Ingredients of TQC

Physics : Topological phases of Matter (2D anyons) (TPMs)

Defn: A system is in topological phase if at low energies, long distances (length), the effective field theory is a Topological Quantum Field Theory.
(due to Nayak, ...)

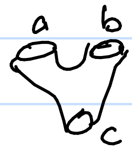
Bosonic (2+1) TQFTs $\longleftrightarrow$ Modular $\otimes$ categories

State spaces? $\mathcal{H}(M, \ell)$

where $\ell \subseteq \mathcal{L} = \{0, 1, \ldots, k-1\}$
we have an involution $*: \mathcal{L} \to \mathcal{L}$ s.t. $0^* = 0$.

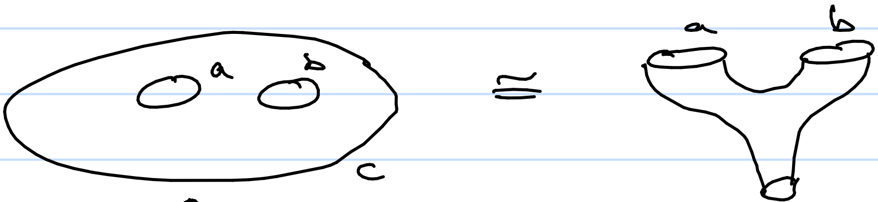
- $\mathcal{H}$ is a 2D topological modular functor.

Recall, we had defined

$N_{ab}^{c} = \dim\left(\mathcal{H}\left(\right.\right.$  $\left.\left.\right)\right)$ these have symmetries

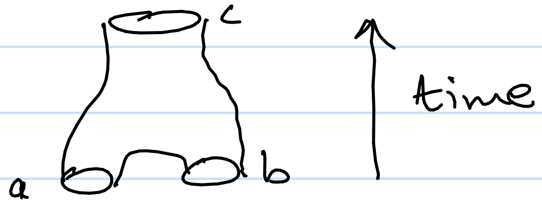
It helps: if $N_{ab}^{c} \in \{0, 1\}$
: also if $a^* = a$
(These don't always happen)

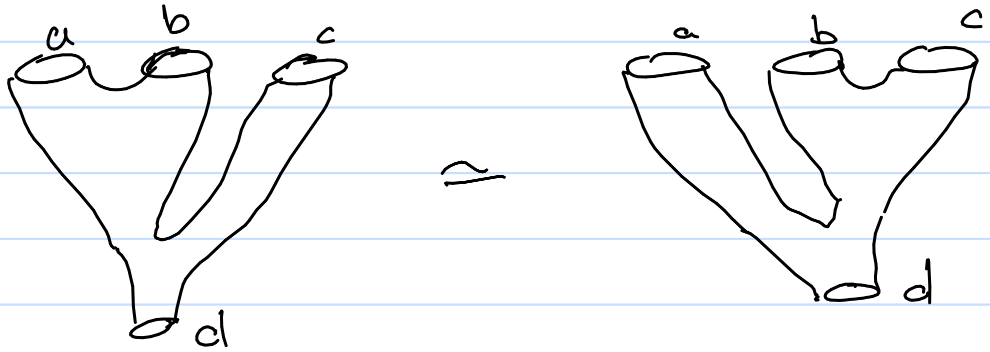
Input data: $\{N_{ab}^{c}\}_{(a,b,c) \in \mathcal{L}^3}$

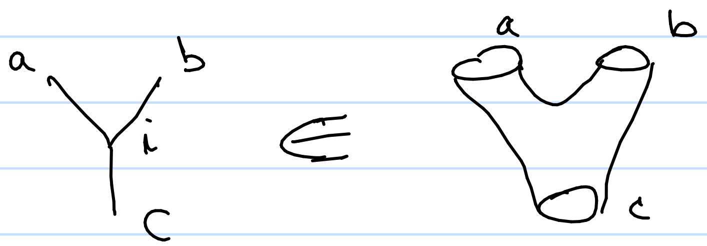


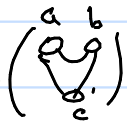
Flat surface at low energy a, b anyons. they can swim

