An introduction to the calculus of variations

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January 10, 2007

Contents

Chapter 1

Overview

1.1 Foreword

In this chapter we quickly describe the content of these lecture notes. To start with we state some problems which will serve as paradigms as we go along. Here they are stated loosely and they will be treated in more details in the coming chapters. Next we touch upon the main themes that will occur on our way: necessary conditions (Euler equations), sufficient conditions, existence theory.

1.2 Path of shortest length

Let $A, B \in \mathbb{R}^2$ be two distinct points. We look for a *path of shortest length* joining A and B. In order to state the problem more precisely we need to define what we mean by "path", by "joining A and B", and by "length of a path". There are several possibilities of doing so.

1.2.1 First model: curves

One can model a *path joining A and B* by a continuous function $u : [0,1] \to \mathbb{R}^2$ such that $u(0) = A$ and $u(1) = B$. We insist that u be continuous so that it really joins its endpoints A and B. In defining the *length* of such path we must require more than the mere continuity of u (see chapter ... for a detailed treatment of this question). Here we will content ourselves with defining the length of u when $u \in C^1$ (that is u is differentiable at each $t \in [0,1]$ and its derivative $u' : [0,1] \to \mathbb{R}^2$ is continuous) as is done in a course on curves and surfaces.

$$\text{length}(u) = \int_0^1 \|u'(t)\| dt\,.$$

Here $\|u'(t)\|$ denotes the Euclidean norm of the vector $u'(t)$, i.e.

$$\|u'(t)\| = \sqrt{u_1'(t)^2 + u_2'(t)^2}\,.$$

We can now express the minimization problem in the following abbreviated form:

$$(\mathscr{P}_1)\begin{cases}\text{minimize } \int_0^1 \|u'(t)\|dt\\ \text{among } u \in C^1([0,1],\mathbb{R}^2) \text{ such that } u(0)=A \text{ and } u(1)=B\end{cases}$$

Of course we expect the solution to be the straight line segment with endpoints A and B. This requires a proof however. Now we state the corresponding Theorem.

thm.1 **1.2.1 Theorem.** *A function $u \in C^1([0,1],\mathbb{R}^2)$ such that $u(0) = A$ and $u(1) = B$ minimizes its length if and only if it is a one-to-one parametrization of the straight line segment*

$$[\![A,B]\!] = \mathbb{R}^2 \cap \{A + t(B-A) : t \in [0,1]\}\,.$$

This shows that minimizers for the minimization problem $(\mathscr{P}_1)$ are *not* unique even though we expect "the unique solution to be the straight line segment". This is due, of course, to the way we have set up the problem and the fact that there are many distinct ways to parametrize the straight line segment as a curve.

1.2.2 Second model: graphs

In the previous paragraph we thought of a path as a continuous *image* of $[0,1]$. Here we will think of a path as a continuous *graph* instead. The points A and B have Cartesian coordinates, say, $(a,\overline{a})$ and $(b,\overline{b})$. Renaming A as B and B as A if necessary we may assume that $a \leq b$. We will subsequently assume that $a < b$. A continuous graph joining A and B is then defined as a continuous function $u : [a,b] \to \mathbb{R}$ such that $u(a) = \overline{a}$ and $u(b) = \overline{b}$. In accordance with the previous paragraph the length of the graph of u is defined as

$$L(u) = \int_a^b \sqrt{1 + u'(x)^2}dx\,.$$

In this setting the minimizations problem reads

$$(\mathscr{P}_2)\begin{cases}\text{minimize } \int_a^b \sqrt{1+u'(x)^2}dx\\ \text{among } u \in C^1([a,b],\mathbb{R}) \text{ such that } u(a)=\overline{a} \text{ and } u(b)=\overline{b}\end{cases}$$

In the present set up a minimizer exists and is *unique*.

thm.2 **1.2.2 Theorem.** *A function $u \in C^1([a,b],\mathbb{R})$ such that $u(a) = \overline{a}$ and $u(b) = \overline{b}$ minimizes L if and only if it is the (unique) affine function in that class.*

1.2.3 A third possible model

A third way to model this problem is to consider a path joining A and B to be a (say compact) connected set in the plane containing the pair $\{A, B\}$. The concept of connectedness for a set plays the role of that of continuity for a function in the two previous models: it ensures that indeed the set *joins* A and B in some sense. This more general viewpoint encompasses the two preceding ones. There is some added difficulty, though, which consists in defining the length of a subset of the plane. This turns out to be one of the motivational challenges of the foundation of Geometric Measure Theory. We refer the reader to ... for the particular case evoked here.

