ANALYSIS QUALIFYING EXAM

September 1997

Justify answers as completely as you can. Give careful statements of theorems you are using. **Time limit – 3 HOURS.**

1. Let P, Q be complex polynomials with the degree of Q at least two more than the degree of P. Prove there is an $r > 0$ such that if C is a closed curve outside $|z| = r$, then

$$\int_C \frac{P(z)}{Q(z)} dz = 0.$$

2. Let λ be a real number with $\lambda > 1$. Prove that the equation $\lambda - z - e^{-z} = 0$ has exactly one root z_0 with $Re z_0 > 0$.

3. Let B be the open unit ball in $\mathbb{R}^n$ and $f : B \to \mathbb{R}$ a differentiable function whose partial derivatives are uniformly bounded but not necessarily continuous. Prove that f has a unique continuous extension to the closure of B.

4. Let $f : [0, 1] \to \mathbb{R}$ be Riemann integrable over $[b, 1]$ for all b such that $0 < b \leq 1$.

a. If f is bounded, prove that f is Riemann integrable over $[0, 1]$.

b. What if f is not bounded?

5a.. Find a counter example to the following assertion:

If $g_n(z)$ is an entire function having only real zeros for $n = 1, 2, \ldots$ and if

$$\lim_{n\to\infty} g_n(z) = g(z)$$

uniformly on compact sets in $\mathbb{C}$, then $g(z)$ has only real zeros.

5b. Add an additional hypothesis about g to make the assertion true, and prove the result.

6. Prove that for each open set $U \subset \mathbb{R}^n$, there exists a countable family $\mathfrak{F}$ of closed, disjoint cubes, each contained in U such that the Lebesgue measure

$$m(U - \cup\{C : C \in \mathfrak{F}\}) = 0.$$