An introduction to the calculus of variations

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Contents

Chapter 1

Paradigms

1.1 Foreword

In this chapter we quickly describe the content of these lecture notes. To start with we state some problems which will serve as paradigms as we go along. Here they are stated loosely and they will be treated in more details in the coming chapters. Next we touch upon the main themes that will occur on our way: necessary conditions (Euler equations), sufficient conditions, existence theory.

1.2 Path of shortest length

Let $A, B \in \mathbb{R}^2$ be two distinct points. We look for a *path of shortest length* joining A and B. In order to state the problem more precisely we need to define what we mean by "path", by "joining A and B", and by "length of a path". There are several possibilities of doing so.

1.2.1 First model: curves

One can model a *path joining A and B* by a continuous function $u : [0,1] \to \mathbb{R}^2$ such that $u(0) = A$ and $u(1) = B$. We insist that u be continuous so that it really joins its endpoints A and B. In defining the *length* of such path we must require more than the mere continuity of u (see chapter ... for a detailed treatment of this question). Here we will content ourselves with defining the length of u when $u \in C^1$ (that is u is differentiable at each $t \in [0,1]$ and its derivative $u' : [0,1] \to \mathbb{R}^2$ is continuous) as is done in a course on curves and surfaces.

$$\text{length}(u) = \int_0^1 \|u'(t)\| dt \,.$$

Here $\|u'(t)\|$ denotes the Euclidean norm of the vector $u'(t)$, i.e.

$$\|u'(t)\| = \sqrt{u_1'(t)^2 + u_2'(t)^2} \,.$$

We can now express the minimization problem in the following abbreviated form:

$$(\mathscr{P}_1) \begin{cases} \text{minimize } \int_0^1 \|u'(t)\| dt \\ \text{among } u \in C^1([0,1],\mathbb{R}^2) \text{ such that } u(0) = A \text{ and } u(1) = B \end{cases}$$

Of course we expect the solution to be the straight line segment with endpoints A and B. This requires a proof however. Now we state the corresponding Theorem.

`thm.1` **1.2.1 Theorem.** *A function $u \in C^1([0,1],\mathbb{R}^2)$ such that $u(0) = A$ and $u(1) = B$ minimizes its length if and only if it is a one-to-one parametrization of the straight line segment*

$$[\![A, B]\!] = \mathbb{R}^2 \cap \{A + t(B - A) : t \in [0,1]\} .$$

This shows that minimizers for the minimization problem $(\mathscr{P}_1)$ are *not* unique even though we expect "the unique solution to be the straight line segment". This is due, of course, to the way we have set up the problem and the fact that there are many distinct ways to parametrize the straight line segment as a curve.

1.2.2 Second model: graphs

In the previous paragraph we thought of a path as a continuous *image* of $[0,1]$. Here we will think of a path as a continuous *graph* instead. The points A and B have Cartesian coordinates, say, $(a, \overline{a})$ and $(b, \overline{b})$. Renaming A as B and B as A if necessary we may assume that $a \leq b$. We will subsequently assume that $a < b$. A continuous graph joining A and B is then defined as a continuous function $u : [a,b] \to \mathbb{R}$ such that $u(a) = \overline{a}$ and $u(b) = \overline{b}$. In accordance with the previous paragraph the length of the graph of u is defined as

$$L(u) = \int_a^b \sqrt{1 + u'(x)^2} dx .$$

In this setting the minimizations problem reads

$$(\mathscr{P}_2) \begin{cases} \text{minimize } \int_a^b \sqrt{1 + u'(x)^2} dx \\ \text{among } u \in C^1([a,b],\mathbb{R}) \text{ such that } u(a) = \overline{a} \text{ and } u(b) = \overline{b} \end{cases}$$

In the present set up a minimizer exists and is *unique*.

`thm.2` **1.2.2 Theorem.** *A function $u \in C^1([a,b],\mathbb{R})$ such that $u(a) = \overline{a}$ and $u(b) = \overline{b}$ minimizes L if and only if it is the (unique) affine function in that class.*

1.2.3 A third possible model

A third way to model this problem is to consider a path joining A and B to be a (say compact) connected set in the plane containing the pair $\{A, B\}$. The concept of connectedness for a set plays the role of that of continuity for a function in the two previous models: it ensures that indeed the set *joins* A and B in some sense. This more general viewpoint encompasses the two preceding ones. There is some added difficulty, though, which consists in defining the length of a subset of the plane. This turns out to be one of the motivational challenges of the foundation of Geometric Measure Theory. We refer the reader to ... for the particular case evoked here.

1.2.4 Vocabulary

The two minimization problems introduced so far are particular cases of the following general setting. We let $\mathscr{C}$ denote a set and $\mathscr{F} : \mathscr{C} \to \mathbb{R}$ a function. We then consider the general minimization problem

$$(\mathscr{P}) \begin{cases} \text{minimize } \mathscr{F}(u) \\ \text{among } u \in \mathscr{C} \end{cases}$$

The elements of $\mathscr{C}$ will usually be called the *competitors*. The *infimum* of the problem is defined as follows:

$$\inf(\mathscr{P}) = \inf\{\mathscr{F}(u) : u \in \mathscr{C}\}\,.$$

A *minimizer* is by definition a competitor $u^* \in \mathscr{C}$ that achieves the infimum of the problem:

$$\mathscr{F}(u^*) = \inf(\mathscr{P})$$

For example problem $(\mathscr{P}_1)$ above corresponds to the case when

$$\mathscr{C} = C^1([0,1], \mathbb{R}^2) \cap \{u : u(0) = A \text{ and } u(1) = B\}$$

and

$$\mathscr{F}(u) = \int_0^1 \|u'(t)\| dt\,,$$

whereas problem $(\mathscr{P}_2)$ corresponds to

$$\mathscr{C} = C^1([a,b], \mathbb{R}) \cap \{u : u(a) = \overline{a} \text{ and } u(b) = \overline{b}\}$$

and

$$\mathscr{F}(u) = \int_a^b \sqrt{1 + u'(x)^2} dx\,.$$

It turns out that often $\mathscr{C}$ will consist in a class of functions, that is the variables of $\mathscr{F}$ are functions. To avoid confusion between these functions u and $\mathscr{F}$ itself it is customary to call $\mathscr{F}$ a *functional*.

1.2.1 Exercise – Let $\mathscr{C} = \mathbb{R}$ and $\mathscr{F}(u) = \exp(u)$. Does $\mathscr{F}$ have a minimizer in $\mathscr{C}$?

ex.1 1.2.2 Exercise – This exercise is to the extent that problem $(\mathscr{P}_1)$ is "more general" that problem $(\mathscr{P}_2)$.

(1) Given a competitor u for problem $(\mathscr{P}_2)$, associate with u a competitor $\overline{u}$ for problem $(\mathscr{P}_1)$ such that $\operatorname{graph}(u) = \operatorname{im}(\overline{u})$ and $\overline{u}$ is one-to-one.

(2) Show that $\operatorname{length}(\overline{u}) = L(u)$.

(3) Show that $\inf(\mathscr{P}_1) \leq \inf(\mathscr{P}_2)$.

1.2.3 Exercise – Using Exercise 1.2.2 and assuming that Theorem 1.2.1 holds true, prove Theorem 1.2.2.

1.2.4 Exercise – Without assuming that Theorem 1.2.1 holds, prove Theorem 1.2.2 in the special case when $A = (0,0)$ and $B = (1,0)$.

