of $\{v_{a+1}, \ldots, v_b\}$. Then $\tilde{\gamma} \in S_4$. There are $2^{b-a} - 1$ such $\tilde{\gamma}$'s, and the $\tilde{\gamma}$'s corresponding to different γ's are disjoint. Thus

$$\#S_4 \geq \left(2^{b-a} - 1\right) \#S_3.$$

(Again we have equality, but it does not matter). In fact, we can say precisely what $\#S_3$ is. Namely, each a-simplex α has precisely $2^{a+1} - 2$ nonempty proper faces. Thus

$$\begin{aligned} \#S_3 &= 2^{a+1} - 2 \\ \#S_4 &= \left(2^{b-a} - 1\right)\left(2^{a+1} - 2\right). \end{aligned}$$

Now suppose $\gamma \in S_2$. Label the vertices of β such that

$$\alpha = [v_0, \ldots, v_a] \quad \beta = [v_0, \ldots, v_a, \ldots, v_b].$$

Then

$$\gamma = [v_0, \ldots, v_a, (v_{a+1}, \ldots, v_b)]$$

where $(v_{a+1}, \ldots, v_b)$ is a nonempty proper subset of $\{v_{a+1}, \ldots, v_b\}$. Hence

$$\#S_2 = 2^{b-a} - 2.$$

We now consider S_5. Label the vertices of β so that

$$\alpha = [v_0, \ldots, v_a] \quad \beta = [v_0, \ldots, v_a, \ldots, v_b].$$

The elements of S_5 are precisely those of the form

$$\gamma = [(v_{a+1}, \ldots, v_b)]$$

where $(v_{a+1}, \ldots, v_b)$ denotes any nonempty (not necessarily proper) subset. Thus

$$\#S_5 = 2^{b-a} - 1.$$

Lastly, we consider S_7. Given $\beta^{(b)} > \alpha^{(a)}$ there is at least one n-simplex $\gamma^{(n)} > \alpha$ such that $\gamma \not> \beta$. Thus

$$\#S_7 \geq 1.$$

Piecing it all together, we find

$$\begin{aligned} \operatorname{Ric}\left[\beta^{(b)} > \alpha^{(a)}\right] &\leq \#S_1 + \left(2^{b-a} - 2\right) + \left(2^{a+1} - 2\right) + 2 \\ &\quad - \left(2^{b-a} - 1\right)\left(2^{a+1} - 2\right) - \left(2^{b-a} - 1\right) - \left(2^{b-a} - 1\right)\#S_1 - 1 \\ &\leq \left(2 - 2^{b-a}\right)\#S_1 + \left(2 - 2^{b-a}\right)\left(2^{a+1} - 2\right) \\ &\leq 0 \end{aligned}$$

since $b-a \geq 1$ and $a \geq 0$. Thus, every edge of M' has nonpositive Ricci curvature.

The next step is to subdivide each n-simplex of M' in a stellar manner. That is, for each n-simplex σ of M', add a vertex v_σ to the interior of σ, and join v_σ to each face of σ. Thus, in the interior of σ there is a $(p+1)$-simplex for each p-face of σ. Let M^* be the resulting simplicial complex. Every edge of M^* is either an edge of M', or is the join, for some n-simplex σ of M', of v_σ and a vertex of σ.

To complete the proof of Theorem 7.2 we will show that for every edge e of M'

$$\mathrm{Ric}^*(e) < \mathrm{Ric}'(e)$$

where $\mathrm{Ric}^*(e)$ denotes the Ricci curvature of e considered as an edge in M^*, and $\mathrm{Ric}'(e)$ is the Ricci curvature of e considered as an edge in M'. We will then show that if e is the join of some v_σ and a vertex of σ then

$$\mathrm{Ric}(e) < 0.$$

Let e be an edge of M'. If $n=2$ then

$$\#\{f^{(2)} > e \text{ in } M^*\} = \{f^{(2)} > e \text{ in } M'\} = 2.$$

If $n>2$ then there is a new (i.e., in M^* but not in M') 2-simplex $f > e$ for each n-simplex $\sigma^{(n)}$ of M' with $\sigma > e$ namely the join of e and v_σ. There is a new parallel neighbor for each n-simplex $\sigma^{(n)}$ of M' such that $\sigma > v$ for some vertex v of e but $\sigma \not> e$, namely the join of v and v_σ. Therefore

$$\mathrm{Ric}^*(e) \leq \mathrm{Ric}'(e) + \#\left\{\sigma_{M'}^{(n)} > e\right\} - \#\left\{\sigma_{M'}^{(n)} \text{ s.t. } \exists\, v < e \text{ with } \sigma > v \text{ and } \sigma \not> e\right\}.$$

