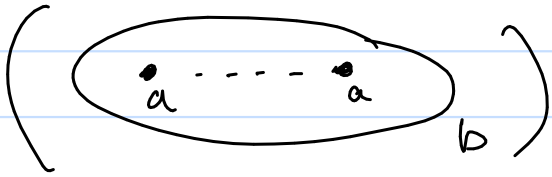
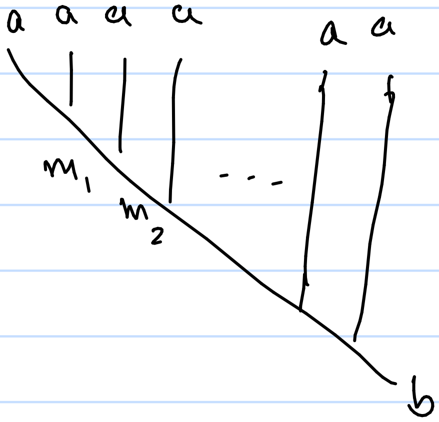
Last time:

$\mathcal{H}$ 

Choose a basis & label it



<u>Example</u>: Fib (Fibonacci theory)

$r = 5$ $\quad k = r - 2 = 3$

We look at $\{0, 2\} \subseteq \{0, 1, 2, 3\} = \mathcal{L}$

$0 \to \mathbb{1}$, $2 \to \tau$

<u>Example</u>: Ising $\quad r = 4, \; k = 2$

$\{0, 1, 2\}$

$\{\mathbb{1}, \sigma, \psi\}$

ψ: Majorana Fermion (non abelian anyon)

<u>Setup</u>: $V_n = \mathrm{Hom}^5(\mathbb{1}, \tau^{\otimes n})$

n Fibonacci anyons in a disc with bdry labelled by 0

These support qubits!

Thm: The (Universal) Quantum Circuit Model can be efficiently simulated on a TQC modelled on Fib.

(We need to be able to, given an arbitrary unitary operator on Hilbert space of size 2^k (say), approximate it)

Recall: gates are braiding

Ingredients: B_n acts on $V_n = \mathrm{Hom}(1\!\!1, \tau^{\otimes n})$ & $V_n' = \mathrm{Hom}(\tau, \tau^{\otimes n})$

one checks that

($V_{n+1} \simeq V_n'$ as B_n reps)

→ it acts irreducibly on both (because $\varphi: B_n \to TL_n$ is surjective)

(look up Bratelli diagrams)

Notice: $H(\tau^{\otimes n}, \tau^{\otimes n})$ is an algebra it has V_n & V_n' as simple modules

→ it acts unitarily

→ [Freedman, Larsen, Wang]

$\rho(B_n) \subset U(V_n)$

$\rho'(B_n) \subset U(V_n')$

(ρ is the above mentioned representation)

but

$\overline{\rho(B_n)} \supseteq SU(V_n)$, $\overline{\rho'(B_n)} \supseteq SU(V_n')$

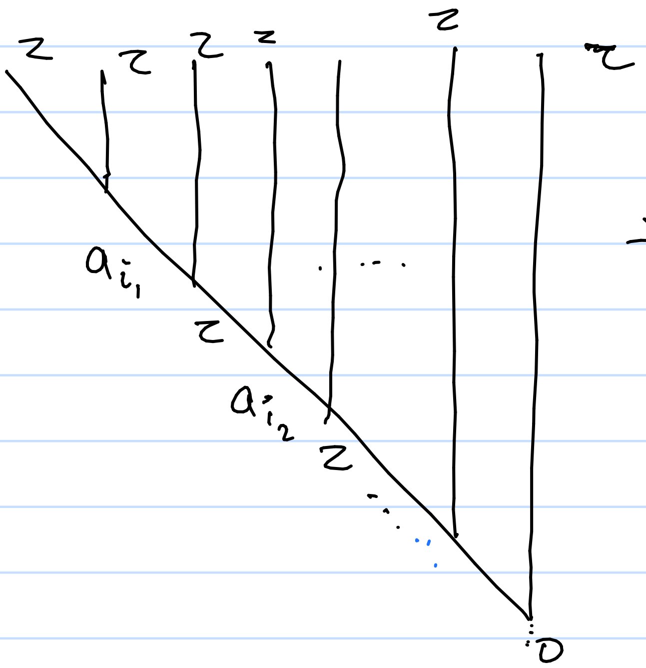
→ $\dim(V_n) = \mathrm{Fib}(n)$

$\dim(V_n') = \mathrm{Fib}(n+1)$

these dimensions grow exponentially

We are now going to realize k-qubits in V_{2k+2}.

