ANALYSIS QUALIFYING EXAM

January 1998

Justify answers as completely as you can. Give careful statements of theorems you are using. **Time limit – 3 HOURS.**

1. Show that there is a complex analytic function defined on the set $U = \{z \in \mathbb{C} | \ |z| > 4\}$ whose derivative is

$$\frac{z}{(z-1)(z-2)(z-3)} \ .$$

Is there a complex analytic function defined on the set U whose derivative is

$$\frac{z^2}{(z-1)(z-2)(z-3)} \ ?$$

2. Let f be a Lebesgue integrable function on $[a, b]$. Show that

$$\frac{d}{dx}\int_a^x |f(t) - r|dt = |f(x) - r|$$

for almost all x in $[a, b]$ and all rational number r.

3. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in the open unit disk $|z| < 1$ where

$$\sum_{n=2}^{\infty} n|a_n| \leq |a_1| \quad \text{and} \quad a_1 \neq 0 \ .$$

Prove f is injective (i.e. one-to-one).

4. Find a sequence of measurable functions $f_n : [0, 1] \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f_0(x)$ for all $x \in [0, 1]$ but such that for any measurable subset E of $[0, 1]$ with the Lebesgue measure $m([0, 1] - E) = 0$, f_n **does not** converge to f_0 *uniformly* on E.

5. Prove that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos nx \, dx = 0$$

for any Lebesgue integrable function f on the real line.

6. Let

$$f(z) = z + z^2 + z^4 + \cdots + z^{2^n} + \cdots$$

in the open unit disk $|z| < 1$. Prove that every point $z_\circ$ on the circle $|z| = 1$ is a singular point of $f(z)$, i.e. there is no holomorphic function $\phi(z)$ in a neighborhood of $z_\circ$ such that $\phi(z) = f(z)$ in $|z| < 1$. [Hint: Note that $f(z^2) = f(z) - z$ and that $z_0 = 1$ is a singular point.]