1.3 Vocabulary

The two minimization problems introduced so far are particular cases of the following general setting. We let $\mathscr{C}$ denote a set and $\mathscr{F} : \mathscr{C} \to \mathbb{R}$ a function. We then consider the general minimization problem

$$(\mathscr{P}) \begin{cases} \text{minimize } \mathscr{F}(u) \\ \text{among } u \in \mathscr{C} \end{cases}$$

The elements of $\mathscr{C}$ will usually be called the *competitors*. The *infimum* of the problem is defined as follows:

$$\inf(\mathscr{P}) = \inf\{\mathscr{F}(u) : u \in \mathscr{C}\}\,.$$

A *minimizer* is by definition a competitor $u^* \in \mathscr{C}$ that achieves the infimum of the problem:

$$\mathscr{F}(u^*) = \inf(\mathscr{P})$$

For example problem $(\mathscr{P}_1)$ above corresponds to the case when

$$\mathscr{C} = C^1([0,1], \mathbb{R}^2) \cap \{u : u(0) = A \text{ and } u(1) = B\}$$

and

$$\mathscr{F}(u) = \int_0^1 \|u'(t)\| dt\,,$$

whereas problem $(\mathscr{P}_2)$ corresponds to

$$\mathscr{C} = C^1([a,b], \mathbb{R}) \cap \{u : u(a) = \overline{a} \text{ and } u(b) = \overline{b}\}$$

and

$$\mathscr{F}(u) = \int_a^b \sqrt{1 + u'(x)^2} dx\,.$$

It turns out that often $\mathscr{C}$ will consist in a class of functions, that is the variables of $\mathscr{F}$ are functions. To avoid confusion between these functions u and $\mathscr{F}$ itself it is customary to call $\mathscr{F}$ a *functional*.

1.3.1 EXERCISE – Let $\mathscr{C} = \mathbb{R}$ and $\mathscr{F}(u) = \exp(u)$. Does $\mathscr{F}$ have a minimizer in $\mathscr{C}$?

ex.1 1.3.2 EXERCISE – This exercise is to the extent that problem $(\mathscr{P}_1)$ is "more general" that problem $(\mathscr{P}_2)$.

(1) Given a competitor u for problem $(\mathscr{P}_2)$, associate with u a competitor $\overline{u}$ for problem $(\mathscr{P}_1)$ such that $\operatorname{graph}(u) = \operatorname{im}(\overline{u})$ and $\overline{u}$ is one-to-one.

(2) Show that $\operatorname{length}(\overline{u}) = L(u)$.

(3) Show that $\inf(\mathscr{P}_1) \leq \inf(\mathscr{P}_2)$.

1.3.3 EXERCISE – Using Exercise 1.3.2 and assuming that Theorem 1.2.1 holds true, prove Theorem 1.2.2.

1.3.4 EXERCISE – Without assuming that Theorem 1.2.1 holds, prove Theorem 1.2.2 in the special case when $A = (0, 0)$ and $B = (1, 0)$.

1.4 Paths of shortest weighted length

1.4.1 Fermat's principle and Snell's law

sec.snell.1

Here again we consider paths between two points A and B and we seek to determine along which path light is going to travel. The material between A and B may not be homogeneous, though, so that it may take more time to travel along the straight line segment. Fermat's principle states that "nature always acts in the shortest way". In the context of geometrical optics this means that the light (locally) minimizes its "optical length" between its endpoints. The optical length depends upon the material in which the light travels: the speed of light is inversely proportional to the *optical density of the medium.* The nature of the material visited may change along the way (i.e. the optical density is a function of the space position). We interpret this mathematically by saying that light minimizes a *weighted length*, the weight w being a real-valued function of the ambient space, say $w : \mathbb{R}^2 \to \mathbb{R}$