An edge of M' corresponds to a pair of simplices $\beta^{(b)} > \alpha^{(a)}$ of M. Suppose $\gamma^{(n)}$ is an n-cell of M with $\gamma \geq \beta$. Label the vertices of γ so that

$$\alpha = [v_0, \ldots, v_a] \quad \beta = [v_0, \ldots, v_b] \quad \gamma = [v_0, \ldots, v_n].$$

Note that there are $(a+1)!\ (b-a)!\ (n-b)!$ such labelings. For any such labeling, there is an n-cell γ' of M' containing the edge $e = [\beta > \alpha]$. Namely, consider

$$\gamma' = \left[\gamma^{(n)} > \gamma^{(n-1)} > \cdots > \gamma^{(b)} = \beta > \cdots > \gamma^{(a)} = \alpha > \cdots > \gamma^{(0)}\right] \tag{7.2}$$

where $\gamma^{(k)} = [v_0, \ldots, v_k]$.

Every n-cell of M' with e as a face arises in this fashion. Thus

$$\#\left\{\sigma_{M'}^{(n)} > e\right\} = \#\left\{\sigma_M^{(n)} \geq \beta^{(b)}\right\} a!(b-a)!(n-b)!$$

Each $\gamma_M^{(n)} \geq \beta$ also gives rise to n-simplices $\sigma_{M'}^{(n)}$ such that $\sigma > v$ for some vertex v of e, but $\sigma \not> e$. Namely, the vertices of e correspond to the simplices α and β of M. An n-cell of M' which contains β but not α, is a sequence of simplices of M

$$\gamma^{(n)} > \gamma^{(n-1)} > \cdots > \gamma^{(0)}$$

such that

$$\gamma^{(b)} = \beta, \quad \gamma^{(a)} \neq \alpha.$$

Given $\sigma_M^{(n)} \geq \beta$, label the vertices $\{v_0, v_1, \ldots, v_a, \ldots, v_b, \ldots, v_n\}$. If $\beta = \operatorname{span}\{v_0, \ldots, v_b\}$ but $\alpha \neq \operatorname{span}\{v_0, \ldots, v_a\}$ then we get a desired n-cell of M' as in (7.2). To find the number of such orderings we note that there are $(n-b)!$ possible orderings of $\{v_{b+1}, \ldots, v_n\}$ and $(b+1)!$ possible orderings of $\{v_0, \ldots, v_b\}$. From this last number, we must subtract those orderings with $\alpha = \operatorname{span}\{v_0, \ldots, v_a\}$. There are $(a+1)!\,(b-a)!$ such orderings. Hence, σ gives rise to $(n-b)!\,((b+1)! - (a+1)!(b-a)!)$ edges $\tilde{e}$ of M^* which share the vertex β with e, but do not share a 2-simplex in M^*.

Similarly, to find edges $\tilde{e}$ of M^* which share the vertex α with e, but do not share a 2-simplex in M^* we need to find orderings of the vertices of σ so that $\alpha = \operatorname{span}\{v_0, \ldots, v_a\}$ but $\beta \neq \operatorname{span}\{v_0, \ldots, v_b\}$. There are $(a+1)!\,((n-a)! - (b-a)!(n-b)!)$ such orderings.

Piecing it all together, we find

$$\begin{aligned}
\operatorname{Ric}^*(e) \leq \operatorname{Ric}'(e) + \#\{\sigma^{(n)} \geq \beta\}& \\
[(a+1)!(b-a)!(n-b)! - (n-b)!\,((b+1)! - (a+1)!(b-a)!)& \\
- (a+1)!\,((n-a)! - (b-a)!(n-b)!)]&.
\end{aligned}$$

Since $\#\{\sigma^{(n)} \geq \beta\} > 0$, it is sufficient to show

$$\begin{aligned}
&(a+1)!(b-a)!(n-b)! - (n-b)!\,((b+1)! - (a+1)!(b-a)!) \\
&- (a+1)!\,((n-a)! - (b-a)!(n-b)!) < 0.
\end{aligned} \tag{7.3}$$

