Solutions to Analysis Exam, August, 2003

1. Suppose $0 < \alpha < 2$.

(a) The *principal value integral*

$$\int_0^\infty \frac{x^\alpha}{x - x^3}\,dx \;=\; \lim_{\epsilon\to 0}\Big(\int_0^{1-\epsilon} + \int_{1+\epsilon}^\infty\Big)\frac{x^\alpha}{x-x^3}\,dx\ .$$

Here it is necessary to consider a limit (called an improper integral) because the function $\frac{x^\alpha}{x-x^3}$ is *not* integrable near $x = 1$. It is locally integrable near 0 and ∞. The choice of $1 - \epsilon$ and $1 + \epsilon$ for approximating integration limits is the simplest (the principal value) as opposed to other choices such as $1 - \epsilon$, $1 + \epsilon^2$.

(b) Here one can apply the Cauchy integral formula to the principal branch $f(z)$ of $-\frac{z^\alpha}{z(z-1)(z+1)}$ on the domain Ω_ϵ in the upper half plane bounded by the 4 oriented intervals $[-\frac{1}{\epsilon}, -1-\epsilon]$, $[-1+\epsilon, -\epsilon]$, $[-\epsilon, \epsilon]$, $[\epsilon, 1-\epsilon]$, $[1+\epsilon, \frac{1}{\epsilon}]$, the upper counterclockwise oriented semicircle $C_{1/\epsilon}$ of radius $\frac{1}{\epsilon}$, and the 3 clockwise oriented upper semicircles $D_\epsilon^{-1}, D_\epsilon^0, D_\epsilon^{+1}$ of radius ϵ centered at $-1, 0, 1$. As $\epsilon \to 0$, the integral over the big semicircle $C_{1/\epsilon}$ approaches 0 because there $|f(z)|(\frac{1}{\epsilon}) \le C(\frac{1}{\epsilon})^{1+\alpha-3} \to 0$. Also the integral over D_ϵ^0 approaches 0 as $\epsilon \to 0$ because there $|f(z)|\epsilon \le C\epsilon^\alpha \to 0$. The integrals over D_ϵ^{+1} and D_ϵ^{-1} approach $-\pi i$ times the residues of f at $+1$ and -1, which are $-\frac{1}{2}$ and $-\frac{1}{2}e^{\pi i\alpha}$. On the positive X-axis $f(z) = \frac{x^\alpha}{x-x^3}$ and on the negative X-axis $f(z) = \frac{e^{\pi i\alpha}|x|^\alpha}{x-x^3}$. Changing variables $x \mapsto -x$ for the integrals on the negative X-axis now gives

$$0 \;=\; \int_{\Omega_\epsilon} f(z)\,dz \;= o(\epsilon) + \frac{\pi i}{2}(1 + e^{\pi i\alpha}) + (1 - e^{\pi i\alpha})\Big(\int_\epsilon^{1-\epsilon} + \int_{1+\epsilon}^{1/\epsilon}\Big)\frac{x^\alpha}{x-x^3}\,dx\ .$$

Letting $\epsilon \to 0$ the principal value integral equals

$$\int_0^\infty \frac{x^\alpha}{x-x^3}\,dx \;=\; \Big(\frac{\pi i}{2}\Big)\frac{e^{\pi i\alpha}+1}{e^{\pi i\alpha}-1} \;=\; \frac{\pi}{2}\cot\Big(\frac{\pi\alpha}{2}\Big)\ .$$

(c)

$$\begin{aligned} I(\alpha) \;&=\; \int_0^\infty \frac{x^\alpha}{x-x^3}\,dx \;=\; \int_\infty^0 \frac{y^{-\alpha}}{\frac{1}{y}-\frac{1}{y^3}}\frac{-dy}{y^2} \;=\; \int_0^\infty \frac{y^{-\alpha}}{y-\frac{1}{y}}\,dy \\ &=\; \int_0^\infty \frac{y^{2-\alpha}}{y^3-y}\,dy \;=-\int_0^\infty \frac{y^{2-\alpha}}{y-y^3}\,dy \;=\; -I(2-\alpha) \end{aligned}$$

and $\cot\frac{\pi(2-\alpha)}{2} = \cot\left(\pi - \frac{\pi\alpha}{2}\right) = -\cot\frac{\pi\alpha}{2}$ agrees.

2. Suppose that $f(x,y)$ is continuous on the plane and that there is finite M so that $|f(x,y) - f(x,z)| \le M|y-z|$ for all $x, y, z \in \mathbf{R}$.