In the case of paths being considered as C^1 images in $\mathbb{R}^2$ the corresponding weighted length is defined as

$$\operatorname{length}_w(u) = \int_0^1 \|u'(t)\| w(u(t)) dt \,.$$

In case of a C^1 graph the weighted length is defined by

$$L_w(u) = \int \sqrt{1 + u'(t)^2}\, w(t, u(t)) dt \,.$$

In both cases some assumptions need be made (about w) in order that the integrand is indeed an integrable function. It is worth noticing that w might be discontinuous in some cases, for instance when the light is going to be refracted by a lens. We now consider this example in some detail. Suppose that A and B lie in two different media separated by the line $y = 0$, and that $\overline{a} > 0$ and

$\bar{b} < 0$. We denote the optical densities of these media by $n_1 > 0$ and $n_2 > 0$. Therefore

$$w(x, y) = \begin{cases} n_1 & \text{if } y \geq 0 \\ n_2 & \text{if } y < 0\,. \end{cases} \tag{1.1}$$

eq.w.snell

Intuitively a ray of light will follow a straight path in the medium $y \geq 0$ (because its optical density is constant) and a straight path in the medium $y < 0$ (for the same reason). However these two paths need not have the same direction. We will subsequently derive the angle between these two line segments depending upon n_1 and n_2 – this relation is known as *Snell's law* in geometrical optics. In a classic analogy by Richard Feynman, the area of lower optical density is replaced by a beach, the area of higher optical density by the sea, and the fastest way for a rescuer on the beach to get to a drowning person in the sea is to run along a path that follows Snell's law.

We now derive Snell's law under the additional assumption that

(H) *Light travels straight paths in each of the media $y \geq 0$ and $y < 0$.*

Let denote by $P = (x, 0)$ the point on the axis $y = 0$ where light possibly chooses a different direction. We also let $u_x : [a, b] \to \mathbb{R}$ denote the piecewise affine function joining A to P and then P to B. Then

$$\begin{aligned} L_w(u_x) &= n_1 \|A - P\| + n_2 \|P - B\| \\ &= n_1 \sqrt{(x - a)^2 + \bar{a}^2} + n_2 \sqrt{(x - b)^2 + \bar{b}^2}\,. \end{aligned}$$

We notice that the above formula defines a function $f(x)$ of one real variable $x \in \mathbb{R}$.

prop.1 **1.4.1 Proposition.** *Let $f(x) = L_w(u_x)$ be defined as above. Then f has exactly one minimum $x^* \in \mathbb{R}$ characterized by the equation*

$$n_1 \frac{x^* - a}{\sqrt{(x^* - a)^2 + \bar{a}^2}} = n_2 \frac{b - x^*}{\sqrt{(x^* - b)^2 + \bar{b}^2}}\,.$$

Proof. Notice that $f(x) \to \infty$ as $x \to \infty$ and also as $x \to -\infty$. Therefore, as f is continuous it has at least one minimum. Since f is also derivable everywhere, each such minimum x must be so that $f'(x) = 0$. Furthermore we observe that

$$f'(x) = n_1 \frac{x - a}{\sqrt{(x - a)^2 + \bar{a}^2}} + n_2 \frac{x - b}{\sqrt{(x - b)^2 + \bar{b}^2}}\,.$$

It remains to show that the equation $f'(x) = 0$ has at most one solution (Exercise 1.4.1) $\square$

We see that under assumption (H) our minimization problem admits exactly one minimizer consisting of two pieces of straight line segments. This configuration (corresponding to $P^* = (x^*, 0)$) is characterized by two angles θ_1 and θ_2.

The angle θ_1 is the angle at vertex P^* of the triangle AP^*O (where $O = (a, 0)$) and θ_2 is the angle at vertex P^* of the triangle $O'P^*B$ (where $O' = (b, 0)$). With this notation Proposition 1.4.1 reads

$$\frac{n_1}{n_2} = \frac{\cos\theta_2}{\cos\theta_1}\,.$$

A few comments are in order.