1.2.5 Piecewise affine functions

In this section we will proceed to proving Theorem 1.2.2. In fact we will provide many different proofs of this result as we go along in these lecture notes. The proof we offer here roughly runs as follows. We approximate the graph of a C^1 competing function $u : [a,b] \to \mathbb{R}$ by a "polygonal curve" (joining A and B) with nearly the same length. Then we observe that the length of the said polygonal curve is greater than or equal to the length of the straight line segment from A to B according to the triangle inequality for the Euclidean norm. Since the accuracy of the approximation is arbitrary the conclusion follows.

In order to make the previous argument rigorous we need a definition. Recall that a function $u : [c,d] \to \mathbb{R}$ is *affine* if there are constants p and q such that $u(x) = px + q$. Given real numbers $\overline{c}, \overline{d} \in \mathbb{R}$ (and assuming that $c < d$) there exists a unique affine function u on $[c,d]$ such that $u(c) = \overline{c}$ and $u(d) = \overline{d}$.

1.2.3 Definition. A function $u : [a,b] \to \mathbb{R}$ is called *piecewise affine* if there are $a = a_0 < a_1 < \ldots < a_{\kappa-1} < a_\kappa = b$ and $\overline{a}_0, \overline{a}_1, \ldots, \overline{a}_{\kappa_1}, \overline{a}_\kappa \in \mathbb{R}$ such that for each $k = 1, \ldots, \kappa$, u is affine on $[a_{k-1}, a_k]$, $u(a_{k-1}) = \overline{a}_{k-1}$ and $u(a_k) = \overline{a}_k$. We let $A_{PW}([a,b])$ denote the collection of piecewise affine functions on $[a,b]$.

Notice that piecewise affine functions are continuous, and in general are not C^1 (that is u as in the definition above is not necessarily differentiable at the points a_k, $k = 1, \ldots, \kappa - 1$).

prop.approx.pa **1.2.4 Proposition.** *Let $u \in C^1([a,b])$ and $\varepsilon > 0$. There exists a piecewise affine function $\tilde{u} \in A_{PW}([a,b])$ such that $\tilde{u}(a) = u(a)$, $\tilde{u}(b) = u(b)$ and*

$$|u'(x) - \tilde{u}'(x)| < \varepsilon$$

for each $x \in [a,b]$ at which $\tilde{u}$ is differentiable.

Proof. Since the derivative u' is a continuous function on $[a,b]$ there exists $\delta > 0$ such that if $x_1, x_2 \in [a,b]$ and $|x_1 - x_2| < \delta$ then $|u'(x_1) - u'(x_2)| < \varepsilon$. We choose a partition of $[a,b]$, $a = a_0 < \ldots < a_\kappa = b$, such that $|a_{k-1} - a_k| < \delta$ for each $k = 1, \ldots, \kappa$. Next we consider the piecewise affine function $\tilde{u}$ on $[a,b]$ which is affine on each interval $[a_{k-1}, a_k]$, $k = 1, \dots, \kappa$, and such that $\tilde{u}(a_k) = u(a_k)$, $k = 0, \dots, \kappa$. Clearly $\tilde{u}(a) = u(a)$ and $\tilde{u}(b) = u(b)$. Finally if $x \in [a_{k-1}, a_k]$, $k = 1, \dots, a_\kappa$, then

$$\tilde{u}'(x) = \frac{u(a_k) - u(a_{k-1})}{a_k - a_{k-1}} .$$

On the other hand there exists $x_0 \in [a_k, a_{k-1}]$ such that

$$u'(x_0) = \frac{u(a_k) - u(a_{k-1})}{a_k - a_{k-1}} ,$$

according to the Mean Value Theorem. Therefore $\tilde{u}'(x) = u'(x_0)$ for every $x \in [a_{k-1}, a_k]$. Since also $|u'(x) - u(x_0)| < \varepsilon$ for such x the conclusion follows. □

ex.8 1.2.5 Exercise – Let $u : [c, d] \to \mathbb{R}$ be an affine function. Prove that $L(u) = \|u(c) - u(d)\|$.

1.2.6 A fundamental theorem

We now turn to proving Theorem 1.2.2, i.e. the fact that the function whose graph is the straight line segment joinging A and B does minimize the length L functional and is its unique minimizer.

Proof. Let u_* be the unique affine function on $[a, b]$ such that $u_*(a) = \overline{a}$ and $u_*(b) = \overline{b}$. We want to show that u_* is a minimizer for problem $(\mathscr{P}_2)$ and is the unique minimizer for that problem.

We now prove that u_* is a minimizer. Let u be any competitor for problem $(\mathscr{P}_2)$, and $\varepsilon > 0$. Since $u \in C^1([a, b])$ we infer from Proposition 1.2.4 that there exists a piecewise affine function $\tilde{u}$ on $[a, b]$ verifying the conclusions of that Proposition. Let $a = a_0 < \ldots < a_\kappa = b$ be a partition of $[a, b]$ associated with $\tilde{u}$ in the definition of piecewise affine. We denote by $P_k \in \mathbb{R}^2$ the point in the plane with coordinates $(a_k, u(a_k))$, $k = 0, \dots, \kappa$. Then Exercise 1.2.5 implies that

$$\begin{aligned} \sum_{k=1}^{\kappa} \int_{a_{k-1}}^{a_k} \sqrt{1 + \tilde{u}'(x)^2} dx &= \sum_{k=1}^{\kappa} \|P_k - P_{k-1}\| \\ &\geq \|P_\kappa - P_0\| \\ &= \|A - B\| \\ &= \int_a^b \sqrt{1 + u_*'(x)^2} dx . \end{aligned} \tag{1.1}$$

eq.7

On the other hand we infer from Exercise 1.2.6 and the choice of $\tilde{u}$ that for each $k = 1, \dots, \kappa$ and each $x \in [a_{k-1}, a_k]$ one has

$$\left| \sqrt{1 + \tilde{u}'(x)^2} - \sqrt{1 + u'(x)^2} \right| < \varepsilon .$$

Therefore it follows from (1.1) that

$$\begin{aligned} L(u) &= \int_a^b \sqrt{1+u'(x)^2}dx \\ &= \sum_{k=1}^{\kappa} \int_{a_{k-1}}^{a_k} \sqrt{1+u'(x)^2}dx \\ &\geq \sum_{k=1}^{\kappa} \int_{a_{k-1}}^{a_k} \left(\sqrt{1+\tilde{u}'(x)^2} - \varepsilon\right) dx \\ &\geq \int_a^b \sqrt{1+u_*'(x)^2} - \int_a^b \varepsilon \\ &= L(u_*) - \varepsilon(b-a)\,. \end{aligned}$$

From the arbitrariness of ε we infer that $L(u) \geq L(u_*)$ and the proof that u_* is a minimizer is complete.

Next we prove that u_* is the unique minimizer. In order to do so we introduce some convenient notation. The problem under consideration is

$$(\mathscr{P}_{A,B}) \begin{cases} \text{minimize } \int_a^b \sqrt{1+u'(x)^2}dx \\ \text{among } u \in C^1([a,b]) \text{ such that } u(a) = \overline{a} \text{ and } u(b) = \overline{b}\,. \end{cases}$$

The first part of this proof established in particular that $\inf(\mathscr{P}_{A,B}) = \|A-B\|$. This holds of course regardless of the choice of A and B.

