ANALYSIS QUALIFYING EXAM

September 1998

Justify answers as completely as you can. Give careful statements of theorems you are using. **Time limit – 3 HOURS.**

1. (a) For what real numbers α is $|x|^\alpha$ integrable on $\{x \in \mathbb{R}^n \ : \ |x| < 1\}$?

(b) For what real numbers β is $|x|^\beta$ integrable on $\{x \in \mathbb{R}^n \ : \ |x| > 1\}$?

2. (a) How many roots does $p(z) = 2z^5 + 4z^2 + 1$ have in the disk $|z| < 1$?

(b) How many roots does the same polynomial have on the real axis?

3. Let $P_n(x) = a_n x^2 + b_n x + c_n$ be a sequence of quadratic polynomials which converges pointwise on $0 \le x \le 1$.

(a) Prove that P_n converges uniformly on $[0, 1]$.

(b) Does P_n converge uniformly on $[0, 2]$?

4. Evaluate the improper integral

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx.$$

[Hint: Express $\sin^2 x$ in terms of e^{2ix}.]

5. Suppose $f : \mathbb{R} \to \mathbb{R}$, $0 < f(0)$ and $f(100) < 100$.

(a) Prove that $f(x) = x$ for some x if f is continuous.

(b) Prove that $f(x) = x$ for some x if f is monotonically increasing (though possibly discontinuous).

6. What is the general form of an entire function which has absolute value 1 on the circle $|z| = 1$ and has no zero inside the circle? Prove your result.