Things of anyon c breaking up to give 2 anyons



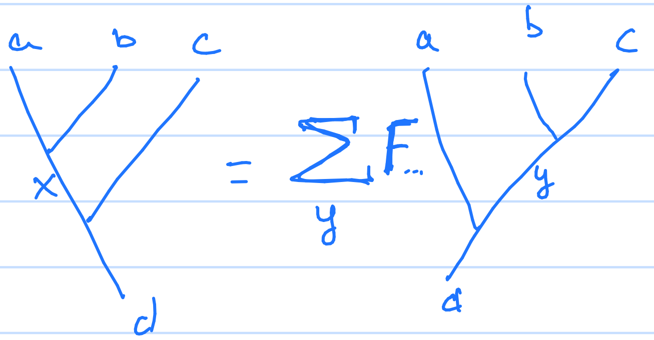




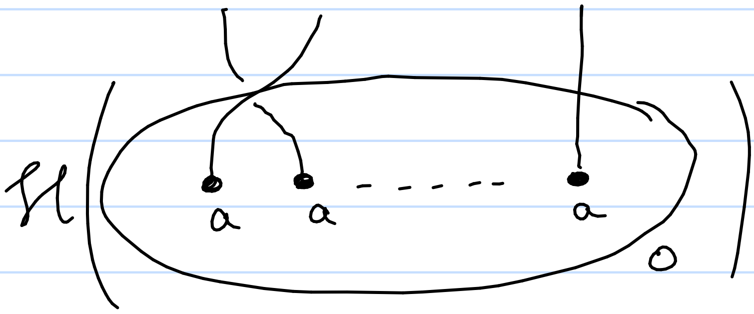
↳ think of it as an element in the Hilbert space $\mathcal{H}($  $)$

So,

$\Rightarrow$



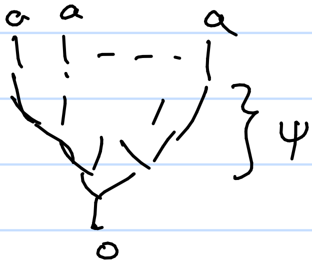
associativity

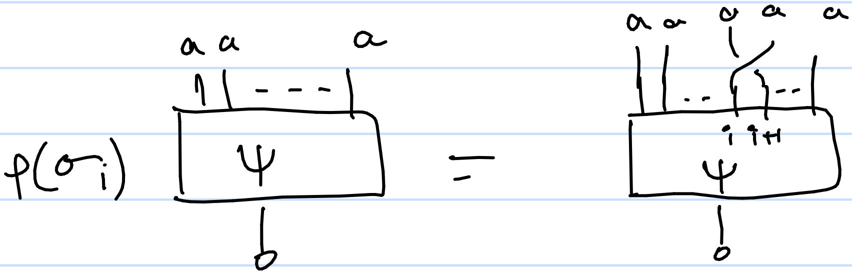


$= \mathcal{H}^{0}_{a,a,\ldots,a}$

B_n acts by particle exchange

a vector in $\mathcal{H}^{0}_{a,\ldots,a}$ =





There are unitary operators on $\mathcal{H}(\ldots)$
$\Rightarrow$ These are gates
(Braiding)

$\{\rho(\sigma_i)\} \longrightarrow$ gate set

<u>Idea</u>: Construct a MTC from explicit algebras

Temperley - Lieb algebras:

$TL_n(A)$: $\quad u_1, u_2, \ldots, u_{n-1}$ generators such that

1) $u_i^2 = d\, u_i$ $\quad$ where $A = -(A^2 + A^{-2})$
2) $u_i u_{i\pm 1} u_i = u_i$
3) $[u_i, u_j] = 0$ $\quad$ if $|i-j| \neq 1$

This is an algebra over $\mathbb{C}(A)$
But working over $\mathbb{Q}[A, A^{-1}]$ suffices.

Group algebra of B_n = $\quad \mathbb{Q}[A, A]^{-1}\, B_n$

$$\varphi : \mathbb{Q}[A, A^{-1}]\, B_n \longrightarrow TL_n(A)$$

$$\varphi(\sigma_i) = \beta_i := A^{-1} u_i + A\, 1$$

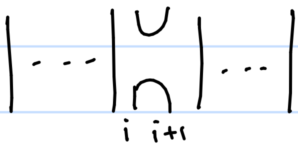
<u>Exercise</u>: Check that this is a surjective homomorphism.

