ANALYSIS QUALIFYING EXAM

August 2003

Please explain *all* your answers and indicate which theorems you are using.

1. Suppose $0 < \alpha < 2$.

(a) Explain what is meant by the *principal value integral* $\int_0^\infty \frac{x^\alpha}{x-x^3}\,dx$.

(b) Compute this integral.

(c) Show that the answer you obtained in (b) is in agreement with the change of variable $x = \frac{1}{y}$ in the integral.

2. Suppose that $f(x,y)$ is continuous on the plane and that there is finite M so that $|f(x,y) - f(x,z)| \le M|y-z|$ for all $x, y, z \in \mathbf{R}$.

(a) Show that, for any $x \in \mathbf{R}$, the partial derivative $\frac{\partial f}{\partial y}(x,y)$ exists for almost all $y \in \mathbf{R}$.

(b) Prove that $\frac{d}{dy}\int_0^1 f(x,y)\,dx = \int_0^1 \frac{\partial f}{\partial y}(x,y)\,dx$.

(c) Express $\frac{d}{dy}\int_0^{y^2} f(x,y)\,dx$ in terms of integrals of f and $\frac{\partial f}{\partial y}$.

3.(a) Show that the direct analog of Rolle's theorem does not apply to holomorphic functions. Do this by exhibiting an entire holomorphic function f such that $f(0) = f(1)$ and yet $f'(z)$ never takes the value 0.

(b) Suppose f is a holomorphic function on the unit disk $\{z : |z| < 1\}$. Show that f must be constant if $f(a_i) = f(b_i)$ for two sequences a_i, b_i of positive real numbers that satisfy the inequalities

$$0 < \ldots < a_{i+1} < b_{i+1} < a_i < b_i < \ldots < a_1 < b_1 < 1\,.$$

4. Suppose $0 < M < \infty$ and, for each positive integer j, $f_j : [0,1] \to [-M, M]$ is a monotone increasing function. Prove that there is a subsequence $f_{j'}$ and a countable subset A of $[0,1]$ so that $f_{j'}(t)$ converges, as $j' \to \infty$, for every $t \in [0,1] \setminus A$.

5. (a) Is there a nonconstant real function h that is continuous on the closed disk $\{z : |z| \le 1\}$, harmonic on the open disk $\{z : |z| < 1\}$, and vanishes on the upper unit semi-circle (that is, $h(e^{i\theta}) = 0$ for $0 \le \theta \le \pi$)?

(b) Is there a nonconstant complex function f that is continuous on the closed disk $\{z : |z| \le 1\}$, holomorphic on the open disk $\{z : |z| < 1\}$, and vanishes on the upper unit semi-circle (that is, $f(e^{i\theta}) = 0$ for $0 \le \theta \le \pi$)?

6. Assume that $f(x)$ is a Lebesgue measurable function on $\mathbf{R}$. Prove the function defined on $\mathbf{R}^2$ by $F(x,y) = f(x-y)$ is Lebesgue measurable.