Let $u \in C^1([a,b])$ be such that $u(a) = \overline{a}$ and $u(b) = \overline{b}$. We need to show that if $u \neq u_*$ then $L(u) > L(u_*)$. Assume that $u \neq u_*$. Then there exists $c \in (a,b)$ such that $u(c) \neq u_*(c)$. Let $P = (c, u(c))$. It follows from the observation above that

$$\int_a^c \sqrt{1+u'(x)^2}dx \geq \inf(\mathscr{P}_{A,P}) = \|A-P\|$$

and that

$$\int_c^b \sqrt{1+u'(x)^2}dx \geq \inf(\mathscr{P}_{P,B}) = \|P-B\|\,.$$

Therefore

$$\begin{aligned} \int_a^b \sqrt{1+u'(x)^2}dx &\geq \|A-P\| + \|P-B\| \\ &> \|A-B\| \\ &= \int_a^b \sqrt{1+u_*'(x)^2}dx\,. \end{aligned}$$

□

ex.9 1.2.6 Exercise – Let $p, q \geq 0$. Show that $|\sqrt{1+p} - \sqrt{1+q}| \leq |p-q|/2$.

1.2.7 Exercise – Let $\kappa \geq 2$ be an integer and $P_0, \ldots, P_\kappa \in \mathbb{R}^2$. Prove by induction on κ (and referring to the triangle inequality) that

$$\sum_{k=1}^{\kappa} \|P_k - P_{k-1}\| \geq \|P_\kappa - P_0\| .$$

1.3 Paths of shortest weighted length

1.3.1 Fermat's principle and Snell's law

sec.snell.1

Here again we consider paths between two points A and B and we seek to determine along which path light is going to travel. The material between A and B may not be homogeneous, though, so that it may take more time to travel along the straight line segment. Fermat's principle states that "nature always acts in the shortest way". In the context of geometrical optics this means that the light (locally) minimizes its "optical length" between its endpoints. The optical length depends upon the material in which the light travels: the speed of light is inversely proportional to the *optical density of the medium.* The nature of the material visited may change along the way (i.e. the optical density is a function of the space position). We interpret this mathematically by saying that light minimizes a *weighted length*, the weight w being a real-valued function of the ambient space, say $w : \mathbb{R}^2 \to \mathbb{R}$

In the case of paths being considered as C^1 images in $\mathbb{R}^2$ the corresponding weighted length is defined as

$$\text{length}_w(u) = \int_0^1 \|u'(t)\| w(u(t)) dt .$$

In case of a C^1 graph the weighted length is defined by

$$L_w(u) = \int \sqrt{1 + u'(t)^2}\, w(t, u(t)) dt .$$

In both cases some assumptions need be made (about w) in order that the integrand is indeed an integrable function. It is worth noticing that w might be discontinuous in some cases, for instance when the light is going to be refracted by a lens. We now consider this example in some detail. Suppose that A and B lie in two different media separated by the line $y = 0$, and that $\overline{a} > 0$ and $\overline{b} < 0$. We denote the optical densities of these media by $n_1 > 0$ and $n_2 > 0$. Therefore

$$w(x, y) = \begin{cases} n_1 & \text{if } y \geq 0 \\ n_2 & \text{if } y < 0 . \end{cases} \tag{1.2}$$

eq.w.snell

Intuitively a ray of light will follow a straight path in the medium $y \geq 0$ (because its optical density is constant) and a straight path in the medium $y < 0$ (for the same reason). However these two paths need not have the same direction. We will subsequently derive the angle between these two line segments depending upon n_1 and n_2 – this relation is known as *Snell's law* in geometrical optics. In a classic analogy by Richard Feynman, the area of lower optical density is replaced by a beach, the area of higher optical density by the sea, and the fastest way for a rescuer on the beach to get to a drowning person in the sea is to run along a path that follows Snell's law.

We now derive Snell's law under the additional assumption that

(H) *Light travels straight paths in each of the media $y \geq 0$ and $y < 0$.*

Let denote by $P = (x, 0)$ the point on the axis $y = 0$ where light possibly chooses a different direction. We also let $u_x : [a, b] \to \mathbb{R}$ denote the piecewise affine function joining A to P and then P to B. Then

$$\begin{aligned} L_w(u_x) &= n_1 \|A - P\| + n_2 \|P - B\| \\ &= n_1 \sqrt{(x-a)^2 + \overline{a}^2} + n_2 \sqrt{(x-b)^2 + \overline{b}^2}\,. \end{aligned}$$

We notice that the above formula defines a function $f(x)$ of one real variable $x \in \mathbb{R}$.

prop.1 **1.3.1 Proposition.** *Let $f(x) = L_w(u_x)$ be defined as above. Then f has exactly one minimum $x^* \in \mathbb{R}$ characterized by the equation*

$$n_1 \frac{x^* - a}{\sqrt{(x^* - a)^2 + \overline{a}^2}} = n_2 \frac{b - x^*}{\sqrt{(x^* - b)^2 + \overline{b}^2}}\,.$$

Proof. Notice that $f(x) \to \infty$ as $x \to \infty$ and also as $x \to -\infty$. Therefore, as f is continuous it has at least one minimum. Since f is also derivable everywhere, each such minimum x must be so that $f'(x) = 0$. Furthermore we observe that

$$f'(x) = n_1 \frac{x - a}{\sqrt{(x-a)^2 + \overline{a}^2}} + n_2 \frac{x - b}{\sqrt{(x-b)^2 + \overline{b}^2}}\,.$$

It remains to show that the equation $f'(x) = 0$ has at most one solution (Exercise 1.3.2) □

We see that under assumption (H) our minimization problem admits exactly one minimizer consisting of two pieces of straight line segments. This configuration (corresponding to $P^* = (x^*, 0)$) is characterized by two angles θ_1 and θ_2. The angle θ_1 is the angle at vertex P^* of the triangle AP^*O (where $O = (a, 0)$) and θ_2 is the angle at vertex P^* of the triangle $O'P^*B$ (where $O' = (b, 0)$). With this notation Proposition 1.3.1 reads

$$\frac{n_1}{n_2} = \frac{\cos\theta_2}{\cos\theta_1}\,.$$

A few comments are in order.

(1) We found out heuristically that the minimization problem of this subsection has a solution, if any, which is *piecewise affine* but not C^1 smooth (when $n_1 \neq n_2$). This actually means that the problem has *no* solution in $C^1([a,b],\mathbb{R})$. It shows that we need to take some care in defining the class $\mathscr{C}$ of all competitors to be large enough to ensure that a minimizer actually exists.

(2) The minimizing problem under consideration here was reduced (under assumption (H)) to a *one dimensional minimization problem.* We will justify this reduction later on in section 1.3.5 In order to solve the one dimensional problem we solved the equation $f'(x) = 0$ and used some specific arguments to show that the unique solution x^* corresponds to a minimum of f.

