

ANALYSIS QUALIFYING EXAM

August 2003

Please explain *all* your answers and indicate which theorems you are using.

1. Suppose $0 < \alpha < 1$. Find $\int_0^\infty \frac{x^\alpha}{x-x^3} dx$.

2. Suppose that $f(x, y)$ is continuous on the plane and that there is finite M so that $|f(x, y) - f(x, z)| \leq M|y - z|$ for all $x, y, z \in \mathbf{R}$.

(a) Show that, for any $x \in \mathbf{R}$, the partial derivative $\frac{\partial f}{\partial y}(x, y)$ exists for almost all $y \in \mathbf{R}$.

(b) Prove that $\frac{d}{dy} \int_0^1 f(x, y) dx = \int_0^1 \frac{\partial f}{\partial y}(x, y) dx$.

(c) Express $\frac{d}{dy} \int_0^{y^2} f(x, y) dx$ in terms of integrals of f and $\frac{\partial f}{\partial y}$.

3. Suppose f is a holomorphic function on the unit disk $\{z : |z| < 1\}$.

(a) Show that the set $\{(z, w) \in \mathbf{C}^2 : |z|, |w| \leq \frac{1}{2}, f(z) = f(w)\}$ is either finite or uncountably infinite.

(b) Show that f must be constant if $f(a_i) = f(b_i)$ for two sequences a_i, b_i of positive real numbers that satisfy the inequalities

$$0 < \dots < a_{i+1} < b_{i+1} < a_i < b_i < \dots < a_1 < b_1 < 1.$$

4. Suppose $0 < M < \infty$ and $f_j : [0, 1] \rightarrow [-M, M]$ are monotone increasing. Prove that there is a subsequence $f_{j'}$ and a countable subset A of $[0, 1]$ so that $f_{j'}(t)$ converges, as $j' \rightarrow \infty$, for every $t \in [0, 1] \setminus A$.

5. (a) Is there a nonconstant real function h that is continuous on the closed disk $\{z : |z| \leq 1\}$, harmonic on the open disk $\{z : |z| < 1\}$, and vanishes on the upper unit semi-circle (that is, $h(e^{i\theta}) = 0$ for $0 \leq \theta \leq \pi$)?

(b) Is there a nonconstant complex function f that is continuous on the closed disk $\{z : |z| \leq 1\}$, holomorphic on the open disk $\{z : |z| < 1\}$, and vanishes on the upper unit semi-circle (that is, $f(e^{i\theta}) = 0$ for $0 \leq \theta \leq \pi$)?

6. Assume that $f(x)$ is a Lebesgue measurable function on \mathbf{R} . Prove the function defined on \mathbf{R}^2 by $F(x, y) = f(x - y)$ is Lebesgue measurable.