Instructions: You have **two hours** to complete this exam. You should work alone, without access to the textbook or class notes. You may not use a calculator. Do not discuss this exam with anyone except your instructor.

This exam consists of 6 questions. You must show your work to receive full credit. Be sure to **indicate your final answer clearly** for each question. Pledge your exam when finished, and include your name and section number on the front of the exam. The exam is due by **Friday, 5 p.m.** Good luck!

1. Let $\mathbf{l}_1$ and $\mathbf{l}_2$ be the intersecting lines given by the parameterizations

$$\mathbf{l}_1(t) = (1,0,1) + t(0,2,1),$$
$$\mathbf{l}_2(s) = (3,3,4) + s(-2,1,-1).$$

   (a) Find the angle between $\mathbf{l}_1$ and $\mathbf{l}_2$.

   (b) Find a vector perpendicular to both $\mathbf{l}_1$ and $\mathbf{l}_2$.

Solution 1.

(a). [8 points] Let $\mathbf{v}_1$ be the direction vector for the line $\mathbf{l}_1$, and let $\mathbf{v}_2$ be the direction vector for the line $\mathbf{l}_2$. The angle θ between lines $\mathbf{l}_1$ and $\mathbf{l}_2$ satisfies

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1||\mathbf{v}_2| \cos\theta$$

So

$$\begin{aligned} (0)(-2) + (2)(1) + (1)(-1) &= \sqrt{0^2+2^2+1^2}\sqrt{(-2)^2+1^2+(-1)^2}\cos\theta \\ \cos\theta &= \frac{1}{\sqrt{30}} \\ \theta &= \arccos\frac{1}{\sqrt{30}} \end{aligned}$$

(b). [7 points] Such a vector is $\mathbf{v}_1 \times \mathbf{v}_2$:

$$\begin{aligned} \mathbf{v}_1 \times \mathbf{v}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 1 \\ -2 & 1 & -1 \end{vmatrix} \\ &= \mathbf{i}[(2)(-1) - (1)(1)] + \mathbf{j}[(1)(-2) - (0)(-1)] + \mathbf{k}[(0)(1) - (2)(-2)] \\ &= -3\mathbf{i} + -2\mathbf{j} + 4\mathbf{k} \end{aligned}$$

Any scalar multiple of $(-3,-2,4)$ is acceptable.

2. Let $f(x,y) = 4x^2 + y^2$.

(a) Sketch the level curves of $f(x, y) = k$ for $k = 2,\ 4,\ 8$.

(b) On your graph from part (a), draw a vector at the point $(1, 2)$ which gives the direction of the gradient of f at that point.

(c) Graph and describe the $x = 0$, $y = 0$, and $z = 0$ cross sections of the graph of $f(x, y)$. Each cross section should appear on a separate graph.

(d) Sketch the graph of f.

Solution 2. Separate pdf file.

3. Suppose a particle is travelling in $\mathbb{R}^2$ such that at time t its position is given by

$$\mathbf{c}(t) = \big(t^2 - \pi t, \sin(t)\big).$$

(a) Find the velocity of the particle at time t. At what time(s) t does the particle come to a stop?

(b) Give a parametric equation for the tangent line to $\mathbf{c}(t)$ at time $t = \frac{\pi}{4}$.

Solution 3.

(a). [8 points] The velocity is $\mathbf{c}'(t) = (2t - \pi, \cos t)$. We see $\mathbf{c}'(t) = \vec{0}$ precisely at $t = \pi/2$.

(b). [7 points] Note that $\mathbf{c}(\pi/4) = \left(\frac{-3\pi^2}{16}, \frac{1}{\sqrt{2}}\right)$ and $\mathbf{c}'(t) = \left(\frac{-\pi}{2}, \frac{1}{\sqrt{2}}\right)$. The equation of the tangent line at $t = \pi/4$ is

$$\left(\frac{-3\pi^2}{16}, \frac{1}{\sqrt{2}}\right) + (t - \pi/4)\left(\frac{-\pi}{2}, \frac{1}{\sqrt{2}}\right)$$

4. The temperature at a point (x, y) on a flat metal plate is given by the function

$$T(x, y) = \frac{60}{1 + 2x^2 + y^2},$$

where T is measured in °C and x, y are measured in meters. Find the rate of change of temperature with respect to distance at the point $(2, 1)$ in

(a) the y-direction,

(b) the direction given by the vector $\mathbf{i} - 2\mathbf{j}$,

(c) the direction of the maximum rate of change.

(d) Is there a direction at $(2, 1)$ along which T does not change? If so, find a vector pointing in that direction.

