Solutions to ANALYSIS QUALIFYING EXAM, May 2004

1. (a)

$$\int_1^\infty t^p(\sin^2 t)\,dt \;<\; \infty \iff p<-1$$

because

$$\sum_{j=1}^\infty j^p \;\le\; \sum_{j=1}^\infty (j\pi)^p\frac{\pi}{2} \;\le\; \int_1^\infty t^p(\sin^2 t)\,dt \;<\; \int_1^\infty t^p\,dt$$

and both $\sum_{j=1}^\infty j^p$ and $\int_1^\infty t^p\,dt$ are finite $\iff p<-1$.

(b)

$$\int_0^1 t^q(\sin^2 t)\,dt \;<\; \infty \iff q>-3$$

because

$$\int_0^1 t^q(t/2)^2\,dt \;\le\; \int_0^1 t^q(\sin^2 t)\,dt \;\le\; \int_0^1 t^{q+2}\,dt$$

and $\int_0^1 t^{q+2}\,dt$ is finite $\iff q+2>-1$.

(c)

$$\int_{\mathbf{R}^3\setminus\mathbf{B}_1} |x|^s(\sin^2|x|\,)\,dx_1dx_2dx_3 \;<\; \infty \iff s<-3$$

because, using spherical coordinates,

$$\int_{\mathbf{R}^3\setminus\mathbf{B}_1} |x|^s(\sin^2|x|\,)\,dx_1dx_2dx_3 \;=\; \int_0^\pi\int_0^{2\pi}\int_1^\infty \rho^s(\sin^2\rho)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi\ ,$$

which, as in (a), is finite $\iff s+2<-1$.

2. For $\varepsilon>0$, let $\phi_\varepsilon(t)=\varepsilon^{-1}\phi(t/\varepsilon)$ where $\phi(t)=1-|t|$ for $|t|\le 1$ and $\phi(t)=0$ for $|t|>1$. Then, for $f\in L^1(\mathbf{R})$, let

$$f_\varepsilon(x) \;=\; \int_{-\infty}^\infty f(y)\phi_\varepsilon(x-y)\,dy\ .$$

(a) To estimate $|f_\varepsilon(w)-f_\varepsilon(x)|$, first note that

$$|\phi_\varepsilon(w-y)-\phi_\varepsilon(x-y)| \le \int_{w-y}^{x-y}|\phi_\varepsilon'(t)|\,dt \;\le\; \varepsilon^{-2}|w-x|\ .$$

So

$$|f_\varepsilon(w)-f_\varepsilon(x)| \le \int_{-\infty}^\infty |f(y)\big(\phi_\varepsilon(w-y)-\phi_\varepsilon(x-y)|\,dy \;\le\; \varepsilon^{-2}|w-x|\Big(\int|f(y)|\,dy\Big).$$

(b) Since f is uniformly continuous, we may chose, for any $\eta>0$, a positive δ so that $|f(y)-f(x)|\le\eta$ whenever $|x-y|<\delta$. Note also that $\phi_\varepsilon(x-y)$ is nonzero only for $|x-y|<\varepsilon$. Since $\int_{-\infty}^\infty \phi_\varepsilon(x-y)\,dy=1$, we conclude that, for $\varepsilon<\delta$,

$$|f_\varepsilon(x)-f(x)| \;=\; \Big|\int_{-\infty}^\infty \big[f(y)-f(x)\big]\phi_\varepsilon(x-y)\,dy\Big| \;\le\; \int_{-\infty}^\infty |f(y)-f(x)|\phi_\varepsilon(x-y)\,dy \le\; \eta\cdot 1\ .$$

This is independent of x and gives the desired uniform convergence of f_ε to f.

3. (a) There does exist, for every $\varepsilon > 0$, an open dense subset U of the plane $\mathbf{R}^2$ with 2-dimensional Lebesgue measure less than ε. First make a list $a_1, a_2, a_3, \ldots$ of all the points of the plane whose X and Y coordinates are both rational numbers. Then, the union of open balls

$$U \ = \ \cup_{j=1}^{\infty} \mathbf{B}_{\varepsilon/5^j}(a_j)$$

is a dense open set with Lebesgue measure $\leq \pi \sum_{j=1}^{\infty} \big(\varepsilon/5^j\big)^2 < \varepsilon$.

(b) Suppose, for $\theta \in [0, 2\pi)$, ℓ_θ is the ray $\{(t\cos\theta, t\sin\theta) \in \mathbf{R}^2 \ : \ 0 \leq t < +\infty\}$. If E is a measurable subset of $\mathbf{R}^2$ with positive 2-dimensional Lebesgue measure, then

$$\{\theta \in [0, 2\pi) : \ E \cap \ell_\theta \text{ has positive } 1 - \text{dimensional Lebesgue measure in } \ell_\theta\}$$

has positive 1-dimensional Lebesgue measure. To verify this, we may transform to r, θ polar coordinates and then apply Fubini's theorem. In detail, note that for some positive numbers $R < S$ and $0 < \Theta < 2\pi$, the annular sector