The expression on the left hand side of (7.3) is equal to

$$3(a+1)!(b-a)!(n-b)! - (n-b)!(b+1)! - (a+1)!(n-a)!. \tag{7.4}$$

We now observe

$$\begin{aligned}
&\frac{(n-b)!(b+1)!}{(a+1)!(b-a)!(n-b)!} = \frac{(b+1)!}{(a+1)!(b-a)!} = \frac{(b+1)(b)\ldots(a+2)}{(b-a)(b-a-1)\ldots(1)} \\
&\quad \geq a+2 \geq 2 \\
&\frac{(a+1)!(n-a)!}{(a+1)!(b-a)!(n-b)!} = \frac{(n-a)!}{(b-a)!(n-b)!} = \frac{(n-a)(n-a-1)\ldots(n-b+1)}{(b-a)(b-a-1)\ldots(1)} \\
&\quad \geq n-b+1 \geq 1
\end{aligned} \tag{7.5}$$

Therefore, the expression in (7.4) is ≤ 0. It can equal 0 if and only if $a + 2 = 2$ and $n - b + 1 = 1$, i.e., if $b = n$ and $a = 0$. Now we use the fact that $n \geq 2$ to observe that if $a = 0$ and $b = n \geq 2$ then the quotient in (7.5) is > 2, so that even in this case, the expression in (7.4) < 0.

This completes the proof that for every edge e of M'

$$\mathrm{Ric}^*(e) < 0.$$

We must now show that for every "new" edge e of M^*

$$\mathrm{Ric}^*(e) < 0.$$

The edge is in the interior of an n-simplex $\sigma^{(n)}$ of M' and is the join of a vertex v_σ in the interior of σ and a vertex w of σ. If $\tilde{e}$ is any other edge with v_σ as a vertex, then $\tilde{e}$ lies in the interior of σ, and e and $\tilde{e}$ span a 2-simplex $f^{(2)}_{M^*}$ in the interior of σ. Each 2-simplex of M^* which has e as an edge arises in this manner. There are n such edges $\tilde{e}$. Thus

$$\#\left\{f^{(2)}_{M'} > e\right\} = n$$

and there are no parallel neighbors $\tilde{e}$ of e which share the vertex v_σ.

Now we must count the parallel neighbors of e. If $\tilde{e}$ is an edge in M' which satisfies $w < \tilde{e} \not< \sigma$ then $\tilde{e}$ is a parallel neighbor of e. There are n edges $\tilde{e}$ with $w < \tilde{e} < \sigma$. Hence, there are

$$\#\left\{\tilde{e}^{(1)}_{M'} > w\right\} - n \tag{7.6}$$

parallel neighbors of this type.

Moreover, if $\tilde{\sigma}^{(n)}$ is any n-simplex of M' st $w < \tilde{\sigma}^{(n)} \neq \sigma$, then the stellar subdivision of $\tilde{\sigma}$ adds an edge joining w to $v_{\tilde{\sigma}}$ which is a parallel neighbor to e. There are

$$\#\left\{\tilde{\sigma}^{(n)}_{M'} > w\right\} - 1 \tag{7.7}$$

such parallel neighbors.

We now calculate (7.6) and (7.7). The vertex w of M' can be identified with a simplex $\alpha^{(a)}$ of M. Each edge $\tilde{e} > w$ can be identified with a simplex β such that $\beta > \alpha$ or $\alpha > \beta$. Thus

$$\#\left\{\tilde{e}_{M'} > w\right\} = \#\left\{\beta_M > \alpha\right\} + \#\left\{\beta_M < \alpha\right\}.$$

In any simplicial complex which is a combinatorial n-manifold, the number of b-simplices ($b \leq n$) which have a given a-simplex as a face is at least as large as the corresponding number in an $(n+1)$-simplex, i.e., $\binom{(n+2)-(a+1)}{b-a} = \binom{n-a+1}{b-a}$. Thus

$$\#\left\{\beta_M > \alpha\right\} \geq \sum_{b-a+1}^{n} \binom{n-a+1}{b-a} = \sum_{k=1}^{n-a} \binom{n-a+1}{k} = 2^{n-a+1} - 2.$$

Moreover,

$$\#\{\beta_M < \alpha\} = \sum_{b=0}^{a-1} \binom{a+1}{b+1} = \sum_{k=1}^{a} \binom{a+1}{k} = 2^{a+1} - 2.$$