1.3.1 Exercise – Let $A = (a, \overline{a})$, $B = (b, \overline{b})$ and let $u : [a,b] \to \mathbb{R}$ be the affine function such that $u(a) = \overline{a}$ and $u(b) = \overline{b}$. Show that $L(u) = \|A - B\|$.

ex.2 1.3.2 Exercise – Let $n > 0$ and $p, q \in \mathbb{R}$. We consider the function $f(x) = n\sqrt{(x-p)^2 + q^2}$, $x \in \mathbb{R}$. Compute f' and f'' and show that $f'' > 0$ whenever $q \neq 0$. Use this to finish the proof of Proposition 1.3.1.

1.3.3 Exercise – Find a necessary and sufficient condition on n_1 and n_2 so that the unique minimizer u_{x^*} described in Proposition 1.3.1 is of class C^1. Prove your result.

1.3.2 Continuous piecewise smooth functions

We discovered in the preceding subsection that the natural problem of describing the path of light in nonhomogeneous media leads us to enlarge the class $\mathscr{C} = C^1([a,b],\mathbb{R})$ to a class of functions including at least "piecewise affine" functions. We now proceed to define precisely a class that will be convenient: that of continuous piecewise smooth functions.

def.pwc **1.3.2 Definition.** Let $I \subset \mathbb{R}$ be a closed bounded interval. A function $u : I \to \mathbb{R}$ is said to be *piecewise continuous* whenever the following condition holds. There exists a finite (possibly empty) set $S \subset I$ such that u is continuous on $I \setminus S$ and for every component J of $I \setminus S$ and each endpoint ξ of J the following limit exists

$$\lim_{x \to \xi,\, x \in J} u(x)\,.$$

The collection of piecewise continuous functions on I is denoted $C_{PW}(I)$.

Notice that continuous functions are piecewise continuous (as is checked on letting $S = \emptyset$ in the definition above).

Condition (2) says that I can be decomposed into finitely many intervals on the interior of each of which u is continuous and so that the one sided limits of u exist at the endpoints of these intervals. We can rephrase this by saying that there are $a_0, \ldots, a_\kappa$ such that $a = a_0 < a_1 < \ldots < a_{\kappa-1} < a_\kappa = b$ (here

$I = [a, b]$) and u is continuous on each open interval (a_{k-1}, a_k), $k = 1, \dots, \kappa$ and the following limits exist

$$\lim_{x \to a_k^-} u(x) \text{ and } \lim_{x \to a_k^+} u(x) \,,$$

$k = 1, \dots, \kappa - 1$, as well as

$$\lim_{x \to a_0^+} u(x) \text{ and } \lim_{x \to a_\kappa^-} u(x) \,.$$

Here $S \triangle \{a_1, \dots, a_{\kappa-1}\} \subset \{a, b\}$. We insist that S depends upon the function u. Finally we observe that the values achieved by u on the corresponding set S do not really matter. Indeed if we redefine u on S arbitrarily the resulting function will still be piecewise continuous.

For example the following function is piecewise continuous on $[-1, 1]$.

$$u(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

whereas

$$u(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 3 & \text{if } x = 0 \,. \end{cases}$$

is not.

We now introduce the class of indefinite (Riemann) integral of piecewise continuous functions.

def.pws **1.3.3 Definition.** Let $I = [a, b] \subset \mathbb{R}$ be a closed bounded interval. A function $u : I \to \mathbb{R}$ is said to be *continuous piecewise smooth* on I if there exists $\phi \in C_{PW}(I)$ such that

$$u(x) = u(a) + \int_a^x \phi(t) dt \,,$$

for every $x \in I$. The collection of continuous piecewise smooth functions is denoted $C^1_{PW}(I)$.

1.3.4 Exercise – Show that $u : [-1, 1] \to \mathbb{R} : x \mapsto |x|$ belongs to $C^1_{PW}([-1, 1])$ (you need to exhibit a corresponding function ϕ and show it is piecewise continuous).

1.3.5 Exercise – Show that piecewise continuous functions are bounded.

1.3.6 Exercise – Show that $C_{PW}(I)$ is a linear space.

ex.6 1.3.7 Exercise – Let $[c, d] \subset [a, b]$ and $u \in C_{PW}([a, b])$ (resp. $u \in C^1_{PW}([a, b])$). Show that $u|_{[c,d]} \in C_{PW}([c, d])$ (resp. $u \in C^1_{PW}([c, d])$).

ex.4 1.3.8 Exercise – Prove that each $u \in C_{PW}(I)$ is Riemann integrable on I.

1.3.9 Exercise – Show that $C^1_{PW}(I)$ is a linear space.

1.3.10 Exercise – Prove that if u is continuous piecewise smooth on I then it is indeed continuous on I (This amounts to proving that an indefinite Riemann integral is continuous).

1.3.11 EXERCISE – We say that $u : [0,1] \to \mathbb{R}^2$ is *continuous piecewise affine* whenever u is continuous and there are $0 = t_0 < t_1 < \ldots < t_\kappa = 1$ such that the restriction of u to each interval $[t_{k-1}, t_k]$, $k = 1, \ldots, \kappa$, is affine. Show that if $u \in C^1$ then length(u) equals the supremum of the lengths of continuous piecewise affine functions $\bar{u}$ such that $\bar{u}(t_k) = u(t_k)$ for each $k = 0, \ldots, \kappa$.

1.3.3 More vocabulary

So far we have introduced two problems: that of minimizing length and that of minimizing weighted length. There are several ways of setting up these problems, mainly in the realm of curves on the one hand, and that of graphs on the other hand. For the beginning of these lecture notes we shall stick with the "graph formulation" which corresponds to classical calculus of variations. In that setting, the weighted length of a "graph" $u : [a,b] \to \mathbb{R}$ can be written as

$$\int_a^b \sqrt{1 + u'(x)^2}\, w(x, u(x))dx\,.$$

The case of length actually corresponds to the particular case when $w = 1$. We will be interested in the sequel in functionals $\mathscr{F}$ of the type of that above, only more general in that the integrand is allowed to be an arbitrary function of the triple $(x, u(x), u'(x))$.

1.3.4 Definition. A *Lagrangian* on $[a,b]$ is a function

$$F : [a,b] \times \mathbb{R} \times \mathbb{R}\,.$$

It is customary to write (x, y, p) for the variables of a Lagrangian.

1.3.5 Definition. Let $\mathscr{C} = C^1_{PW}([a,b], \mathbb{R})$ and $\mathscr{F} : \mathscr{C} \to \mathbb{R}$. We call $\mathscr{F}$ a *variational integral* if there exists a Lagrangian F on $[a,b]$ (called the *Lagrangian* associated with $\mathscr{F}$) such that for every $u \in \mathscr{C}$ the function $x \mapsto F(x, u(x), u'(x))$ is Riemann integrable on $[a,b]$ and

$$\mathscr{F}(u) = \int_a^b F(x, u(x), u'(x))dx\,.$$

For instance $L(u)$ is a variational integral with corresponding Lagrangian

$$F(x, y, p) = \sqrt{1 + p^2}$$

and the weighted length $L_w(u)$ is a variational integral corresponding to the Lagrangian

$$F(x, y, p) = w(x, y)\sqrt{1 + p^2}\,.$$

We now turn to discussing the assumption that

$$[a,b] \to \mathbb{R} : x \mapsto F(x, u(x), u'(x))$$

be Riemann integrable whenever u is continuous piecewise smooth. This is going to be the case for instance when F is continuous (see Exercise 1.3.12), e.g. $F(x,y,p) = \sqrt{1+p^2}$. Notwithstanding the reader is invited to read the following small prints kind of bad news.

