OUTLINE

0) Quantum Circuit model
1) Why topology should be involved?
2) Programming Language
3) How powerful (these computers are)?
   - TQCs vs QCM
   - problems / complexity
4) Explicit examples

<u>Quantum Circuit Model:</u>

- State space $\mathcal{H}_n := (\mathbb{C}^2)^{\otimes n}$ ← n qubit state space
- Orthogonal basis for $\mathbb{C}^2$: $\{|0\rangle, |1\rangle\}$

$\Rightarrow$ Basis for $\mathcal{H}_n$: $\{ \underbrace{|i_1 i_2 \cdots i_n\rangle}_{\shortparallel \atop |i_1\rangle \otimes |i_2\rangle \otimes \cdots} : i_j \in \{0,1\}\}$

- Gates $\mathcal{G} = \{G_1, \ldots, G_k\}$ $G_i \in U(\mathcal{H}_{n_i})$ $\}$ some physical operations that can be realized on some physical system

Typically, $n_i << \infty$

$\dim(\mathcal{H}_{n_i}) = 2^{n_i}$

(very often, we want $n_i \leq 2$)

- Examples of gates ① CNOT : $\mathcal{H}_2 \longrightarrow \mathcal{H}_2$

$$\text{CNOT}\,|i_1 i_2\rangle = |i_1 (i_1+i_2)\rangle$$

$$\left[\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & \multicolumn{2}{c}{0} \\ \hline & & 0 & 1 \\ \multicolumn{2}{c|}{0} & 1 & 0 \end{array}\right]$$

② Hadamard gate $H \in U(\mathcal{H}_1)$

$$\left.\begin{array}{l} H\,|0\rangle \longmapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ H\,|1\rangle \longmapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{array}\right\} \text{Bell basis}$$

③ $T \in U(\mathcal{H}_2)$

$$|0\rangle \longrightarrow |0\rangle$$
$$|1\rangle \longrightarrow e^{\pi i/4}|1\rangle$$

(called $\pi/8$ gate)

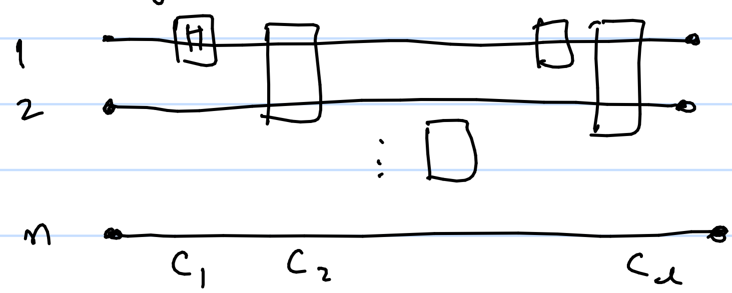
matrix : $\begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i/4} \end{pmatrix}$

- n-qubit, length l, $\mathcal{G}$-circuit $\in \mathcal{G}_l(n)$ is

$$\prod_{i=1}^{l} c_i \quad \text{where } c_i \in \{ I_2^{\otimes q} \otimes G_j \otimes I_2^{\otimes(n-n_j-q)} \} \in U(\mathcal{H}_n)$$

then $\mathcal{G}(n) = \bigcup_{l \geq 1} \mathcal{G}_l(n)$

Think of each qubit as living on some wire



$\in \mathcal{G}_l(n)$

<u>Defn</u>: $\mathcal{G}$ is $\boxed{\text{Universal}}$ if $\forall n$ $\overline{\langle \mathcal{G}(n) \rangle} \subseteq U(\mathcal{H}_n)$ contains $SU(\mathcal{H}_n)$

<u>Example</u>: $\{CNOT, H, T\}$ is universal

<u>KITAEV–SOLOVAY THM</u>: Let $U \in SU(\mathcal{H}_n)$, fix $\varepsilon > 0$, if $\mathcal{G}$ is universal, then $\exists$ $l(\frac{1}{\varepsilon})$ polynomial and a circuit of length $l = X$ s.t. $\|U - X\| < \varepsilon$.

