Suppose U is a bounded smooth domain in $\mathbf{R}^n$ and $0 < R < \infty$. Recall that

$$\partial[U \times (0,R)] \;=\; \partial U \times (0,R) \;\cup\; U \times \{0\} \;\cup\; U \times \{R\} \;.$$

For any function $u \in W^{1,2}(U)$, let $g(x) = u(x)$ for $x \in \partial U$ (in the sense of traces) and

$$u_R : U \times (0,R) \to \mathbf{R} \;, \quad u_R(x,t) = u(x) \;.$$

Also, let

$$g_R : \partial[U \times (0,R)] \to \mathbf{R} \;, \quad g_R(x,t) = g(x) \; for \; x \in \partial U \times (0,R),$$

and

$$g_R(x,0) = g_R(x,1) = u(x) \; for \; x \in U \;.$$

Consider the 2 statements.

(1) u minimizes $\int_U |Dw|^2 \, dx$ among functions $w \in W^{1,2}(U)$ with $w = g$ on ∂U.

(2) u_R minimizes $\int_{U \times (0,R)} |Dv|^2 \, dx$ among functions $v \in W^{1,2}(U \times (0,R))$ with $v = g_R$ on $\partial[U \times (0,R)]$.

(a) Prove that (1) implies (2).

(b) Prove that (2) implies (1) for R sufficiently large.

We will discuss this problem later with a more interesting functional.