Solutions to Analysis Exam, Fall 2005

1. (a) Suppose $f : [0,1] \to \mathbf{R}$ is Lebesgue integrable. Find

$$\lim_{n\to\infty} \int_0^1 \frac{nx^2}{2+nx} f(x)\, dx \ .$$

Noting the bound,

$$|\frac{nx^2}{2+nx} f(x)| \ \leq \ |x||f(x)| \ \leq \ |f(x)| \ ,$$

we use Lebesgue's Dominated Convergence Theorem to see that the limit is $\int_0^1 x\, f(x)\, dx$.

(b) Taking $f \equiv 1$ and noting that $\frac{nx}{2+nx^2}$ is increasing in n and approaches $1/x$ as n approaches ∞, we conclude from the Monotone Convergence Theorem or Fatou's Lemma that

$$\lim_{n\to\infty} \int_0^1 \frac{nx}{2+nx^2}\, dx \ = \ +\infty \ .$$

Alternately one can substitute to compute that

$$\int_0^1 \frac{nx}{2+nx^2}\, dx \ = \ \frac{1}{2}\log(1+\frac{n}{2}) \ \to \ +\infty \ \text{ as } n\to\infty \ .$$

2. Suppose g is holomorphic on $\{z \in \mathbf{C} \ : \ |z| < 2\}$ and $|g(z)| < 1$ whenever $|z| = 1$. The function $h(z) = z - g(z)$ is holomorphic on $\{z \in \mathbf{C} \ : \ |z| < 2\}$ and satisfies

$$|z - h(z)| \ = \ |g(z)| \ < \ 1 \ = \ |z|$$

on the unit circle. By Rouché's Theorem h has, like the function z, a single simple zero w in the unit disk. This is the desired unique point $w \in \mathbf{C}$ with $|w| < 1$ and $g(w) = w$.

3. Suppose $f : \mathbf{R} \to \mathbf{R}$ is infinitely differentiable and $a \in \mathbf{R}$.

(a) The *order of vanishing*

$$N(f,a) \ = \ \sup\{n \ : \ f^{(n)}(a) = 0\} \ .$$

Thus $N(f,a)$ is a positive integer in case some derivative of f at a is nonzero, and $N(f,a) = \infty$ in case the derivatives of f at a of all orders vanish.

(b) For convenience, we also define $N(f,b) = 0$ for a point b with $f(b) \neq 0$. Then for any zero a of f, we immediately verify the relation $N(f',a) = N(f,a) - 1$. Also the Mean Value Theorem implies that, strictly between any 2 consecutive zeros of f, is a zero of f'. It then follows that $N(f') = \infty$ whenever $N(f) = \infty$. So we now may assume that $N(f) < \infty$. We then find that

$$\begin{aligned} N(f') &= \sum_{a \in (f')^{-1}\{0\}} N(f', a) \\ &\geq \#\big[(f')^{-1}\{0\} \setminus f^{-1}\{0\}\big] \; + \sum_{a \in f^{-1}\{0\}} N(f', a) \\ &\geq \big[\# f^{-1}\{0\} - 1\big] \; + \sum_{a \in f^{-1}\{0\}} \big[N(f, a) - 1\big] \; = \; N(f) - 1 \; . \end{aligned}$$

4. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x - \mathbf{i}} \, dx \quad .$$

For R large, consider the upper disk bounded by the interval $[-R, R]$ and the semicircle

$$\Gamma_R \;=\; \{ R e^{\mathbf{i}t} \;:\; 0 \leq t \leq \pi \; \} \; .$$

One is tempted to apply the Residue Theorem with the meromorphic function $\frac{\cos z}{z - \mathbf{i}}$. However the upper boundary integral does not approach 0 as $R \to \infty$. Note that $\cos z = \frac{1}{2}(e^{\mathbf{i}z} + e^{-\mathbf{i}z})$ and that