Recall however that in the case of Snell's law alluded to above we considered a noncontinuous weight function w (se (1.2)) corresponding to a noncontinuous Lagrangian F. Let us write w_{Snell} and F_{Snell} for this particular weight and the corresponding Lagrangian. Perhaps somewhat surprisingly it turns out that in this case there are functions $u \in C^1([a,b])$ such that $x \mapsto F_{\text{Snell}}(x, u(x), u'(x))$ is *not* Riemann integrable on $[a,b]$. We will discuss this below but for now let us mention that it is an unpleasant situation because we run into technical problems to even state properly Snell's law. Indeed we would want to state a Theorem to the effect that the unique minimizer of the minimization problem

$$\begin{cases} \text{minimize } \int_a^b F_{\text{Snell}}(x,u,u'(x))dx \\ \text{among } u \in C^1_{PW}([a,b]) \cap \{u : u(a) = \overline{a} \text{ and } u(b) = \overline{b}\} \end{cases}$$

is the piecewise affine function described in section 1.3.1. It turns out however that we cannot even state the problem in this way because – as we shall explain – there are competitors u for which the integral to be minimized is not defined.

Assume that $u \in C^1([a,b])$. Then

$$F_{\text{Snell}}(x,u(x),u'(x)) = \begin{cases} n_1\sqrt{1+u'(x)^2} & \text{if } u(x) \ge 0 \\ n_2\sqrt{1+u'(x)^2} & \text{if } u(x) < 0\,. \end{cases} \tag{1.3}$$

eq.F.snell

In other words,

$$\frac{F_{\text{Snell}}(x,u(x),u'(x))}{\sqrt{1+u'(x)^2}} = \begin{cases} n_1 & \text{if } u(x) \ge 0 \\ n_2 & \text{if } u(x) < 0\,. \end{cases}$$

That function of x takes only two values, n_1 and n_2. It is equal to n_1 on the closed set $C = [a,b] \cap \{x : u(x) \ge 0\}$. The point is that we can realize *any* closed set $C \subset [a,b]$ in this way. That is, given any C we can find some u such that $C = \{u \ge 0\}$. This means that choosing u properly the integrand of our variational integral can be of the type $f_C(x)\sqrt{1+u'(x)^2}$ where

$$f_C(x) = \begin{cases} n_1 & \text{if } x \in C \\ n_2 & \text{if } x \in [a,b] \setminus C\,, \end{cases} \tag{1.4}$$

eq.bad.snell

for any closed set $C \subset [a,b]$. It turns out that there are sets C so that the function $f_C\sqrt{1+(u')^2}$ is not Riemann integrable on $[a,b]$. We now state the theorems to this effect.

1.3.6 Theorem. *Let $C \subset [a,b]$ be closed. Then there exists $u \in C^1([a,b])$ such that*

$$u(x) = 0 \textit{ if } x \in C$$

and

$$u(x) < 0 \textit{ if } x \in [a,b] \setminus C\,.$$

We recall from the following characterization of Riemann integrable functions.

1.3.7 Definition. A set $S \subset \mathbb{R}$ is called *Lebesgue negligible* if for every $\varepsilon > 0$ there are (finitely many or countable many) intervals $I_1, I_2, \ldots$ in $\mathbb{R}$ such that

$$S \subset \cup_j I_j$$

and

$$\sum_j \operatorname{diam} I_j < \varepsilon\,.$$

1.3.8 Theorem. *A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if the set of points at which f is discontinuous is Lebesgue negligible.*

Since the set of points where f_C is discontinuous is $\operatorname{Bdry} C$, it remains to verify that there exists a set C whose boundary is not Lebesgue negligible. These are "Cantor type sets".

1.3.9 Theorem. *There exists a closed set $C \subset [0,1]$ such that $C = \operatorname{Bdry} C$ and C is not Lebesgue negligible.*

In order to state properly Snell's law we have to find a way around the technical problem we just described. It seems that the more natural way is the following. Given a Lagrangian on $[a,b]$ we will restrict the class of competitors to those functions u such that $x \mapsto F(x,u(x),u'(x))$ is Riemann integrable on $[a,b]$.

1.3.10 Definition. Given a Lagrangian F on $[a,b]$ we call the *domain of* F the following collection of functions:

$$\operatorname{Dom}(F;[a,b]) = C^1_{PW}([a,b]) \cap \{u : \text{the function } x \mapsto F(x,u(x),u'(x)) \text{ is Riemann integrable on } [a,b]\}\,.$$

We identified some continuous piecewise smooth functions $u \notin \operatorname{Dom}(F_{\text{Snell}})$, the obstruction here being related to the possibility of the set $\{x : u(x) = 0\}$ being peculiar (specifically: having non Lebesgue negligible boundary). See Exercise 1.3.15 for the description of a wealth of functions belonging to the domain of F_{Snell}.

Finally let us mention that the problem we have run into is an advocate for using Lebesgue's theory of the integral rather than Riemann's. In fact, $x \mapsto F_{\text{Snell}}(x,u(x),u'(x))$ is Lebesgue integrable whenever $u \in C^1_{PW}([a,b])$ so that we wouldn't have to introduce some artificial restriction on u (that $u \in \operatorname{Dom}(F_{\text{Snell}})$) if we were working with the Lebesgue integral.

ex.3 1.3.12 Exercise – Prove that if $u \in C^1_{PW}([a,b],\mathbb{R})$ and the Lagrangian F is continuous then $x \mapsto F(x,u(x),u'(x))$ is piecewise continuous, and therefore Riemann integrable according to Exercise 1.3.8. In other words, if F is continuous then $\operatorname{Dom}(F;[a,b]) = C^1_{PW}([a,b])$.

1.3.13 Exercise – Let $[c,d] \subset [a,b]$ and let F be a Lagrangian on $[a,b]$. Prove that if $u \in \operatorname{Dom}([a,b])$ then $u|_{[c,d]} \in \operatorname{Dom}([c,d])$.

1.3.14 Exercise – Assume that $n_1 \neq n_2$, $C \subset [a,b]$ is closed, f_C is defined in (1.4) and $u \in C^1([a,b])$. Prove that the set of points where $f_C\sqrt{1+(u')^2}$ is discontinuous is $\operatorname{Bdry} C$.

ex.7 1.3.15 Exercise – Let $u \in C^1_{PW}([a,b])$ be such that the set $[a,b] \cap \{x : u(x) = 0\}$ is *finite*. Show that u belongs to the domain of F_{Snell}.

1.3.16 Exercise – Some interesting functionals are not (generalized) variational integrals. Consider for instance the following minimization problem.

$$(\mathscr{P})\begin{cases}\text{minimize } \sup\{|u'(x)| : x \in [a,b]\}\\ \text{among } u \in C^1([a,b]) \cap \{u : u(a) = \overline{a} \text{ and } u(b) = \overline{b}\}\,.\end{cases}$$

(1) Determine $\inf(\mathscr{P})$.

(2) Show that the affine competitor is a minimizer of $(\mathscr{P})$.

(3) Show that the affine competitor is the unique minimizer of $(\mathscr{P})$ (Hint: use the Mean Value Theorem).