We now calculate $\#\{\tilde{\sigma}^{(n)}_{M'} > w\}$. The n-simplex σ of M' can be identified with a sequence $\gamma^{(n)} > \gamma^{(n-1)} > \cdots > \gamma^{(a)} = \alpha > \cdots > \gamma^{(0)}$ of simplices in M. Label the vertices of $\gamma^{(n)}$ $\{v_0, \ldots, v_n\}$ so that $\gamma^{(k)} = [v_0, \ldots, v_k]$. If we reorder the vertices so that we still have $\alpha = [v_0, \ldots, v_a]$ and let

$$\tilde{\sigma} = \left[\tilde{\gamma}^{(n)} > \tilde{\gamma}^{(n-1)} > \cdots > \tilde{\gamma}^{(a)} = \alpha > \cdots > \tilde{\gamma}^{(0)}\right]$$

where

$$\tilde{\gamma}^{(k)} = [v_0, \ldots, v_k].$$

The number of such orderings is $(a+1)!(n-a)!$ so that

$$\#\left\{\tilde{\sigma}^{(n)}_{M'} > w\right\} \geq (a+1)!(n-a)!.$$

Summarizing,

$$\begin{aligned}\mathrm{Ric}^*(e) &\leq n + 2 - \left[\left((2^{n-a+1} - 2) + (2^{a+1} - 2) - n\right) + ((a+1)!(n-a)! - 1)\right] \\ &= 2n + 7 - \left[2^{n-a+1} + 2^{a+1} + (a+1)!(n-a)!\right].\end{aligned}$$

All that remains is to see that for all $n \geq 2$ and all a, $0 \leq a \leq n$, this expression is negative. Note that

$$\begin{aligned}2^{n-a+1} + 2^{a+1} &\geq 2\left(2^{\frac{n+2}{2}}\right) = 2^{\frac{n}{2}+2} \\ &= 4(2^n)^{\frac{1}{2}}.\end{aligned}$$

Moreover, for $n \geq 2$, $2^n \geq n^2$. Therefore, for $n \geq 2$

$$2^{n-a+1} + 2^{a+1} \geq 4n.$$

If $n \geq 4$, $4n > 2n + 7$, so

$$\mathrm{Ric}^*(e) < 0.$$

If $n = 3$, we note that, for all $0 \leq a \leq 3$ $(a+1)!(n-a)! \geq 3$ so that

$$\begin{aligned}2^{3-a+1} + 2^{a+1} + (a+1)!(3-a)! &\geq 4(3) + 3 \\ &= 15 > 13 = 2(3) + 7\end{aligned}$$

so again

$$\mathrm{Ric}^*(e) < 0.$$

Lastly, if $n = 2$ we examine the 3 possibilities

$$\begin{aligned}2^{2-a+1} + 2^{a+1} + (a+1)!(2-a)! &= \left\{\begin{array}{ll} 12 & \text{if } a = 0, 1 \\ 16 & \text{if } a = 2 \end{array}\right\} > 11 \\ &= 2(2) + 7\end{aligned}$$

so in all cases

$$\mathrm{Ric}^*(e) < 0.$$

This completes the proof. $\square$

§8. Cubed Manifolds.

Of some recent interest are those quasiconvex complexes in which all of the cells are combinatorial cubes ([AR]). In this case, the formulas for curvature simplify somewhat. Our first goal in this section is to present an explicit formula for the curvtaure of a quasiconvex complex of cubes. There is a more classical notion of curvature for manifolds which have been endowed with the structure of a quaiscomvex complex of cubes. We describe this more classical notion, and then compare it to the curvature presented in this paper.

Theorem 8.1. *Let M be a quasiconvex complex of cubes and endow each cell α with the weight $\omega_\alpha = 1$. Then for any p-cell α,*

$$\mathcal{F}_p(\alpha) = 2p(2 + \deg(\alpha)) - \sum_{\gamma^{(p-1)}<\alpha} deg(\gamma),$$

where $\deg(\alpha^{(p)}) = \#\{\beta^{(p+1)} > \alpha\}$.