(1) We found out heuristically that the minimization problem of this subsection has a solution, if any, which is *piecewise affine* but not C^1 smooth (when $n_1 \neq n_2$). This actually means that the problem has *no* solution in $C^1([a, b], \mathbb{R})$. It shows that we need to take some care in defining the class $\mathscr{C}$ of all competitors to be large enough to ensure that a minimizer actually exists.

(2) The minimizing problem under consideration here was reduced (under assumption (H)) to a *one dimensional minimization problem.* We will justify this reduction later on in section **??** In order to solve the one dimensional problem we solved the equation $f'(x) = 0$ and used some specific arguments to show that the unique solution x^* corresponds to a minimum of f.

ex.2

1.4.1 Exercise – Let $n > 0$ and $p, q \in \mathbb{R}$. We consider the function $f(x) = n\sqrt{(x-p)^2 + q^2}$, $x \in \mathbb{R}$. Compute f' and f'' and show that $f'' > 0$ whenever $q \neq 0$. Use this to finish the proof of Proposition 1.4.1.

1.4.2 Exercise – Find a necessary and sufficient condition on n_1 and n_2 so that the unique minimizer u_{x^*} described in Proposition 1.4.1 is of class C^1. Prove your result.

1.4.2 Continuous piecewise smooth functions

We discovered in the preceding subsection that the natural problem of describing the path of light in nonhomogeneous media leads us to enlarge the class $\mathscr{C} = C^1([a, b], \mathbb{R})$ to a class of functions including at least "piecewise affine" functions. We now proceed to define precisely a class that will be convenient: that of continuous piecewise smooth functions.

def.pwc

1.4.2 Definition. Let $I \subset \mathbb{R}$ be a closed bounded interval. A function $u : I \to \mathbb{R}$ is said to be *piecewise continuous* whenever the following condition holds. There exists a finite (possibly empty) set $S \subset I$ such that u is continuous on $I \setminus S$ and for every component J of $I \setminus S$ and each endpoint ξ of J the following limit exists

$$\lim_{x \to \xi,\, x \in J} u(x)\,.$$

The collection of piecewise continuous functions on I is denoted $C_{PW}(I)$.

Notice that continuous functions are piecewise continuous (as is checked on letting $S = \emptyset$ in the definition above).

Condition (2) says that I can be decomposed into finitely many intervals on the interior of each of which u is continuous and so that the one sided limits of u exist at the endpoints of these intervals. We can rephrase this by saying that there are $a_0, \ldots, a_\kappa$ such that $a = a_0 < a_1 < \ldots < a_{\kappa-1} < a_\kappa = b$ (here $I = [a, b]$) and u is continuous on each open interval (a_{k-1}, a_k), $k = 1, \ldots, \kappa$ and the following limits exist

$$\lim_{x \to a_k^-} u(x) \text{ and } \lim_{x \to a_k^+} u(x)\,,$$

$k = 1, \ldots, \kappa - 1$, as well as

$$\lim_{x \to a_0^+} u(x) \text{ and } \lim_{x \to a_\kappa^-} u(x)\,.$$

Here $S \triangle \{a_1, \ldots, a_{\kappa-1}\} \subset \{a, b\}$. We insist that S depends upon the function u. Finally we observe that the values achieved by u on the corresponding set S do not really matter. Indeed if we redefine u on S arbitrarily the resulting function will still be piecewise continuous.

For example the following function is piecewise continuous on $[-1, 1]$.

$$u(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

whereas

$$u(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 3 & \text{if } x = 0\,. \end{cases}$$

is not.

We now introduce the class of indefinite (Riemann) integral of piecewise continuous functions.

`def.pws` **1.4.3 Definition.** Let $I = [a, b] \subset \mathbb{R}$ be a closed bounded interval. A function $u : I \to \mathbb{R}$ is said to be *continuous piecewise smooth* on I if there exists $\phi \in C_{PW}(I)$ such that

$$u(x) = u(a) + \int_a^x \phi(t)dt\,,$$

for every $x \in I$. The collection of continuous piecewise smooth functions is denoted $C^1_{PW}(I)$.