(a) For any $x \in \mathbf{R}$, the function $f(x,\cdot)$ is Lipschitz, hence absolutely continuous. So the partial derivative $\frac{\partial f}{\partial y}(x,y)$ exists for almost all $y \in \mathbf{R}$.

(b) Prove that $\frac{d}{dy}\int_0^1 f(x,y)\,dx \;=\; \int_0^1 \frac{\partial f}{\partial y}(x,y)\,dx$.

For each $y \in \mathbf{R}$, $x \in [0,1]$, and sequence $\epsilon_i \to 0$ let

$$g_i(x,y) \;=\; \frac{f(x,y+\epsilon_i) - f(x)}{\epsilon_i}$$

Then $|g_i(x,y)| \le M$ for all i. So Lebesgue's Dominated Convergence Theorem implies that

$$\begin{aligned}
&\lim_{i\to\infty} \epsilon_i^{-1}\big[\int_0^1 f(x,y+\epsilon_i)\,dx - \int_0^1 f(x,y)\,dx\big] = \lim_{i\to\infty}\int_0^1 g_i(x,y)\,dx \\
&= \int_0^1 \lim_{i\to\infty} g_i(x,y)\,dx \;=\; \int_0^1 \frac{\partial f}{\partial y}(x,y)\,dx\ .
\end{aligned}$$

Since the RHS is independent of the sequence $\epsilon_i \to 0$, one finds that the derivative $\frac{d}{dy}\int_0^1 f(x,y)\,dx$ exists and equals the RHS.

(c) Express $\frac{d}{dy}\int_0^{y^2} f(x,y)\,dx$ in terms of integrals of f and $\frac{\partial f}{\partial y}$. Letting $F(s,t) = \int_0^s f(x,t)\,dx$, we see from the fundamental theorem and (b) that

$$\frac{\partial F}{\partial s}(s,t) = f(s,t) \text{ and } \frac{\partial F}{\partial t}(s,t) = \int_0^s \frac{\partial f}{\partial y}(x,t)\,dx\ .$$

So we use the chain rule to compute

$$\begin{aligned}
\frac{d}{dy}\int_0^{y^2} f(x,y)\,dx \;&=\; \frac{d}{dy}F(y^2,y) \\
&= \frac{\partial F}{\partial s}(y^2,y)\frac{\partial y^2}{\partial y} + \frac{\partial F}{\partial t}(y^2,y)\frac{\partial y}{\partial y} \\
&= 2yf(y^2,y) + \int_0^{y^2}\frac{\partial f}{\partial y}(x,y)\,dx
\end{aligned}$$

3.(a) Show that the direct analog of Rolle's theorem does not apply to holomorphic functions. Do this by exhibiting an entire holomorphic function f such that $f(0) = f(1)$ and yet $f'(z)$ never takes the value 0.

e^z doesn't vanish and $e^{z+2\pi i} = e^z$. So we rotate the domain by 90^o and rescale by letting $f(z) = e^{2\pi iz}$. Then $f(0) = 1 = f(1)$ and $f'(z) = 2\pi ie^{2\pi iz} \neq 0$.

(b) Suppose f is a holomorphic function on the unit disk $\{z : |z| < 1\}$. Show that f must be constant if $f(a_i) = f(b_i)$ for two sequences a_i, b_i of positive real numbers that satisfy the inequalities

$$0 < \ldots < a_{i+1} < b_{i+1} < a_i < b_i < \ldots < a_1 < b_1 < 1 .$$

Both monotone sequences converge to some real number c with $0 \leq c < 1$. Writing $f = u + iv$ we find from Rolle's theorem, points $a_i < c_i < b_i$ so that $\frac{\partial u}{\partial x}(c_i) = 0$. Since $c_i \to c$, we deduce from the real analyticity of $u(\cdot, 0)$ that $\frac{\partial u}{\partial x}(\cdot, 0) \equiv 0$ and so u is constant on the X-axis. Similarly v is also constant on the X-axis. But then the holomorphic function f being constant on the X-axis, must itself be constant.

4. Suppose $0 < M < \infty$ and, for each positive integer j, $f_j : [0, 1] \to [-M, M]$ is a monotone increasing function. Prove that there is a subsequence $f_{j'}$ and a countable subset A of $[0, 1]$ so that $f_{j'}(t)$ converges, as $j' \to \infty$, for every $t \in [0, 1] \setminus A$.