Suppose we have $f : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ classical function

<u>GOAL:</u> compute $f(N)$ N: some binary string

<u>Quantum algorithm takes</u>: $f \overset{\text{magic}}{\rightsquigarrow} \boxed{U_f} \in SU(\mathcal{H}_n)$

$$\text{s.t. } N \rightsquigarrow |N\rangle \in \mathcal{H}_n$$

$$U_f |N\rangle = \underbrace{\sum_{i=1}^{2^n} a_i |X_i\rangle}_{\substack{\text{quantum info.} \\ \text{we have to make} \\ \text{measurement to get} \\ \text{result}}} \quad \& \quad \underbrace{|a_{f(N)}|^2 > 1/2}_{\substack{\text{shows up with} \\ \text{high probability}}}$$

So, after many measurements we get the correct answer.

an example is SHOR's algorithm.

- Few problems with QCM
  → needs powerful quantum computer

KEY PROBLEM : decoherence
(quantum systems are not isolated affected by heat, etc.)

"Reality is the problem"

ER's perspective on decoherence:
LOCAL ERRORS
<u>Ex:</u> bit flip, phase problems

<u>Quantum Error Correction</u> → being used to deal with above

It works!
But very expensive, \$ overhead

So, we look for alternate ways to overcome local errors.

Q: How to overcome local errors?

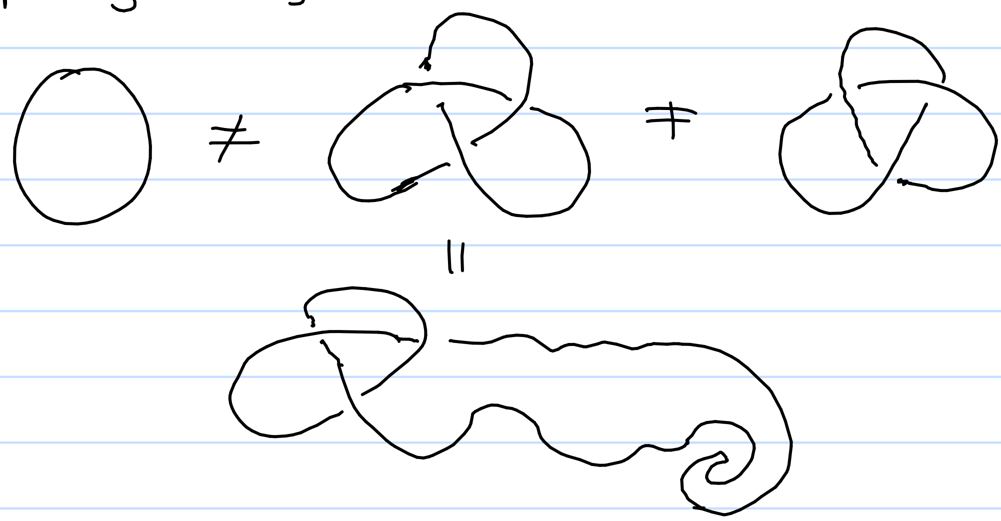
Ans: encode globally
(s.t. local errors don't affect global information)

Enter TOPOLOGY!

- Idea of M. Freedman 1997 (TQFT)
(Topological Quantum Field computer)
- Kitaev : Idea of using Topological Phases of matter)

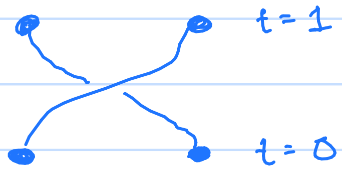
→ Around 2001, they realized that the above 2 ideas are essentially same.
TQFT models TPM

- Errors are local. So, <u>store</u> & <u>manipulate</u> info. globally.



Do Topological Phases of Matter even exist?