1.4.3 Exercise – Show that piecewise continuous functions are bounded.

1.4.4 Exercise – Show that $C_{PW}(I)$ is a linear space.

`ex.6` 1.4.5 Exercise – Let $[c, d] \subset [a, b]$ and $u \in C_{PW}([a, b])$. Show that $u|_{[c,d]} \in C_{PW}([c, d])$.

`ex.4` 1.4.6 Exercise – Prove that each $u \in C_{PW}(I)$ is Riemann integrable on I.

1.4.7 Exercise – Show that $C^1_{PW}(I)$ is a linear space.

1.4.8 Exercise – Prove that if u is continuous piecewise smooth on I then it is indeed continuous on I (This amounts to proving that an indefinite Riemann integral is continuous).

1.4.9 EXERCISE – We say that $u : [0,1] \to \mathbb{R}^2$ is *continuous piecewise affine* whenever u is continuous and there are $0 = t_0 < t_1 < \ldots < t_\kappa = 1$ such that the restriction of u to each interval $[t_{k-1}, t_k]$, $k = 1, \ldots, \kappa$, is affine. Show that if $u \in C^1$ then length(u) equals the supremum of the lengths of continuous piecewise affine functions $\bar{u}$ such that $\bar{u}(t_k) = u(t_k)$ for each $k = 0, \ldots, \kappa$.

1.4.3 Lagrangians

So far we have introduced two problems: that of minimizing length and that of minimizing weighted length. There are several ways of setting up these problems, mainly in the realm of curves on the one hand, and that of graphs on the other hand. For the beginning of these lecture notes we shall stick with the "graph formulation" which corresponds to classical calculus of variations. In that setting, the weighted length of a "graph" $u : [a,b] \to \mathbb{R}$ can be written as

$$\int_a^b \sqrt{1 + u'(x)^2}\, w(x, u(x))dx\,.$$

The case actually corresponds to the particular case when $w = 1$. We will be interested in the sequel in functionals $\mathscr{F}$ of the type of that above, only more general in that the integrand is allowed to be an arbitrary function of the triple $(x, u(x), u'(x))$.

1.4.4 Definition. Let $\mathscr{C} = C^1_{PW}([a,b], \mathbb{R})$ and $\mathscr{F} : \mathscr{C} \to \mathbb{R}$. We call $\mathscr{F}$ a *variational integral* if there exists a function

$$F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\,,$$

called the *Lagrangian* associated with $\mathscr{F}$, such that

$$\mathscr{F}(u) = \int_a^b F(x, u(x), u'(x))dx$$

for every $u \in \mathscr{C}$ (part of the definition of a Lagragian is that for every $u \in \mathscr{C}$ the function $x \mapsto F(x, u(x), u'(x))$ is Riemann integrable over $[a,b]$).

For instance $L(u)$ is a variational integral with corresponding Lagrangian

$$F(x,y,p) = \sqrt{1+p^2}$$

and the weighted length $L_w(u)$ is a variational integral corresponding to the Lagrangian

$$F(x,y,p) = w(x,y)\sqrt{1+p^2}\,.$$

It is of course required that the Lagrangian F verifies suitable assumption so that

$$[a,b] \to \mathbb{R} : x \mapsto F(x, u(x), u'(x))$$

be Riemann integrable. This is going to be the case for instance when F is continuous (see Exercise 1.4.10). Recall however that in the case of Snell's law alluded to above we considered a non continuous weight function w corresponding to a non continuous Lagrangian F.

