ANALYSIS QUALIFYING EXAM

August 2004

1. (a) Find a real-valued function u on the complex plane so that

$$f\big(x+\mathbf{i}y\big) \;=\; u(x+\mathbf{i}y) \;+\; \mathbf{i}\big(x^3+x^2-y^2(3x+1)\big)$$

is holomorphic.

(b) Is your answer unique? If so, prove it. If not, find all the solutions.

2. Suppose that g is twice continuously differentiable and real-valued on $\mathbf{R}^2$. You are to prove that

$$\frac{\partial^2 g}{\partial x \partial y}(0,0) \;=\; \frac{\partial^2 g}{\partial y \partial x}(0,0)\;, \qquad (*)$$

using the following steps:

(a) Compute the integral of $\frac{\partial^2 g}{\partial x \partial y}$ over a rectangle $[0,a]\times[0,b]$.

(b) Do the same for $\frac{\partial^2 g}{\partial y \partial x}$.

(c) Prove that the results are the same.

(d) Show that this implies (*) .

3. Suppose that $D = \{z \in \mathbf{C} :\; |z|<1\}$, $f: D \to D$ is holomorphic, and $z_0 \in D$. Let $w_0 = f(z_0)$.

Show that for every $z \in D$,

(a)

$$\Big|\frac{f(z)-w_0}{1-\bar{w}_0 f(z)}\Big| \le \Big|\frac{z-z_0}{1-\bar{z}_0 z}\Big|\;,$$

(b)

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}\;.$$

4.(a) Suppose that $f: \mathbf{R} \to \mathbf{R}$ is a continuous function such that, for almost all $t \in \mathbf{R}$, $f'(t)$ exists and $|f'(t)| \le 1$. Is it true that, $f(b)-f(a) \;=\; \int_a^b f'(t)\,dt$ for $-\infty < a < b < \infty$? If so, prove it. If not, give a counterexample.

(b) Suppose $g : \mathbf{R} \to \mathbf{R}$ is differentiable at *every* point $t \in \mathbf{R}$. Is g necessarily of *bounded variation* on every closed interval $[a,b] \subset \mathbf{R}$? If so, prove it. If not, give a counterexample.

5. Suppose that f is a holomorphic function on the punctured plane $\mathbf{C}\setminus\{0\}$.

(a) For each positive numbers $\varepsilon < R < \infty$, find a formula for $f(z)$ on the annulus $\{z \in \mathbf{C} \,:\, \varepsilon < |z| < R\}$ in terms of the values of f on the inner boundary circle $\{z \;:\; |z| = \varepsilon\}$ and on the outer boundary circle $\{z \;:\; |z| = R\}$.

(b) Prove that if f is meromorphic and

$$\int_{\{z\,:\,0<|z|<1\}} |f(z)|\,dx\,dy \;<\infty\ , \tag{**}$$

then, at 0, f either has a removable singularity or is meromorphic with a pole of order 1.

(c) Does the integrability assumption (**) alone imply that f is automatically meromorphic at 0. If so, prove it. If not, give a counterexample.

6. Suppose that $E_1, E_2, E_3, \ldots$ is a sequence of Lebesgue measurable subsets of the unit ball $\mathbf{B}$ in $\mathbf{R}^n$, and that each E_k has positive Lebesgue measure $\mu(E_k) > \varepsilon$ for a fixed $\varepsilon > 0$. For each $x \in \mathbf{B}$, let $n(x)$ denote the number of integers k so that $x \in E_k$.

(a) Show that $n(x) \geq 2$ for some $x \in \mathbf{B}$.

(b) Show that $\sup_{x\in\mathbf{B}} n(x) = \infty$.

(c) Show that $n(x) = \infty$ for some $x \in \mathbf{B}$.

[The weaker statements (a) and (b) are not necessarily needed for the proof of (c).]