ANALYSIS QUALIFYING EXAM

AUGUST 1999

Justify answers as completely as you can. Give careful statements of theorems you are using. **Time limit – 3 HOURS.**

1. Suppose that $f_1, f_2, \ldots$ are nonnegative continuous functions on $[0, 1]$ with $\int_0^1 f_n(x)\,dx \ \le\ M$.

(1) Show that there exists a point $a \in [0,1]$ with $f_1(a) \le 2M$ and $f_2(a) \le 2M$.

(2) Does there exist a better estimate? That is, a number $N < M$ so that $\inf_{0\le a\le 1} \max\{f_1(a), f_2(a)\} \le N$ for all such f_1, f_2. If so, find the smallest such N. If not, give a counterexample.

(3) Show that there always exists an $a \in [0,1]$ so that $f_n(a) \le M$ for *infinitely many* n.

2. Suppose f is a holomorphic function on $\{z \mid |z| < 3R\}$, $f(0) = 0$, $M_R = \sup_{|z|\le R} |f(z)|$, and $N_R = \sup_{|z|\le R} |f'(z)|$.

(1) Estimate M_R (from above) in terms of N_R.

(2) Estimate N_R (from above) in terms of M_{2R}.

3. Suppose that $f(x)$ is defined on $[-1, 1]$, and that $f'''(x)$ is continuous. Show that the series

$$\sum_{n=1}^{\infty} (n(f(1/n) - f(-1/n)) - 2f'(0))$$

converges.

4. Prove that there is no one-to-one conformal map of the punctured disc $G = \{z \in \mathbb{C}|\ 0 < |z| < 1\}$ onto the annulus $A = \{z \in \mathbb{C}|\ 1 < |z| < 2\}$.

5. Let f is a meromorphic function on the complex plane such that $f(z) = 1 + z + z^2 + \cdots$ whenever $|z| < 1$. Define a sequence of real numbers $a_0,\ a_1,\ a_2\ \cdots$ by

$$f(z) = \sum_{n=0}^{\infty} a_n (z+2)^n$$

What is the radius of convergence of the new series $\sum_{n=0}^{\infty} a_n z^n$?.

6. A function $g : [0,1] \to \mathbb{R}$ is *concave* if $tg(x) + (1-t)g(y) \le g(tx + (1-t)y)$ for $0 \le t \le 1$. Prove that for any continuous $f : [0,1] \to \mathbb{R}$ with $f(0) = 0$, there is a continuous concave function $g : [0,1] \to \mathbb{R}$ such that $g(0) = 0$ and $g(x) \ge f(x)$ for all $x \in [0,1]$.

(Hint: Show that

$$g(x) = inf\{h(x) : h\, is\, a\, continuous\, concave\, function\, on\, [0,1],\ h(y) \ge f(y)\, for\, y \in [0,1]\}$$

works (in particular, $g(0) = 0$).