We now turn to proving a localization lemma which will be useful when we return to Snell's law more rigorously in the next subsection.

prop.loc **1.4.5 Proposition** (Localization Principle)**.** *Assume that*

(1) $[a,b] \supset [c,d]$ *and* $[a,b] \neq [c,d]$*;*

(2) $\overline{a}, \overline{b} \in \mathbb{R}$*;*

(3) $\mathscr{C}_{a,b} = C^1_{PW}([a,b]) \cap \{u : u(a) = \overline{a} \text{ and } u(b) = \overline{b}\}$*;*

(4) $u_* \in \mathscr{C}_{a,b}$*;*

(5) $\mathscr{C}_{c,d} = C^1_{PW}([c,d]) \cap \{u : u(c) = u_*(c) \text{ and } u(d) = u_*(d)\}$*;*

(6) F *is a Lagrangian on* $[a,b]$ *and we define variational integrals respectively on* $\mathscr{C}_{a,b}$ *and* $\mathscr{C}_{c,d}$ *by the formulas*

$$\mathscr{F}_{a,b}(u) = \int_a^b F(x,u(x),u'(x))dx$$

whenever $u \in \mathscr{C}_{a,b}$ *and*

$$\mathscr{F}_{c,d}(u) = \int_c^d F(x,u(x),u'(x))dx$$

whenever $u \in \mathscr{C}_{c,d}$*;*

(7) u_* *is a minimizer of problem*

$$(\mathscr{P}_{a,b}) \begin{cases} \text{minimize } \mathscr{F}_{a,b}(u) \\ \text{among } u \in \mathscr{C}_{a,b} \, . \end{cases}$$

Then $u_{**} = u_*|_{[c,d]}$ *is a minimizer of problem*

$$(\mathscr{P}_{c,d}) \begin{cases} \text{minimize } \mathscr{F}_{c,d}(u) \\ \text{among } u \in \mathscr{C}_{c,d} \, . \end{cases}$$

Proof. Before we give a fully detailed account of the proof let us briefly state the (easy) idea. If u_{**} were not a minimizer of problem $(\mathscr{P}_{c,d})$ then there would be some $v_{**} \in \mathscr{C}_{c,d}$ such that $\mathscr{F}_{c,d}(v_{**}) < \mathscr{F}_{c,d}(u_{**})$. Defining v_* on $[a,b]$ to coincide with v_{**} on $[c,d]$ and with u_* on $[a,b] \setminus [c,d]$ we would obtain a competitor for $(\mathscr{P}_{a,b})$ such that $\mathscr{F}_{a,b}(v_*) < \mathscr{F}_{a,b}(u_*)$ (according to the additivity of the integral), a contradiction. The proof of course consists in showing that the function v_* constructed piecewise still belongs to the class of competitors (see Exercise 1.4.11).

We first need to show that $u_{**} \in \mathscr{C}_{c,d}$. Since u_{**} clearly verifies the boundary condition stated in the definition of $\mathscr{C}_{c,d}$ we are left with noticing that $u_{**} \in C^1_{PW}([c,d])$. **P** Since $u_* \in C^1_{PW}([a,b])$ there exists $\phi_* \in C_{PW}([a,b])$ such that

$$u_*(x) = u_*(a) + \int_a^x \phi_*(t)dt \tag{1.2}$$

eq.1

whenever $x \in [a,b]$. Letting $\phi_{**} = \phi_*|_{[c,d]}$ we infer from Exercise 1.4.5 that $\phi_{**} \in C_{PW}([c,d])$. Furthermore for every $x \in [c,d]$ one has

$$\begin{aligned} u_{**}(x) &= u_*(x) \\ &= u_*(a) + \int_a^x \phi_*(t)dt \\ &= u_*(a) + \int_a^c \phi_*(t)dt + \int_c^x \phi_{**}(t)dt \\ &= u_{**}(c) + \int_c^x \phi_{**}(t)dt \end{aligned}$$

where the last equality follows from (1.2) applied with $x = c$. **Q**

Next we need to show that $\mathscr{F}_{c,d}(u_{**}) \leq \mathscr{F}_{c,d}(v)$ for every $v \in \mathscr{C}_{c,d}$. Assume if possible that there exists $v_{**} \in \mathscr{C}_{c,d}$ such that

$$\mathscr{F}_{c,d}(v_{**}) < \mathscr{F}_{c,d}(u_{**})\,. \tag{1.3}$$

eq.2

We then define a function $v_* : [a,b] \to \mathbb{R}$ as follows

$$v_*(x) = \begin{cases} u_*(x) & \text{if } x \in [a,b] \setminus (c,d) \\ v_{**}(x) & \text{if } x \in (c,d)\,. \end{cases}$$