1.3.4 Localization principle

We now turn to proving a localization result which will be useful when we return to Snell's law more rigorously in the next subsection. Roughly speaking this says that if u_* minimizes some variational integral on $[a,b]$, and $[c,d] \subset [a,b]$, then the restriction of u_* to $[c,d]$ minimizes the corresponding variational integral on $[c,d]$, with respect to its own boundary values.

prop.loc **1.3.11 Proposition** (Localization Principle)**.** *Assume that*

(1) $[a,b] \supset [c,d]$ *and* $[a,b] \neq [c,d]$*;*

(2) $\overline{a}, \overline{b} \in \mathbb{R}$*;*

(3) F *is a Lagrangian on* $[a,b]$*;*

(4) $\mathscr{C}_{a,b} = \mathrm{Dom}(F;[a,b]) \cap \{u : u(a) = \overline{a} \text{ and } u(b) = \overline{b}\}$*;*

(5) $u_* \in \mathscr{C}_{a,b}$*;*

(6) $\mathscr{C}_{c,d} = \mathrm{Dom}(F;[c,d]) \cap \{u : u(c) = u_*(c) \text{ and } u(d) = u_*(d)\}$*;*

(7) We define variational integrals respectively on $\mathscr{C}_{a,b}$ *and* $\mathscr{C}_{c,d}$ *by the formulas*

$$\mathscr{F}_{a,b}(u) = \int_a^b F(x,u(x),u'(x))dx$$

whenever $u \in \mathscr{C}_{a,b}$ *and*

$$\mathscr{F}_{c,d}(u) = \int_c^d F(x,u(x),u'(x))dx$$

whenever $u \in \mathscr{C}_{c,d}$*;*

(8) u_* *is a minimizer of problem*

$$(\mathscr{P}_{a,b}) \begin{cases} \text{minimize } \mathscr{F}_{a,b}(u) \\ \text{among } u \in \mathscr{C}_{a,b}\,. \end{cases}$$

Then $u_{**} = u_*|_{[c,d]}$ *is a minimizer of problem*

$$(\mathscr{P}_{c,d}) \begin{cases} \text{minimize } \mathscr{F}_{c,d}(u) \\ \text{among } u \in \mathscr{C}_{c,d}\,. \end{cases}$$

Proof. Before we give a fully detailed account of the proof let us briefly state the (easy) idea. If u_{**} were not a minimizer of problem $(\mathscr{P}_{c,d})$ then there would be some $v_{**} \in \mathscr{C}_{c,d}$ such that $\mathscr{F}_{c,d}(v_{**}) < \mathscr{F}_{c,d}(u_{**})$. Defining v_* on $[a,b]$ to coincide with v_{**} on $[c,d]$ and with u_* on $[a,b] \setminus [c,d]$ we would obtain a competitor for $(\mathscr{P}_{a,b})$ such that $\mathscr{F}_{a,b}(v_*) < \mathscr{F}_{a,b}(u_*)$ (according to the additivity of the integral), a contradiction. The proof of course consists in showing that the function v_* constructed piecewise still belongs to the class of competitors (see Exercise 1.3.17).

We first need to show that $u_{**} \in \mathscr{C}_{c,d}$. Since u_{**} clearly verifies the boundary condition stated in the definition of $\mathscr{C}_{c,d}$ we are left with noticing that $u_{**} \in C^1_{PW}([c,d])$ and $x \mapsto F(x, u_{**}(x), u'_{**}(x))$ is Riemann integrable on $[c,d]$. **P** Since $u_* \in C^1_{PW}([a,b])$ there exists $\phi_* \in C_{PW}([a,b])$ such that

$$u_*(x) = u_*(a) + \int_a^x \phi_*(t)dt \tag{1.5}$$ eq.1

whenever $x \in [a,b]$. Letting $\phi_{**} = \phi_*|_{[c,d]}$ we infer from Exercise 1.3.7 that $\phi_{**} \in C_{PW}([c,d])$. Furthermore for every $x \in [c,d]$ one has

$$\begin{aligned} u_{**}(x) &= u_*(x) \\ &= u_*(a) + \int_a^x \phi_*(t)dt \\ &= u_*(a) + \int_a^c \phi_*(t)dt + \int_c^x \phi_{**}(t)dt \\ &= u_{**}(c) + \int_c^x \phi_{**}(t)dt \end{aligned}$$

where the last equality follows from (1.5) applied with $x = c$. This shows that $u_{**} \in C^1_{PW([c,d])}$. Since $f(x) = F(x, u_*(x), u_*(x))$ is Riemann integrable on $[a,b]$, its restriction $(f|_{[c,d]})(x) = F(x, u_{**}(x), u'_{**}(x))$ is Riemann integrable on $[c,d]$. **Q**

Next we need to first show that $\mathscr{F}_{c,d}(u_{**}) \leq \mathscr{F}_{c,d}(v)$ for every $v \in \mathscr{C}_{c,d}$. Assume if possible that there exists $v_{**} \in \mathscr{C}_{c,d}$ such that

$$\mathscr{F}_{c,d}(v_{**}) < \mathscr{F}_{c,d}(u_{**})\,. \tag{1.6}$$ eq.2

We then define a function $v_* : [a,b] \to \mathbb{R}$ as follows

$$v_*(x) = \begin{cases} u_*(x) & \text{if } x \in [a,b] \setminus (c,d) \\ v_{**}(x) & \text{if } x \in (c,d)\,. \end{cases}$$

We claim that $v_* \in \mathscr{F}_{a,b}$, i.e. v_* is a competitor for $\mathscr{P}_{a,b}$. In order to prove this we need to show that

$$v_*(x) = v_*(a) + \int_a^x \psi_*(t)dt \tag{1.7}$$ eq.3

for some $\psi_* \in C_{PW}([a,b])$. We let $\psi_{**} \in C_{PW}([c,d])$ be such that

$$v_{**}(x) = v_{**}(c) + \int_c^x \psi_{**}(t)dt$$

whenever $x \in [c,d]$, and $\phi_* \in C_{PW}([a,b])$ be such that

$$u_*(x) = u_*(a) + \int_a^x \phi_*(t)dt$$

whenever $x \in [a,b]$. We define

$$\psi_*(x) = \begin{cases} \phi_*(x) & \text{if } x \in [a,b] \setminus (c,d) \\ \psi_{**}(x) & \text{if } x \in (c,d)\,, \end{cases}$$

and we observe that $\psi_* \in C_{PW}([a,b])$. **P** If $S_* \subset [a,b]$ is associated with ϕ_* in the definition of piecewise continuity and $S_{**} \subset [c,d]$ is associated with ψ_{**}, then $S = S_* \cup S_{**}$ witnesses the piecewise continuity of ψ_*. **Q** Clearly (1.7) holds for $a \leq x \leq c$. Next if $c < x < d$ then, by definition of v_*,

$v_*(x) = v_{**}(x)$

which, according to the choice of ψ_{**}, equals

$$= v_{**}(c) + \int_c^x \psi_{**}(t)dt$$

and since $v_{**}(c) = v_*(c)$ **P** $v_{**}(c) = u_*(c)$ because $v_{**} \in \mathscr{C}_{c,d}$ and $v_*(c) = u_*(c)$ by definition of v_* **Q** one has

$$\begin{aligned} &= v_*(c) + \int_c^x \psi_*(t)dt \\ &= v_*(a) + \int_a^c \psi_*(t)dt + \int_c^x \psi_*(t)dt \\ &= v_*(a) + \int_a^x \psi_*(t)dt\,. \end{aligned}$$