$$\begin{aligned} \Big| \int_{\Gamma_R} \frac{e^{\mathbf{i}z}}{z - \mathbf{i}} \, dz \Big| \; &\leq \; \int_0^{2\pi} \Big| \frac{e^{\mathbf{i}Re^{\mathbf{i}t}}}{Re^{\mathbf{i}t} - \mathbf{i}} \Big| |R\mathbf{i}e^{\mathbf{i}t}| \; dt \\ &\leq \frac{2R}{R} \int_0^{2\pi} e^{-R \sin t} \, dt \;\; \to \;\; 0 \text{ as } \; R \to \infty \; , \end{aligned}$$

by Dominated Convergence. Since the function $\frac{e^{\mathbf{i}z}}{z - \mathbf{i}}$ has only one pole at $\mathbf{i}$ with residue e^{-1}, the Residue Theorem gives that

$$\int_{-\infty}^{\infty} \frac{e^{\mathbf{i}x}}{x - \mathbf{i}} \, dx \;\; = 2\pi \mathbf{i} e^{-1} \; .$$

To find the other term $\int_{-\infty}^{\infty} \frac{e^{-\mathbf{i}x}}{x - \mathbf{i}} \, dx$, we use the lower half disk with lower boundary $\gamma_R \;=\; \{ -Re^{\mathbf{i}t} \;:\; 0 \leq t \leq \pi \; \}$ and see that, just like above,

$$\Big| \int_{\gamma_R} \frac{e^{-\mathbf{i}z}}{z - \mathbf{i}} \, dz \Big| \;\; \to \;\; \infty \text{ as } \; R \to \infty \; .$$

The meromorphic function $\frac{e^{-\mathbf{i}z}}{z - \mathbf{i}}$ has no pole in the lower half plane. So the Residue Theorem now gives

$$\int_{-\infty}^{\infty} \frac{e^{-\mathbf{i}x}}{x - \mathbf{i}} \, dx \;\; = 0 \; ,$$

and we conclude that $\int_{-\infty}^{\infty} \frac{\cos x}{x - \mathbf{i}} \, dx = \pi \mathbf{i} e^{-1}$.

5. Suppose $f : \mathbf{R} \to \mathbf{R}$ is infinitely differentiable.

(a) $\int_0^x f''(t)(x-t)\,dt$ is the integral form for the remainder term in a degree 1 Taylor approximation of f and so equals $f(x) - f(0) - f'(0)x$. One can also prove this directly by integrating by parts:

$$\begin{aligned}\int_0^x f''(t)(x-t)\,dt \;&=\; f'(x)(x-x) - f'(0)(x-0) - \int_0^x f'(t)\frac{d}{dt}(x-t)\,dt \\ &=\; -f'(0)x + \int_0^x f'(t)\,dt \;=\; -f'(0)x + f(x) - f(0)\ .\end{aligned}$$

(b) From (a)

$$\begin{aligned}|f(x)| \;&\le\; |f(0) + f'(0)x + \int_0^x f''(t)(x-t)\,dt\,| \\ &\le\; |f(0)| + |f'(0)||x| + M\int_0^x (x-t)\,dt \;=\; |f(0)| + |f'(0)||x| + \frac{1}{2}M|x|^2\end{aligned}$$

for $x > 0$. For $x < 0$ one gets the same estimate by symmetry.

6. Suppose g is holomorphic on $A = \{z \in \mathbf{C} \;:\; 0 < |z| < 1\,\}$ and

$$\limsup_{|z|\to 0} |g(z) - \lambda| \;>\; 0$$

for every $\lambda \in \mathbf{C}$. Show that either

(I) $\lim_{|z|\to 0} |z|^{1/2}|g(z)| \;=\; \infty$ or

(II) $g(A)$ is dense in $\mathbf{C}$.

The hypothesis implies that the singularity of f is *not removable* and thus is either (I) a pole or (II) an essential singularity. In case (I), g is meromorphic at 0 and so, near 0, $|g(z)| \ge \frac{c}{|z|^j}$ for some $c > 0$ and $j \in \{1, 2, \ldots\}$, hence,

$$\lim_{|z|\to 0} |z|^{1/2}|g(z)| \ge \frac{c}{|z|^{j-1/2}} \;=\; \infty\ .$$

In any neighborhood of an essential singularity, g becomes arbitrarily close to any complex number so that $g(A)$ is dense in case (II).

A specific example of a g satisfying (II) is $g(z) = e^{1/z}$.