We claim that $v_* \in \mathscr{F}_{a,b}$, i.e. v_* is a competitor for $\mathscr{P}_{a,b}$. In order to prove this we need to show that

$$v_*(x) = v_*(a) + \int_a^x \psi_*(t)dt \tag{1.4}$$

eq.3

for some $\psi_* \in C_{PW}([a,b])$. We let $\psi_{**} \in C_{PW}([c,d])$ be such that

$$v_{**}(x) = v_{**}(c) + \int_c^x \psi_{**}(t)dt$$

whenever $x \in [c,d]$, and $\phi_* \in C_{PW}([a,b])$ be such that

$$u_*(x) = u_*(a) + \int_a^x \phi_*(t)dt$$

whenever $x \in [a,b]$. We define

$$\psi_*(x) = \begin{cases} \phi_*(x) & \text{if } x \in [a,b] \setminus (c,d) \\ \psi_{**}(x) & \text{if } x \in (c,d)\,, \end{cases}$$

and we observe that $\psi_* \in C_{PW}([a,b])$. **P** If $S_* \subset [a,b]$ is associated with ϕ_* in the definition of piecewise continuity and $S_{**} \subset [c,d]$ is associated with ψ_{**}, then $S = S_* \cup S_{**}$ witnesses the piecewise continuity of ψ_*. **Q** Clearly (1.4) holds for $a \le x \le c$. Next if $c < x < d$ then, by definition of v_*,

$$v_*(x) = v_{**}(x)$$

which, according to the choice of ψ_{**}, equals

$$= v_{**}(c) + \int_c^x \psi_{**}(t)dt$$

and since $v_{**}(c) = v_*(c)$ **P** $v_{**}(c) = u_*(c)$ because $v_{**} \in \mathscr{C}_{c,d}$ and $v_*(c) = u_*(c)$ by definition of v_* **Q** one has

$$\begin{aligned}
&= v_*(c) + \int_c^x \psi_*(t)dt \\
&= v_*(a) + \int_a^c \psi_*(t)dt + \int_c^x \psi_*(t)dt \\
&= v_*(a) + \int_a^x \psi_*(t)dt\,.
\end{aligned}$$

The proof of (1.4) when $d \le x \le b$ is analogous.

Finally,

$$\begin{aligned}
\mathscr{F}_{a,b}(v_*) &= \int_a^b F(x, v_*(x), v_*'(x))dx \\
&= \int_a^c F(x, v_*(x), v_*'(x))dx \\
&\quad + \int_c^d F(x, v_*(x), v_*'(x))dx \\
&\quad + \int_d^b F(x, v_*(x), v_*'(x))dx \\
&= \int_a^c F(x, u_*(x), u_*'(x))dx \\
&\quad + \int_c^d F(x, v_{**}(x), v_{**}'(x))dx \\
&\quad + \int_d^b F(x, u_*(x), u_*'(x))dx \\
&= \int_a^c F(x, u_*(x), u_*'(x))dx \\
&\quad + \mathscr{F}_{c,d}(v_{**}) \\
&\quad + \int_d^b F(x, u_*(x), u_*'(x))dx \\
&< \int_a^c F(x, u_*(x), u_*'(x))dx \\
&\quad + \mathscr{F}_{c,d}(u_{**}) \\
&\quad + \int_d^b F(x, u_*(x), u_*'(x))dx \\
&= \int_a^c F(x, u_*(x), u_*'(x))dx \\
&\quad + \int_c^d F(x, u_*(x), u_*'(x))dx \\
&\quad + \int_d^b F(x, u_*(x), u_*'(x))dx \\
&= \int_a^b F(x, u_*(x), u_*'(x))dx \\
&= \mathscr{F}_{a,b}(u_*)\,,
\end{aligned}$$

contradicting the minimality of u_*. □

`ex.3` 1.4.10 Exercise – Prove that if $u \in C^1_{PW}([a,b],\mathbb{R})$ and the Lagrangian F is continuous then $x \mapsto F(x, u(x), u'(x))$ is piecewise continuous, and therefore Riemann integrable according to Exercise 1.4.6.

ex.5 1.4.11 EXERCISE – Explain why the proof of Proposition 1.4.5 fails if in the definitions of $\mathscr{C}_{a,b}$ and $\mathscr{C}_{c,d}$ the space C^1_{PW} is replaced with C^1.