Proof : Suppose $\mathbf{Q} \cap [0, 1] = \{a_1, a_2, \ldots\}$. A subsequence $f_{\alpha_1(1)}(a_1), f_{\alpha_1(2)}(a_1), \ldots$ of the bounded sequence of numbers $f_1(a_1), f_2(a_1), \ldots$ converges to a number $f(a_1)$. Inductively, choose a subsequence $f_{\alpha_j(1)}(a_j), f_{\alpha_j(2)}(a_j), \ldots$ of the sequence $f_{\alpha_{j-1}(1)}(a_j), f_{\alpha_{j-1}(2)}(a_j), \ldots$ convergent to a number $f(a_j)$.

Let $j' = \alpha_j(j)$ and $f(x) = \sup_{a_i<x} f(a_i) = \lim_{\epsilon\downarrow 0} \sup_{x-\epsilon<a_i<x} f(a_i)$. Then f is monotone increasing and the set A of discontinuities of f is at most countable. To see that $\lim_{j\to\infty} f_{j'}(x) = f(x)$ for any $x \in (0, 1) \setminus A$, we choose, for $\epsilon > 0$, numbers $a_i < x < a_{\tilde{i}}$ so that $f(a_{\tilde{i}}) - \epsilon < f(x) < f(a_i) + \epsilon$, and then J so that

$$|f_{j'}(a_i) - f(a_i)| < \epsilon \text{ and } |f_{j'}(a_{\tilde{i}}) - f(a_{\tilde{i}})| < \epsilon$$

for $j \geq J$. For such j it follows that

$$f(x) - 2\epsilon < f(a_i) - \epsilon < f_{j'}(a_i) < f_{j'}(x) < f_{j'}(a_{\tilde{i}}) < f(a_{\tilde{i}}) + \epsilon < f(x) + 2\epsilon .$$

Thus $|f_{j'}(x) - f(x)| < 2\epsilon$.

5. (a) Is there a nonconstant real function h that is continuous on the closed disk $\{z : |z| \leq 1\}$, harmonic on the open disk $\{z : |z| < 1\}$, and vanishes on the upper unit semi-circle (that is, $h(e^{i\theta}) = 0$ for $0 \leq \theta \leq \pi$)?

The Poisson integral formula shows that, for any continuous function g on the unit circle, one may find a harmonic function on the open ball which is continuous on the closed ball and has boundary values g. So it suffices to chose any nonconstant g which vanishes on the upper semi-circle.

(b) Is there a nonconstant complex function f that is continuous on the closed disk $\{z : |z| \leq 1\}$, holomorphic on the open disk $\{z : |z| < 1\}$, and vanishes on the upper unit semi-circle (that is, $f(e^{i\theta}) = 0$ for $0 \leq \theta \leq \pi$)?

There is a conformal map from the unit disk to the upper half plane. This takes the upper semi-circle to an interval on the X-axis. Composing with this conformal map thus gives a holomorphic map on the upper half plane which vanishes on this interval. Schwarz reflection about this interval then extends this function to be a holomorphic function whose domain contains the interval and vanishes on the interval. The identity theorem implies that this function, and hence the original function, must vanish identically.

6. Assume that $f(x)$ is a Lebesgue measurable function on $\mathbf{R}$. Prove the function defined on $\mathbf{R}^2$ by $F(x, y) = f(x - y)$ is Lebesgue measurable

We need to show that $F^{-1}\big((a, b)\big)$ is measurable in $\mathbf{R}^2$ for any interval $(a, b) \subset \mathbf{R}$. Note that $F = f \circ P$ where $P(x, y) = x+y$. Also note that $P = p \circ \sqrt{2} \cdot \phi$ where ϕ is a 45^o rotation of the plane and $p(x, y) = x$. So

$$F^{-1}\big((a, b)\big) = (\sqrt{2} \cdot \phi)^{-1}\big(p^{-1}[f^{-1}\big((a, b)\big)]\big)$$

$E = f^{-1}\big((a, b)\big)$ is measurable in $\mathbf{R}$ by the measurability of f, and $p^{-1}(E) = E \times \mathbf{R}$ is measurable by the definition of Lebesgue measure as a product measure. Moreover, since Lebesgue measurability is preserved under rotation and homothety $F^{-1}\big((a, b)\big) = (\sqrt{2} \cdot \phi)^{-1}(E \times \mathbf{R})$ is measurable.