Solution 4. Note that

$$\mathbf{D}T(x, y) = \left(\frac{-240x}{(1 + 2x^2 + y^2)^2}, \frac{-120y}{(1 + 2x^2 + y^2)^2} \right)$$

(a). [5 points] The derivative in the y-direction at $(2, 1)$ is

$$\nabla T|_{(2,1)} \cdot (0, 1) = \frac{-120}{(1 + 2 \cdot 2^2 + 1^2)^2} = \frac{-6}{5} \frac{{}^\circ \text{ C}}{\text{m}}$$

(b). [5 points] First, we must normalize $(1, -2)$ to $\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$. The directional derivative at $(2, 1)$ is

$$\nabla T|_{(2,1)} \cdot \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right) = \left(\frac{-24}{5}\right)\left(\frac{1}{\sqrt{5}}\right) + \left(\frac{-6}{5}\right)\left(\frac{-2}{\sqrt{5}}\right) = \frac{-12\sqrt{5}}{25} \frac{{}^\circ \text{ C}}{\text{m}}$$

(c). [5 points] The direction of maximum rate of change is $\nabla T|_{(2,1)}$. First, normalize this to $\frac{\nabla T|_{(2,1)}}{||\nabla T|_{(2,1)}||}$. Thus, the rate of change in this direction is

$$\nabla T|_{(2,1)} \cdot \frac{\nabla T|_{(2,1)}}{||\nabla T|_{(2,1)}||} = \frac{||\nabla T|_{(2,1)}||^2}{||\nabla T|_{(2,1)}||} = ||\nabla T|_{(2,1)}|| = \sqrt{\left(\frac{-24}{5}\right)^2 + \left(\frac{-6}{5}\right)^2} = \frac{\sqrt{612}}{5} = \frac{6\sqrt{17}}{5}$$

(d). [5 points] We must find $\vec{v} = (v_1, v_2)$ such that $\frac{-24}{5} v_1 + \frac{-6}{5} v_2 = 0$. $\vec{v} = (1, -4)$ works.

5. Let $f(x, y, z) = x^3 y^2 z$, and let $\mathbf{c}(t) = \big(e^t, \sin(t), g(t)\big)$, where $g(t)$ is a differentiable function. Use a Chain Rule from vector calculus to find

$$\frac{d}{dt} f\big(\mathbf{c}(t)\big).$$

Your final answer will be in terms of $g(t)$, $g'(t)$ and t.

Solution 5. [10 points] The chain rule says that $\frac{d}{dt}f(\mathbf{c}(t)) = \mathbf{D}f(\mathbf{c}(t))\mathbf{D}\mathbf{c}(t)$. Since $\mathbf{D}f(x,y,z) = (3x^2y^2z, 2x^3yz, x^3y^2)$ and

$$\mathbf{D}\mathbf{c}(t) = \begin{pmatrix} e^t \\ \cos t \\ g'(t) \end{pmatrix}$$

we have

$$\begin{aligned} \frac{d}{dt}f(\mathbf{c}(t)) &= \left(3e^{2t}\sin^2(t)g(t), 2e^{3t}\sin(t)g(t), e^{3t}\sin^2(t)\right)\begin{pmatrix} e^t \\ \cos t \\ g'(t) \end{pmatrix} \\ &= 3e^{3t}\sin^2(t)g(t) + 2e^{3t}\sin(t)\cos(t)g(t) + e^{3t}\sin^2(t)g'(t) \end{aligned}$$

You must remember to evaluate $\mathbf{D}f$ at $\mathbf{c}(t)$.

6. Let

$$f(x,y) = x^3 + 3x^2y^2 + y^3.$$

(a) Suppose a particle is sitting on the graph of f at the point $(-1,1)$. In which direction should the particle move in order for its height to decrease most rapidly?

(b) Give the equation for the tangent plane of $f(x,y)$ at the point $(-1,1)$.

(c) Give the quadratic Taylor polynomial for $f(x,y)$ at the point $(-1,1)$.

Solution 6.

(a). [7 points] In order to find the direction for the height to decrease most rapidly, we need to find $-\nabla f$, which is $-\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$, and then evaluate at $(-1,1)$. We have

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2 + 6xy^y \\ \frac{\partial f}{\partial y} &= 6x^2y + 3y^2 \end{aligned}$$

So $-\nabla f\big|_{(-1,1)} = (3, -9)$.

(b). [6 points] A normal vector for the tangent plane is $(3,-9,1)$, so the tangent plane is given by

$$3(x+1) - 9(y-1) + (z-3) = 0$$

(c). [7 points] Take the second order derivatives and evaluate at $(-1,1)$.

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2}\Big|_{(-1,1)} &= 6x + 6y^2\Big|_{(-1,1)} = 0 \\ \frac{\partial^2 f}{\partial y^2}\Big|_{(-1,1)} &= 6x^2 + 6y\Big|_{(-1,1)} = 12 \\ \frac{\partial^2 f}{\partial x \partial y}\Big|_{(-1,1)} &= 12xy\Big|_{(-1,1)} = -12 \end{aligned}$$

So the Taylor expansion is

$$T(f(x,y)) = 3 + (-3)(x+1) + 3(y-1) + 6(y-1)^2 + (-12)(x+1)(y-1)$$