(For homo: show β_i satisfy braid relation
surj: show β_i generate TL_n)

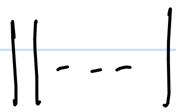
<u>Theorem</u>: TL_n is f.d. s.s. (for generic A)

<u>Diagramatic version</u> : $\mathcal{TL}_n(A)$

Define U_i :



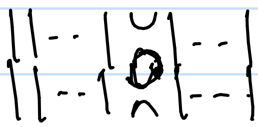
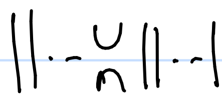
We Make a monoid with generators U_i

Unit : $1_n \rightarrow$ 

product operation : stacking pictures

elements are up to "d-isotopy"

- loops : remove loops & multiply by $d^{\# \text{loops}}$
- $+$: formal sums, action of $\mathbb{Q}[A, A^{-1}]$ as coeff.

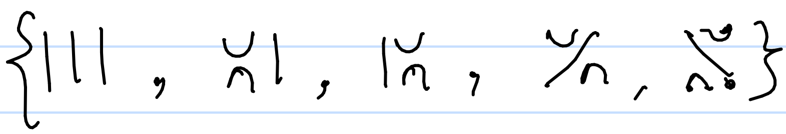
Example : $U_i^2 =$  $= d$  $= d\, U_i$

<u>Thm</u> : $TL_n(A) \simeq \mathcal{TL}_n(A)$

$u_i \longmapsto U_i$

A basis for $TL_n(A)$ is noncrossing perfect matchings on A.

$TL_3(A)$ has basis



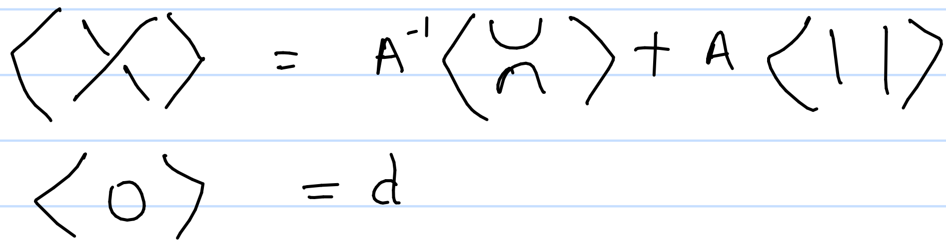
Thm: $\dim(TL_n(A)) = C_n$ catalan number

$TL_3(A) \cong \mathbb{C} \oplus M_2$

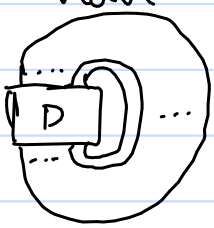
We have Kauffman bracket

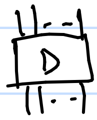
$$\sigma_i \longmapsto A^{-1}u_i + A$$

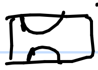
(Skien relations)

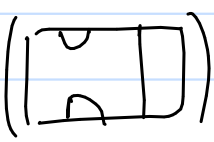
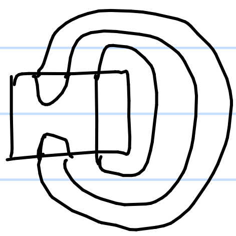


On $\coprod_{n\geq 1} TL_n(A)$, we have a Markov trace

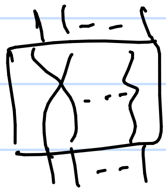
$Tr(D) =$  $= d^{\#\text{loops}}$



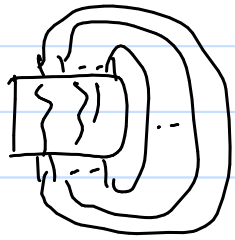
Example: $Tr(U_1)$ ($\in TL_2(A)$) $= Tr($  $) =$  $= d$