$$\mathcal{H}_k = (\mathbb{C}^2)^{\otimes k} \hookrightarrow V_{2k+2}$$



$$=: |i_1 i_2 \cdots i_k\rangle$$

$a_{i_j} \in \{\mathbb{1}, \tau\}$

$a_1 = \tau$

$a_0 = \mathbb{1}$

$i_j \in \{0, 1\}$

$$\begin{tikzcd}
\mathcal{H}_k \arrow[r, hook] \arrow[d, "U"'] & V_{2k+2} \arrow[d, "\rho(\beta) = X"] \\
\mathcal{H}_k \arrow[r, hook] & V_{2k+2}
\end{tikzcd}$$

we want to simulate this operator

$\rightsquigarrow$ want this to approximate U in a certain way

$\exists$ poly time alg. to find this

choose $\beta \in B_n$ s.t. $(n = 2k+2 ?)$

$$\| X - \underbrace{U \oplus I_{\mathcal{H}_k^\perp}}_{\in\, U(V_{2k+2})} \| < \varepsilon$$

this approximation leads to leakage

This proves the thm

<u>OPEN Q</u>: For given model, which ones can be realized in leakage free way?

Fib TQC is universal.

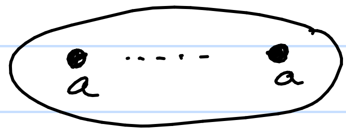
- Another way of proving the thm is that T, H, CNOT are realizable on this model.

- For the Ising model, $\rho(B_n)$ turns out to be finite.

  So $\overline{\rho(B_n)} \not\supset SU(V_n)$

(• What kind of computations can be done on the Ising model?)

Thm:- Any TQC can be simulated on (U)QCM efficiently

Pf: Let $S_n = \mathcal{H}\Big($  $\Big)$

In order to simulate a TQC, we want to

$$\forall \quad S_n \xrightarrow{\rho(\beta)} S_n \qquad \beta \to \text{braid}$$

want to approximate $\rho(\beta)$ by U_β

idea :

$$\begin{tikzcd} S_n \arrow[r, "\rho(\beta)"] \arrow[d, hook] & S_n \arrow[d, hook] \\ Y^{\otimes(n-1)} \arrow[r, "U_\beta"'] & Y^{\otimes n-1} \end{tikzcd}$$

$$Y = \bigoplus_{(a,b,c)\in \mathcal{L}^3} \mathcal{H}(P, a, b, c)$$

$\rightsquigarrow$ pair of pants with labels a, b, c

Note: $\dim(Y) \neq 2$

* But qudit model is polynomially equivalent to qubit model.

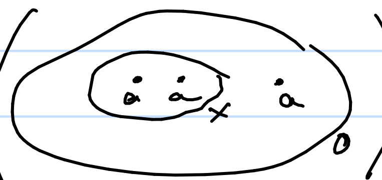
qu(d)it

$d = \dim(Y)$

$$Y^{\otimes n-1} = \left[\bigoplus_{(a,b,c)\in\mathcal{L}^3} \mathcal{H}(P,a,b,c)\right]^{\otimes(n-1)}$$

distribute tensor product over direct sums

Example:

$$\mathcal{H}\left(\right.$$  $$\left.\right) = \bigoplus_x \left[\mathcal{H}((P,a,a,x)) \otimes \mathcal{H}(P,x,a,0)\right]$$

This shows that

$$Y^{\otimes(n-1)} \supseteq S_n$$

i.e. $$Y^{\otimes(n-1)} = S_n \oplus S_n^{\perp}$$

$S_n^{\perp}$: non computational part

Find U_β s.t. $U_\beta|_{S_n} \approx \rho(\beta)$

& $U_\beta|_{S_n^\perp} \approx I_{S_n^\perp}$

Actual statement : Any 2-dim Topological Quantum functor can be simulated on the QCM.

- $\dim(Y_{fib}) = 5$

- What if this theory comes from Hopf algebra? Then we have R-matrix for braiding
In fact for Ising we can do this

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

is unitary solution to YBE

it can be used to simulate the Ising model.

- There's poly alg. to find U_β.