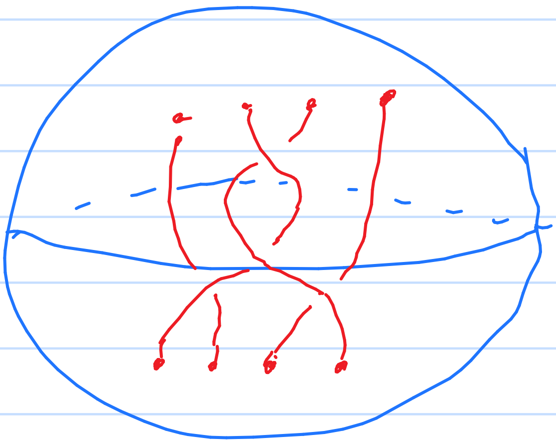
In $\mathbb{R}^3$ particles are bosons / fermions



$$(\pm 1)\ \psi_0 = \psi_1 \quad \left(\text{because of Physics}\right)$$

$-1 \rightarrow$ fermions

$+1 \rightarrow$ bosons



$\longrightarrow \pm 1$

- Makes sense because $\pi_1(S^3 \setminus \{p_1, \dots, p_n\}) = 1$

- But $\pi_1(\mathbb{R}^2 \setminus \{p_1, \dots, p_k\}) \cong F_n$

ANYONS: (1-dim state space)

(abelian)

$$e^{i\theta}\psi(z_1, z_2) = \psi(z_2, z_1)$$

wave function corresponding to identical particles

anything which is rational multiple of π (for Physical reasons)

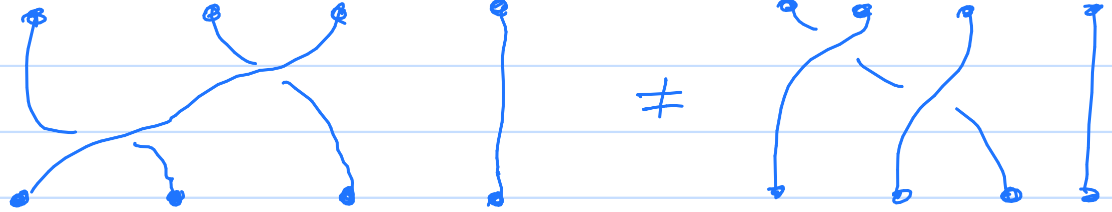
Could exist in "$\mathbb{R}^2_{D^2}$"

Non-abelian anyons:

Take some basis $\psi_1, \dots, \psi_k$ of the state space of configurations of n anyons in D^2.

$\sigma : z_i \leftrightarrow z_{i+1}$

$$\sigma \psi_1(z_1, \dots, z_i, z_{i+1}, \dots z_n) = \sum_{i=1}^{k} \alpha_i \psi_i$$



So, there is possibility of having non-abelian anyons.

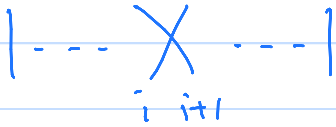
BRAID GROUP:

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} : \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \; [\sigma_i, \sigma_j] = 1 \text{ if } |i-j| > 1 \rangle$$

Mathematically, it is the Mapping Class Group of $\mathbb{D}^2 \setminus \{p_1, \ldots, p_n\}$

OR

It's the motion group $\mathrm{Mot}(\mathbb{D}^2, \{p_1, \ldots, p_n\})$

($\sigma_i \rightarrow$ 

composition by stacking diagrams read from bottom to top)

We are going to model Topological Properties of Top. Phases of Matter (Anyons)

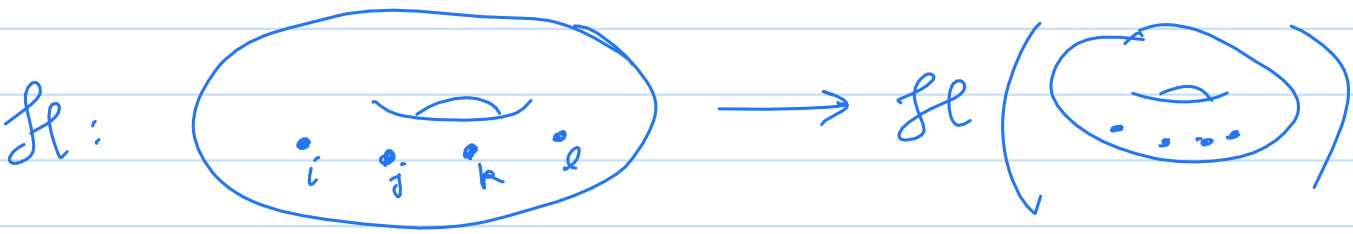
Take Quantum Mechanics + Topology of surfaces with bdry ↔ marked pts