$Tr(U_1)$ ($\in TL_3(A)$) $= Tr($  $) =$  $= d^2$

$\beta \in B_n$



$\longrightarrow \quad \hat{\beta} = L$



$L = \hat{\beta}$ a link

We can write $\beta = \sigma_{i_1}^{\alpha_1} \cdots \sigma_{i_k}^{\alpha_k}$

$$e(\beta) := \sum_{j=1}^{k} \alpha_j$$

The Jones polynomial

$$J(L,q) = \left.\frac{(-A^{-3})^{e(\beta)}\ \mathrm{Tr}(\varphi(\beta))}{d}\right|_{A = q^{-1/4}} \in \mathbb{Q}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$$

- Jones poly. is defined for oriented links.
- We can oriented all lines upwards to get oriented links.

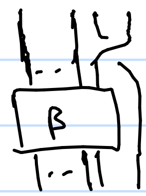
<u>Alexander's Thm</u>: Any oriented link is the closure of some oriented braid.

<u>Markov's Thm</u>: Two braids β and γ have $\hat{\beta} = \hat{\gamma}$ iff $\exists$ a sequence of Markov moves

$$\beta \xrightarrow{M_{i_1}} \cdots \xrightarrow{M_{i_k}} \gamma$$

(M1) $\beta \mapsto$

(M2) $\beta \mapsto$



$\in B_{n+1}$

Example: $J(\underset{\underset{B_2}{\cap}}{\hat{\sigma}_1^3}, q) = q + q^3 - q^4$

Theorem [Vertigan] Exact computation of $J(L,q)\big|_{q=e^{2\pi i/\ell}}$ is $FP^{\#P}$-hard in # of crossings of L for $\ell \neq 1,2,3,4,6$

$FP^{\#P} \to$ function version of NP-hard

Not known: Fully Polynomial Randomized Approximation Scheme.

Q Can we approximate $J(L,q)$ on Q.C. in polynomial time?
(with high accuracy & high confidence)

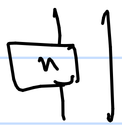
YES (Technically $|J(L,q)|^2$)

JONES - WENZL PROJECTIONS

$\Delta_0(d) = 1$, $\Delta_1(d) = d$, $\Delta_{n+1} = d\Delta_n - \Delta_{n-1}$

$P_n \in TL_n(A)$

$P_1 = | =: \boxed{1}$, $P_2 = || - \frac{1}{d} \, {}^{\cup}_{\cap} =: \boxed{2}$

$P_{n+1} =$  $- \dfrac{\Delta_{n-1}}{\Delta_n}$ 

Exercise: Draw P_3

Lemma:
1) $P_n^2 = P_n$
2) $P_n U_i = U_i P_n = 0, \quad 1 \le i \le n-1$
3) $Tr(P_n) = \Delta_n$

Exercise: Prove this

Theorem: Let $r \geq 3$ integer

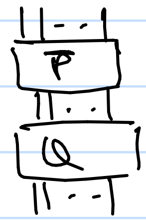
set
$$A = \begin{cases} i e^{-2\pi i/4r} & r \text{ even} \\ i e^{\pm 2\pi i/2r} & r \text{ odd} \end{cases}$$
$\pm \rightsquigarrow r \bmod 4$

$TL_n(A)$ bad things happen
(we can have zero divisors)

Thm: $\dfrac{TL_n(A)}{\langle P_{n-1} \rangle}$ is s.s. and composing with φ, we get a unitary B_m representation $\forall n$

need sesquilinear form for unitary

Use $\langle\ \rangle$ on $TL_n(A)$ defined by

$\langle P, Q \rangle = Tr\Bigg($  $\Bigg)$ $\quad \bar{P}$ is P upside down

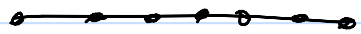
This is a sesquilinear form &

$$\Big\langle \rho(\beta)|\psi\rangle, \rho(\beta)|\pi\rangle \Big\rangle = \langle |\psi\rangle, |\pi\rangle \rangle$$

on $\dfrac{TL_n(A)}{\langle P_{n-1} \rangle}$ is positive definite

We want to come up with a category

<u>Next time</u>: construct a category

<u>objects</u>:  intervals with points on them

<u>Morphisms</u>: linear combinations of TL-diagrams