The proof of (1.7) when $d \leq x \leq b$ is analogous. In order to finish the proof that $v_* \in \mathscr{C}_{a,b}$ we need to establish that $x \mapsto F(x, v_*(x), v_*'(x))$ is Riemann integrable on $[a,b]$. Letting f denote the latter, we see this is the case because f is Riemann integrable on $[a,c]$ **P** letting $g(x) = F(x, u_*(x), u_*'(x))$, $x \in [a,b]$, we see that g is Riemann integrable on $[a,b]$ (because $u_* \in \mathrm{Dom}(F;[a,b])$) and therefore $f|_{[a,c]} = g|_{[a,c]}$ is Riemann integrable on $[a,c]$ **Q** , on $[c,d]$ **P** letting $h = F(x, v_{**}(x), v_{**}'(x))$, $x \in [c,d]$, we see that $f|_{[c,d]} = h$ is Riemann integrable on $[c,d]$ because $v_{**} \in \mathrm{Dom}(F;[c,d])$ **Q** , and on $[d,b]$ **P** because $f|_{[d,b]} = g|_{[d,b]}$ is Riemann integrable on $[d,b]$.

Finally,

$$\begin{aligned}
\mathscr{F}_{a,b}(v_*) &= \int_a^b F(x, v_*(x), v_*'(x))dx \\
&= \int_a^c F(x, v_*(x), v_*'(x))dx \\
&\quad + \int_c^d F(x, v_*(x), v_*'(x))dx \\
&\quad + \int_d^b F(x, v_*(x), v_*'(x))dx \\
&= \int_a^c F(x, u_*(x), u_*'(x))dx \\
&\quad + \int_c^d F(x, v_{**}(x), v_{**}'(x))dx \\
&\quad + \int_d^b F(x, u_*(x), u_*'(x))dx \\
&= \int_a^c F(x, u_*(x), u_*'(x))dx \\
&\quad + \mathscr{F}_{c,d}(v_{**}) \\
&\quad + \int_d^b F(x, u_*(x), u_*'(x))dx \\
&< \int_a^c F(x, u_*(x), u_*'(x))dx \\
&\quad + \mathscr{F}_{c,d}(u_{**}) \\
&\quad + \int_d^b F(x, u_*(x), u_*'(x))dx \\
&= \int_a^c F(x, u_*(x), u_*'(x))dx \\
&\quad + \int_c^d F(x, u_*(x), u_*'(x))dx \\
&\quad + \int_d^b F(x, u_*(x), u_*'(x))dx \\
&= \int_a^b F(x, u_*(x), u_*'(x))dx \\
&= \mathscr{F}_{a,b}(u_*)\,,
\end{aligned}$$

contradicting the minimality of u_*. □

`ex.5` 1.3.17 EXERCISE – Explain why the proof of Proposition 1.3.11 fails if in the definitions of $\mathscr{C}_{a,b}$ and $\mathscr{C}_{c,d}$ the space C^1_{PW} is replaced with C^1.

1.3.18 Exercise – Let $a < p < b$, $\overline{p} \in \mathbb{R}$ and let F be a Lagrangian on $[a,b]$. Generalize the localization principle to problems of the type

$$\begin{cases} \text{minimize } \int_a^b F(x,u(x),u'(x))dx \\ \text{among } u \in \overline{C}^1_{PW}([a,b]) \cap \{u : u(a) = \overline{a},\ u(p) = \overline{p} \text{ and } u(b) = \overline{b}\}\,. \end{cases}$$

1.3.19 Exercise – Let $a < p < b$ and $\overline{p} \in \mathbb{R}$. Define $\mathscr{C}$ to be the collection of functions $u : [a,b] \to \mathbb{R}$ such that $u|_{[a,p]} \in C^1([a,p])$, $u|_{[p,b]} \in C^1([p,b])$, $u(a) = \overline{a}$, $u(p) = \overline{p}$ and $u(b) = \overline{b}$. We consider the minimization problem

$$\begin{cases} \text{minimize } \sup\{|u'(x)| : x \in [a,b] \setminus \{p\}\} \\ \text{among } u \in \mathscr{C}\,. \end{cases}$$

(1) Show that this problem admits a minimizer.

(2) Find values of $a, \overline{a}, p, \overline{p}, b, \overline{b}$ such that the localization principle *does not* hold for this problem.

sec.snell.2

1.3.5 Return to Snell's law

Here we will derive Snell's law rigorously in the setting of continuous piecewise smooth functions. The main point will be to justify hypothesis (H) introduced in subsection 1.3.1. Our argument will based on the following strengthening of Theorem 1.2.2.

thm.3

1.3.12 Theorem. *Let $A = (a,\overline{a}), B = (b,\overline{b}) \in \mathbb{R}^2$ be such that $a < b$. Then the minimization problem*

$$(\mathscr{P}^{\text{Length}}_{A,B}) \begin{cases} \textit{minimize } \int_a^b \sqrt{1+u'(x)^2}dx \\ \textit{among } u \in C^1_{PW}([a,b]) \cap \{u : u(a) = \overline{a} \textit{ and } u(b) = \overline{b}\} \end{cases}$$

admits a unique minimizer which is the affine competitor.

We are now ready to prove the main result of this subsection.

thm.snell

1.3.13 Theorem. *Let $A = (a,\overline{a}), B = (b,\overline{b}) \in \mathbb{R}^2$ be such that $a < b$ and $\overline{b} < 0 < \overline{a}$. Let also $n_2 > n_1 > 0$ and define the weight w by formula (1.2). Then the minimization problem*

$$(\mathscr{P}^{\text{Snell}}_{A,B}) \begin{cases} \textit{minimize } \int_a^b \sqrt{1+u'(x)^2}w(x,u(x))dx \\ \textit{among } u \in \mathrm{Dom}(F_{\text{Snell}};[a,b]) \cap \{u : u(a) = \overline{a} \textit{ and } u(b) = \overline{b}\} \end{cases}$$

admits a unique minimizer which is the piecewise affine competitor described in section 1.3.1.

Proof. We recall that $x_* \in [a,b]$ was characterized in Proposition 1.3.1 and we denote by u_* the competitor for our problem which is affine on $[a,x_*]$ and affine on $[x_*,b]$ and such that $u_*(a)=\overline{a}$, $u_*(x_*)=0$ and $u_*(b)=\overline{b}$. The goal is to prove that u_* is the unique minimizer of $(\mathscr{P}^{\mathrm{Snell}}_{A,B})$. In order to keep the notations short we let $\mathscr{F}$ denote the weighted length L_w.

We start by proving that u_* is a minimizer. This will be achieved in the following way. With any competitor u we will associate another competitor $\tilde{u}$ such that $\mathscr{F}(u) \geq \mathscr{F}(\tilde{u})$ and $\tilde{u}$ is affine in $[a,x_u]$ and on $[x_u,b]$, with $\tilde{u}(x_u)=0$, for some $x_u \in [a,b]$ (depending on u). Referring to Proposition 1.3.1 we see that $\mathscr{F}(\tilde{u}) \geq \mathscr{F}(u_*)$ and therefore $\mathscr{F}(u) \geq \mathscr{F}(u_*)$ showing that u_* minimizes $\mathscr{F}$.