(Anyons can live in surface only)

Mathematically, we start with a set
$\mathcal{L} = \{0, 1, \dots, r-1\}$ finite set
correspond 1-1 to anyon types

$0 \longleftrightarrow$ vacuum (empty anyon type)

<u>QM</u> : (1) Superposition principle (PRINCIPLE 1)
means that state space of QM system is a Hilbert space



marked surface

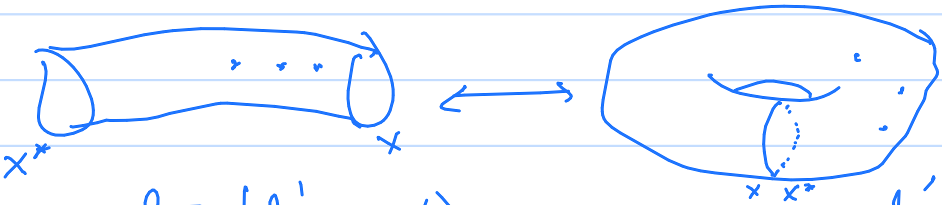
A Hilbert space depending on top. info & marked pts.

(2) <u>Entanglement</u> : (PRINCIPLE 2)

means $\mathcal{H}(M_1, \ell_1) \amalg \mathcal{H}(M_2, \ell_2)$
$= \mathcal{H}(M_1, \ell_1) \otimes \mathcal{H}_2(M_2, \ell_2)$

(3) Locality (path integral)

$$(M, \ell) \underset{\text{cutting}}{\overset{\text{gluing}}{\rightleftarrows}} (M', \ell')$$



$\ell = (\ell', x, x')$

$$\mathcal{H}(M', \ell') = \bigoplus_{x \in \mathcal{L}} \mathcal{H}(M, (\ell', x, x^*))$$

(Hilbert space of system is sum of H.S. of its histories)

4) <u>Schrödinger</u> :

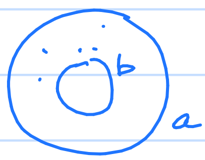
$$U_t |\psi(0)\rangle = |\psi(t)\rangle$$

U_t unitary

(time evolution has to be unitary)

(Only care about effective systems)

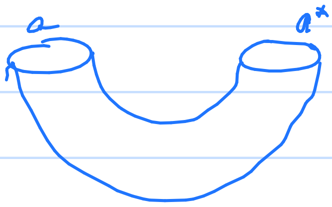
More local axioms :

$\mathcal{H}\Big($  $\Big) = \begin{cases} 0 & a \neq b^* \\ \mathbb{C} & a = b^* \end{cases}$

annulus

$\mathcal{L}$ has an involution $*$, $0^* = 0$

particle / anti-particle duality



$\mathcal{H}\Big($  $\Big) = \begin{cases} 0 & a \neq 0 \\ \mathbb{C} & a = 0 \end{cases}$

$\mathcal{H}\Big($  $\Big) = \mathbb{C}^{N_{ab}^{c}}$

$$\mathcal{H}(-M) = \mathcal{H}(M)^{\perp}$$

$-M$: M with opposite orientation

Upshot: QM + surfaces with ∂ are modelled by (2+1) TQFTs (Synder's talks)

Example: (1) $\mathcal{L} = \{0, 1\}$

$0^* = 0 \Rightarrow 1^* = 1$

$N_{11}^{1} = 1$

$N_{01}^{1} = 1 \quad \dots$

(2) Fibonacci

$\mathcal{L} = \{0, 1, 2\}$ $\quad 1^* = 1, \; 2^* = 1$

$N_{11}^{2} = 1 \qquad N_{22}^{1} = N_{22}^{2} = 0$

$N_{11}^{1} = 0$

we have complete symmetry in a, b, c in this examples