$$A \ = \ \{z \ : \ R \leq |z| \leq S, \ 0 \leq \arg(z) \leq \Theta\}$$

will have positive measure intersection with E. The polar coordinate transformation $T(z) = \big(|z|, \arg(z)\big)$ maps A to the rectangle $[R, S] \times [0, \Theta]$ and is Lipschitz with Lipschitz inverse there. It also maps the interval $A \cap \ell_\theta$ to the horizontal interval $[R, S] \times \{\theta\}$. Since T is bilipschitz on A, it preserves measurability, positive 2- dimensional measuredness in A and positive 1-dimensional measuredness in each interval $A \cap \ell_\theta$. So $T(A \cap E)$ has positive 2-dimensional measure. By Fubini's Theorem, the slices $S_\theta = T(A \cap E) \cap \big([R, S] \times \{\theta\}\big)$, and hence $E \cap A \cap \ell_\theta = T^{-1}(S_\theta)$, each have positive measure for a positive measure set of $\theta \in [0, \Theta]$.

4. To compute $\int_0^\infty \frac{dx}{x^2 + \mathbf{i}}$, let $f(z) = 1/(z^2 + \mathbf{i}) = 1/\big(z^2 - (-\mathbf{i})\big) = 1/(z + \alpha)(z - \alpha)$ where $\alpha = (-1 + \mathbf{i})/\sqrt{2}$. Since

$$\int_0^M \frac{dx}{x^2 + \mathbf{i}} \ = \ \frac{1}{2} \int_{-M}^M \frac{dx}{x^2 + \mathbf{i}} \ ,$$

we will use the Cauchy residue formula on the upper semi-disk with base $[-M, M]$. As $M \to \infty$, the boundary integral over the upper semi-circle C_M satisfies

$$\Big|\int_{C_M} f(z)\, dz\Big| \ \leq \ \int_0^\pi |f(z)| M \, d\theta \ \leq \ \pi \frac{M}{M^2 - 1} \ \ \to \ 0 \ .$$

Only the pole α lies in the upper semi-disks, and the residue of $f(z)$ at α is $\frac{1}{\alpha + \alpha} = \frac{\overline{\alpha}}{2}$. We conclude that

$$\int_0^\infty \frac{dx}{x^2 + \mathbf{i}} \ = \ \frac{1}{2} \cdot 2\pi \mathbf{i} \frac{\overline{\alpha}}{2} \ = \ \frac{\pi}{2\sqrt{2}}(1 - \mathbf{i}) \ .$$

5. Suppose that D is the unit disk $\{z \in \mathbf{C} : |z| < 1\}$ and f is a nonconstant holomorphic function on some connected open neighborhood of $\overline{D}$ and that $|f(z)| = 1$ whenever $|z| = 1$. By the maximum modulus principle $f(D) \subset \overline{D}$. Moreover, $f(D) \subset D$ because f, being nonconstant, is an open mapping so that $f(D)$ is open.

We readily check that $\partial f(D) \subset \partial D$. In fact, for any $b \in \partial f(D)$, any sequence of points $a_i \in D$ with $f(a_i) \to b$ has a subsequence convergent to some $a \in \overline{D}$. Since $b = f(a)$ and f is an open mapping, $a \notin D$. Thus $a \in \partial D$, and $b = f(a) \in \partial D$.

We conclude that $f(D)$ is closed as well as open relative to D. Since, being the continuous image of a connected set, $f(D)$ is connected, and so $f(D) = D$.

6. (a) The *simple connectivity* of Ω is a necessary and sufficient condition that *every* holomorphic function f on Ω will admit a holomorphic function F on Ω with $F' = f$.

If Ω is simply-connected, then we can define F in each component Ω_i of Ω by fixing one point $p \in \Omega_i$ and defining $F(z)$, for $z \in \Omega_i$, as the path integral of f along any path in Ω_i from z_0 to z. The independence of the path follows from the simple connectivity of Ω.

If Ω is not simply connected, then we choose a non-contractible loop Γ in Ω and a point $a \in \mathbf{C} \setminus \Omega$ *inside* Γ, i.e. the winding number of Γ about a is nonzero. The function $f(z) = 1/(z - a)$ is holomorphic in Ω. If f admitted a primitive F with $F' = f$, then the path integral of f along Γ would be zero, by the fundamental theorem of calculus. However this integral is nonzero because a is inside Γ.

(b) Suppose A is a finite subset of the unit disk D and $U = D \setminus A$. A necessary and sufficient condition on a holomorphic function f on U to have a holomorphic F with $F' = f$ on U is the vanishing of every path integral of f along any circle $\partial \mathbf{B}_\varepsilon(a)$ for all $a \in A$ and sufficiently small positive ε. (For meromorphic f, one says simply that its residue at each $a \in A$ is zero.)

Under this condition, $F(z)$ is well-defined by taking an arbitrary path integral of f in U from a fixed point z_0 to z.

If the path integral of f is nonzero on some circle, then it is impossible that $f = F'$, by the fundamental theorem of calculus.

It is *not* necessary that the singularities be removable. One gets examples by starting with F and differentiating. e.g. $f(z) = \big(1/z\big)' = -1/z^2$ or $f(z) = \big(e^{-z}\big)' = -e^{-z}$.