Let u be a competitor for $(\mathscr{P}^{\mathrm{Snell}}_{A,B})$. Define

$$X = [a,b] \cap \{u : u(x) = 0\}\,.$$

Since u is continuous and $u(a) > 0 > u(b)$ we infer that $X \neq \emptyset$ according to the intermediate value Theorem. Put

$$\tilde{b} = \inf\, X \text{ and } \tilde{a} = \sup X\,.$$

Notice that $u(\tilde{b}) = u(\tilde{a}) = 0$ by the continuity of u, and therefore $a < \tilde{b} \leq \tilde{a} < b$.

Define a function $u_1 : [a,\tilde{b}]$ such that $u_1(a) = \overline{a}$, $u_1(\tilde{b}) = 0$ and u_1 is affine. We claim that

$$\int_a^{\tilde{b}} \sqrt{1+u'(x)^2}\,w(x,u(x))dx \geq \int_a^{\tilde{b}} \sqrt{1+u_1'(x)^2}\,w(x,u_1(x))dx\,. \tag{1.8}$$ eq.4

(The length of u_1, $L(u_1)$, is understood to be its length on its domain $[a,\tilde{b}]$). In order to establish this inequality we observe that $u(x) > 0$ for every $x \in [a,\tilde{b})$ by the definition of $\tilde{b}$, so that

$$\begin{aligned}
\int_a^{\tilde{b}} \sqrt{1+u'(x)^2}\,w(x,u(x))dx &= n_1 \int_a^{\tilde{b}} \sqrt{1+u'(x)^2}\,dx \\
&\geq n_1 L(u|_{[a,\tilde{b}]}) \\
&\geq n_1 L(u_1) \\
&= \int_a^{\tilde{b}} \sqrt{1+u_1'(x)^2}\,w(x,u_1(x))dx\,.
\end{aligned}$$

according to Theorem 1.3.12 and Exercise 1.3.7

Next we define a function $u_2 : [\tilde{a},b]$ so that $u_2(\tilde{a}) = 0$, $u_2(b) = \overline{b}$ and u_2 is affine. An argument analogous to the one above show that

$$\int_{\tilde{a}}^{b} \sqrt{1+u'(x)^2}\,w(x,u(x))dx \geq \int_{\tilde{a}}^{b} \sqrt{1+u_2'(x)^2}\,w(x,u_2(x))dx\,. \tag{1.9}$$ eq.5

In case $\tilde{b} = \tilde{a}$ we let x_u be the common value and we define $\tilde{u}$ to coincide with u_1 on $[a, \tilde{b}]$ and with u_2 on $[\tilde{a}, b]$. It then follows from (1.8) and (1.9) that

$$\begin{aligned}\mathscr{F}(u) &= \int_a^{\tilde{b}} \sqrt{1 + u'(x)^2} w(x, u(x)) dx \\ &\qquad + \int_{\tilde{a}}^{b} \sqrt{1 + u'(x)^2} w(x, u(x)) dx \\ &\geq \int_a^{\tilde{b}} \sqrt{1 + \tilde{u}'(x)^2} w(x, \tilde{u}(x)) dx \\ &\qquad + \int_{\tilde{a}}^{b} \sqrt{1 + \tilde{u}'(x)^2} w(x, \tilde{u}(x)) dx \\ &= \mathscr{F}(\tilde{u}) \,.\end{aligned}$$

In case $\tilde{b} < \tilde{a}$ we define $u_3 : [a, \tilde{a}]$ such that $u_3(a) = \overline{a}$, $u_3(\tilde{a}) = 0$ and u_3 is affine. We infer from Theorem 1.3.12 and Exercise 1.3.7 that

$$\begin{aligned}\int_a^{\tilde{a}} \sqrt{1 + u'(x)^2} w(x, u(x)) dx &\geq n_1 \int_a^{\tilde{a}} \sqrt{1 + u'(x)^2} dx \\ &= n_1 L(u|_{[a,\tilde{a}]}) \\ &\geq n_1 L(u_3) \\ &= \int_a^{\tilde{a}} \sqrt{1 + u_3'(x)^2} w(x, u_3(x)) dx \,.\end{aligned} \tag{1.10}$$

eq.6

Now we set $x_u = \tilde{a}$ and we define $\tilde{u}$ to coincide with u_3 on $[a, \tilde{a}]$ and with u_2 on $[\tilde{a}, b]$. It follows from (1.10) and (1.9) that

$$\begin{aligned}\mathscr{F}(u) &= \int_a^{\tilde{a}} \sqrt{1 + u'(x)^2} w(x, u(x)) dx \\ &\qquad + \int_{\tilde{a}}^{b} \sqrt{1 + u'(x)^2} w(x, u(x)) dx \\ &\geq \int_a^{\tilde{a}} \sqrt{1 + \tilde{u}'(x)^2} w(x, \tilde{u}(x)) dx \\ &\qquad + \int_{\tilde{a}}^{b} \sqrt{1 + \tilde{u}'(x)^2} w(x, \tilde{u}(x)) dx \\ &= \mathscr{F}(\tilde{u}) \,.\end{aligned}$$

This completes the proof that u_* minimizes.

We now turn to proving that u_* is the unique minimizer. Assume that u is a minimizer of $(\mathscr{P}^{\text{Snell}}_{A,B})$ and define X, $\tilde{a}$, u_2 and u_3 as in the first part of the proof. According to Proposition 1.3.1 we need only to show that u is affine on $[a, \tilde{a}]$ and affine also on $[\tilde{a}, b]$. Suppose that u is not affine on $[a, \tilde{a}]$. Since $u|_{[a,\tilde{a}]}$ minimizes $\mathscr{F}$ restricted to $[a, \tilde{a}]$ (according to Proposition 1.3.11), Theorem 1.3.12 would imply that

$$\begin{aligned}\int_a^{\tilde{a}} \sqrt{1+u'(x)^2}w(x,u(x))dx &\geq n_1 \int_a^{\tilde{a}} \sqrt{1+u'(x)^2}dx \\ &> n_1 \int_a^{\tilde{a}} \sqrt{1+u_3'(x)^2}dx \\ &= \int_a^{\tilde{a}} \sqrt{1+u_3'(x)^2}w(x,u_3(x))dx\,,\end{aligned}$$

contradicting the local minimality of u. Similarly if u were not affine on $[\tilde{a}, b]$ then we would infer that

$$\begin{aligned}\int_{\tilde{a}}^{b} \sqrt{1+u'(x)^2}w(x,u(x))dx &= n_2 \int_{\tilde{a}}^{b} \sqrt{1+u'(x)^2}dx \\ &> n_2 \int_{\tilde{a}}^{b} \sqrt{1+u_2'(x)^2}dx \\ &= \int_{\tilde{a}}^{b} \sqrt{1+u_2'(x)^2}w(x,u_2(x))dx\,,\end{aligned}$$

contradicting the minimality of $u|_{[\tilde{a},b]}$. □

1.3.20 Exercise – Prove Theorem 1.3.12.

1.4 Paths of shortest length on a sphere

1.5 More examples

1.5.1 Catenary

1.5.2 Brachistochrone

1.5.3 Elastic string

Chapter 2

First and second variation