# ANALYSIS QUALIFYING EXAM

## January 2005

Please explain *all* your answers and indicate which theorems you are using.

**1.** Suppose $f : \mathbf{C} \to \mathbf{C}$ is continuous and the complex derivative $f'(z)$ exists for all $z \in \mathbf{C}$.

(a) What is the Cauchy integral formula for $f$ on the disk $|z| < R$ ? (No proof necessary)

(b) Using (a), show that every complex derivative $f''(z), f'''(z), \cdots, f^{(n)}(z), \cdots$ exists.

(c) Find an estimate for $|f^{(n)}(0)|$ in terms of $n$, $R$, and $M_R = \sup_{|z|=R} |f(z)|$.

**2.** For $0 < \alpha \leq 1$, a function $f : [0,1] \to [0,1]$ is $\alpha$-Hölder continuous if there is a positive constant $C$ so that

$$|f(x) - f(y)| \;\leq\; C\,|x-y|^{\alpha} \quad \text{for } 0 \leq x < y \leq 1 \,.$$

(a) Show that $g(x) = \sqrt{x}$ is $\frac{1}{2}$-Hölder continuous.

(b) Show that $g(x) = \sqrt{x}$ <u>is not</u> 1-Hölder continuous.

**3.** (a) Show that if $f$ is meromorphic (but not holomorphic) at 0, then, for some $n \in \{1, 2, \cdots\}$,

$$\lim_{r \to 0} r^n \int_0^{2\pi} |f(re^{i\theta})|\, d\theta \quad \text{exists and is nonzero}\,.$$

(b) Show that if $g$ is an entire holomorphic function, and

$$\lim_{r \to \infty} r^{-1/2} \int_0^{2\pi} |g(re^{i\theta})|\, d\theta \;<\; \infty \;, \quad \text{then } g \text{ is a constant} \;.$$

**4.** Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuously differentiable with $\int_0^\infty |f(t)|\, dt \;<\; \infty$.

(a) Find $\lim_{\varepsilon \to 0} \int_0^\infty f(t) e^{-\varepsilon t^2}\, dt$.

(b) Find $\lim_{\varepsilon \to 0} \int_0^\infty f(t) e^{-t^2/\varepsilon}\, dt$.

(c) Find $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\infty f(t)\, t\, e^{-t^2/\varepsilon}\, dt$. (Hint: Integrate by parts.)

**5.** (a) Suppose that $A$ is a (possibly uncountable) set. Prove that if $f_a : \mathbf{R} \to [0,1]$ is a continuous function for each $a \in A$, then $f(x) = \sup_{a \in A} f_a(x)$ is Lebesgue measurable. (Hint: Consider $\{x \;:\; f(x) > t\}$.)

(b) Show that there exists a set $A$ and a family $\{g_a \;:\; a \in A\}$ of Lebesgue measurable functions $g_a : \mathbf{R} \to [0,1]$ so that $g(x) = \sup_{a \in A} g_a(x)$ is <u>*not*</u> Lebesgue measurable. (You may assume the existence of some unmeasurable subset of $\mathbf{R}$.)

**6.** (a) For what complex numbers $z$ is the series $\sum_{k=0}^{\infty} 2^{-k}\, e^{kz}$ absolutely convergent?

(b) For these $z$, find a formula for the sum of this series.

(c) For what complex numbers $z$ is the series $\sum_{k=0}^{\infty} 2^{-k} \cos kz$ absolutely convergent?