Homework 9, due Thursday, Nov.20

Problem 1. Suppose $\| \ \|$ is a norm on a vector space X.

(a) Prove that $|\,\|x\| - \|y\|\,| \leq \|x - y\|$ for all $x, y \in X$.

(b) Show that $\| \ \|$ is continuous on X.

Problem 2. Suppose S is a subset of a Hilbert space X and

$$S^{\perp} = \{y \in X \,:\, \langle x, y \rangle = 0 \text{ for all } \ x \in S\} \ .$$

(a) Show that $S^{\perp}$ is a closed subspace of X.

(b) Show that $S \cap S^{\perp} = \{0\}$.

(c) Show that, for any subspace Y of X, $(Y^{\perp})^{\perp} = \overline{Y}$, the closure of Y (i.e. the set of all limit points of Cauchy sequences in Y).

Problem 3. Show that, for any buunded domain $\Omega \subset \mathbf{R}^n$, the number

$$\lambda_1 \ = \ \inf_{u \in H_0^1(\Omega)} \frac{\int_{\Omega} |Du|^2 \, dx}{\int_{\Omega} u^2 \, dx}$$

is positive.

Problem 4. Let ℓ^2 be the set of all sequences $(a_1, a_2, a_3, \cdots)$ of real numbers such that $\sum_{i=1}^{\infty} a_i^2 < \infty$. Show that ℓ^2 is a Hilbert space with the inner product

$$\langle (a_1, a_2, \cdots), (b_1, b_2, \cdots) \rangle \ = \ \sum_{i=1}^{\infty} a_i b_i \ .$$