1.4.4 Return to Snell's law

Here we will derive Snell's law rigorously in the setting of continuous piecewise smooth functions. The main point will be to justify hypothesis (H) introduced in subsection 1.4.1. Our argument will based on the following strengthening of Theorem 1.2.2 (we assume for now that Theorem 1.4.6 holds true – it will be proved in ...).

thm.3 **1.4.6 Theorem.** *Let $A = (a, \overline{a}), B = (b, \overline{b}) \in \mathbb{R}^2$ be such that $a < b$. Then the minimization problem*

$$(\mathscr{P}^{\text{Length}}_{A,B}) \begin{cases} \textit{minimize } \int_a^b \sqrt{1 + u'(x)^2} dx \\ \textit{among } u \in C^1_{PW}([a,b]) \cap \{u : u(a) = \overline{a} \textit{ and } u(b) = \overline{b}\} \end{cases}$$

admits a unique minimizer which is the affine competitor.

We are now ready to prove the main result of this subsection.

thm.snell **1.4.7 Theorem.** *Let $A = (a, \overline{a}), B = (b, \overline{b}) \in \mathbb{R}^2$ be such that $a < b$ and $\overline{b} < 0 < \overline{a}$. Let also $n_2 > n_1 > 0$ and define the weight w by formula* (1.1). *Then the minimization problem*

$$(\mathscr{P}^{\text{Snell}}_{A,B}) \begin{cases} \textit{minimize } \int_a^b \sqrt{1 + u'(x)^2} w(x, u(x)) dx \\ \textit{among } u \in C^1_{PW}([a,b]) \cap \{u : u(a) = \overline{a} \textit{ and } u(b) = \overline{b}\} \end{cases}$$

admits a unique minimizer which is the piecewise affine competitor described in section 1.4.1.

Proof. We recall that $x_* \in [a, b]$ was characterized in Proposition 1.4.1 and we denote by u_* the competitor for our problem which is affine on $[a, x_*]$ and affine on $[x_*, b]$ and such that $u_*(a) = \overline{a}$, $u_*(x_*) = 0$ and $u_*(b) = \overline{b}$. The goal is to prove that u_* is the unique minimizer of $(\mathscr{P}^{\text{Snell}}_{A,B})$. In order to keep the notations short we let $\mathscr{F}$ denote the weighted length L_w.

We start by proving that u_* is a minimizer. This will be achieved in the following way. With any competitor u we will associate another competitor $\tilde{u}$ such that $\mathscr{F}(u) \geq \mathscr{F}(\tilde{u})$ and $\tilde{u}$ is affine in $[a, x_u]$ and on $[x_u, b]$, with $\tilde{u}(x_u) = 0$, for some $x_u \in [a, b]$ (depending on u). Referring to Proposition 1.4.1 we see that $\mathscr{F}(\tilde{u}) \geq \mathscr{F}(u_*)$ and therefore $\mathscr{F}(u) \geq \mathscr{F}(u_*)$ showing that u_* minimizes $\mathscr{F}$. □

1.5 Geodesics on spheres

One way one can modify the problem of the previous section is to restrict the paths joining a and b to lie on a specified surface S itself containing a and b.

For instance one could require that the path is contained in a sphere. More specifically let $S = \mathbb{R}^n \cap \{x : \|x\| = 1\}$ and choose $a, b \in S$. The problem now consists in finding a continuous image u with endpoints a and b, contained in S, and minimizing its length among all the paths meeting the said conditions. As we shall show the solution is an arc of great circle through a and b. Since generically there are two such arcs, the present problem illustrates the fact that one critical point of the length functional may not be a minimum, not even a local minimum. On the other hand, if a and b are antipodal then there are infinitely many critical arcs which are all minimizing their length.

1.6 Catenary

1.7 Brachistochrone

1.8 Elastic string and its higher dimensional analog