Back to Jones Polynomial:

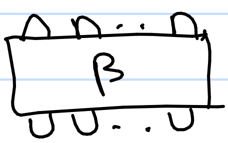
Thm: $\left| \dfrac{J(L, q)}{d^n} \right|$ ← computation of this is Bounded error Quantum Poly B. Q. P.

$L = \hat{\beta}$, $\beta \in B_n$

plat

For $q = A^{-4} = e^{-8\pi i / 2r\alpha}$ $\quad \alpha = \begin{cases} 2 & r \text{ even} \\ 1 & r \text{ odd} \end{cases}$

$r = 5$: $q = e^{-4\pi i/5}$

$\beta \in B_{2n}$ $\quad \hat{\beta}^{pl} =$ 

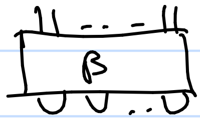
Alexander thm still holds

- Every link is plat closure of some braid.

Algorithm:

(1) initialize: $\cup \cdots \cup =: |cup\rangle$

$$|cup\rangle \in \mathrm{Hom}(\mathbb{1}, 1^{\otimes 2n})$$

(2) $\rho(\beta)\,|\overline{cup}\rangle =$ 

(3) Measure against $\langle cap| := \cap \cdots \cap$

(the above is one run of a QC)

Probability of the outcome $= \cap \cdots \cap$

is $\| \langle \overrightarrow{cap} \,|\, \rho(\beta) \,|\, \overrightarrow{cup} \rangle \|^2$

Output : random variable $\hat{Z}(\beta) \in \{0,1\}$ for each run

(interpretation) output 0 if $\hat{Z}(\beta) \neq 0$ (true)

1 else

$$\mathrm{prob}\,(0) = \| \langle \overrightarrow{cap} \,|\, \rho(\beta) \,|\, \overrightarrow{cup} \rangle \|$$

$$= \left| \frac{J(L,q)}{d^n} \right|^2$$

$L = \hat{\beta}^{\,plat}$

Let $Z(\beta) =$ average of $\hat{Z}(\beta)$ for some poly no. of tries

in δ

then $Z(\beta) \in [0,1]$

Then : $\text{Prob}\left(\left| \frac{J(L,q)}{d^n} \right|^2 - Z(\beta) \right| < \delta \Big) \geq \frac{3}{4}$

$q = e^{\pi i / l}$ $\quad l = 1, 2, 3, 4, 6$

$\mathbb{Z}_2$ (for $l = 3$); Ising (for $l = 4$); metaplectic cat. obj of dim $\{1, 2, \sqrt{3}\}$ (for $l = 6$)

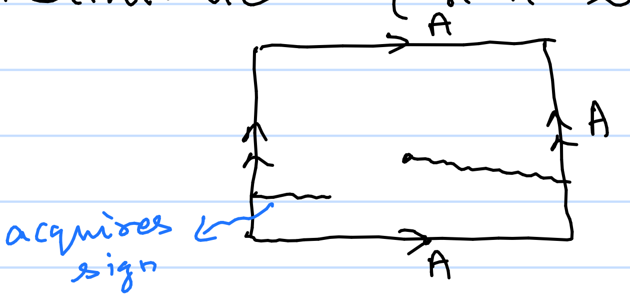
there is classical easy algorithm

$(l=3)$ Ising theory → gives link invariant called Arf invariant

$(l=2)$ → counts rank of homology of some cover

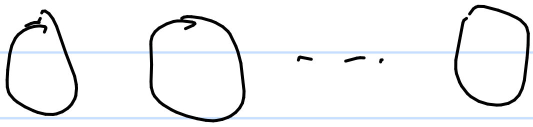
Beyond 2D bosonic theories

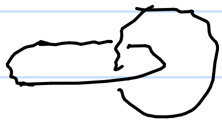
- Fermionic (non-local)



- 3D : pointlike particle : boson/fermion

can consider looplike particles





(3+1) TQFTs

Invariant associated with the double of a finite group $D(G)$

$\rightsquigarrow$ it counts no. of homomorphisms

$\pi_1(\tilde{K})$ to G $/\sim$

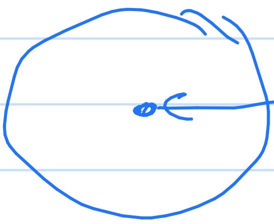
knot complement

(eg $\tilde{K} = B_3$ for K trefoil

It's open if there is a poly. time alg. for counting such homo.

One expects to have such an alg.

Gapped phases of matter



instead of just simple object, we put an algebra in the category

→ Iris Cong