Proof: For any p-cell α

$$\mathcal{F}_p(\alpha) = \#\{\beta^{(p+1)} > \alpha\} + \#\{\gamma^{(p-1)} < \alpha\} - \#\{\text{parallel neighbors of } \alpha\}.$$

Say a parallel neighbor α' of α is an $(p+1)$-neighbor if $\exists\beta^{(p+1)}$ with $\beta > \alpha$, $\beta > \alpha'$, and a $(p-1)$-neighbor if $\exists\gamma^{(p-1)}$ with $\gamma < \alpha$, $\gamma < \alpha'$. Then

$$\#\{\text{parallel neighbors of } \alpha\} = \#\{(p+1)\text{-neighbors of } \alpha\} + \#\{(p-1)\text{-neighbors of } \alpha\}.$$

Each $\beta^{(p+1)} > \alpha$ is an $(p+1)$-cube. The cell α is one p-face of β, and the opposite p-face α' is an $(p+1)$-neighbor of α. No other face of β is a parallel neighbor of α, and all $(p+1)$-neighbors occur in this fashion. Therefore,

$$\#\{\beta^{(p+1)} > \alpha\} = \#\{(p+1)\text{-neighbors of } \alpha\}.$$

This yields the identity

$$\mathcal{F}_p(\alpha) = \#\{\gamma^{(p-1)} < \alpha\} - \#\{(p-1)\text{-neighbors of } \alpha\}.$$

For each $\gamma^{(p-1)} < \alpha$, say a parallel neighbor α' of α is a γ-neighbor if $\gamma < \alpha'$. Then

$$\#\{(p-1)\text{-neighbors of } \alpha\} = \sum_{\gamma^{(p-1)}<\alpha} \#\{\gamma\text{-neighbors of } \alpha\}$$

so that

$$\mathcal{F}_p(\alpha) = \sum_{\gamma^{(p-1)}<\alpha} 1 - \#\{\gamma\text{-neighbors of } \alpha\}.$$

Fix a $\alpha^{(p-1)} < \alpha$. Let $\tilde{\alpha}^{(p)} \neq \alpha$ be a p-cell which has γ as a face. Then $\tilde{\alpha}$ is a γ-neighbor of α unless there is a $\beta^{(p+1)}$ which has both α and $\tilde{\alpha}$ as faces. On the other hand, for any $\beta^{(p+1)} > \alpha$ there is exactly one p-cell $\tilde{\alpha}$ such that $\beta > \tilde{\alpha} > g$. Therefore, the number of γ-neighbors of α is $\deg(\gamma) - (\deg(\alpha) + 1)$. Therefore,

$$\mathcal{F}_p(\alpha) = \sum_{\gamma^{(p-1)}<\alpha} 1-[\deg(\gamma)-(\deg(\alpha)+1)] = (\#\{\gamma^{(p-1)} < \alpha\})(2+\deg(\alpha)) - \sum_{\gamma^{(p-1)}<\alpha} \deg(\gamma)$$

$$= 2p(2 + \deg(\alpha)) - \sum_{\gamma^{(p-1)}<\alpha} \deg(\gamma).$$

□

If M is a quasiconvex complex of cubes which is homeomorphic to a manifold, we say that M is a cubed manifold. As mentioned earlier, there is a more classical notion of the curvature of a cubed manifold. The idea is that the combinatorial curvature of M should reflect how $\deg(\alpha)$ compares with one one would see in the standard cubic decomposition of $\mathbb{R}^n$. Let M be a cubed manifold. Say M has positive (resp. nonnegative, nonpositive, negative) cube curvature at α if $\deg(\alpha) < 2(n-p)$ (resp. $\leq 2(n-p), \geq (n-p), > 2(n-p)$). Say M has positive (nonnegative …) cube curvature if M has positive (nonnegative …) curvature at each cell.

It is easy to check that if $n = 2$ then M has a cubing with positive (nonnegative, …) cube curvature if and only if M has a Riemannian metric with Gauss curvature everywhere positive (nonnegative, …) curvature. In [AR] Aitchison and Rubinstein have shown that cubed 3-manifolds with negative cube curvature share many properties with Riemannian manifolds with negative curvature.

Theorem 8.1 easily yields the following corollary.

Corollary 8.2. *Let M be a cubed manifold. Suppose that M has positive (resp. nonnegative, nonpositive, negative) cube curvature at each $(n-2)$-cell. Then $\mathcal{F}_{n-1}$ is positive (resp. nonnegative, nonpositive, negative).*

We can now apply the Bochner theorems to learn

Corollary 8.3. *Let M be a cubed manifold.*

(i) *If M has positive cube curvature at each $(n-2)$-cell then*

$$H_{n-1}(M, \mathbb{R}) = 0.$$

(ii) *If M has nonnegative cube curvature at each $(n-2)$-cell then*

$$\dim H_{n-1}(M, \mathbb{R}) \leq n.$$

By Poincaré duality, the conclusions in this theorem are equivalent to $H_1(M, \mathbb{R}) = 0$ and $\dim H_1(M, \mathbb{R}) \leq n$ respectively.

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