$\mathrm{Bord}_{n,n-1}$ is a symmetric monoidal category.

$\mathrm{Bord}_{n,n-1,n-2}$ is symm. monoidal $\boxed{\begin{array}{l}\text{2-category} \\ \text{bicategory}\end{array}}$.

- Recall the 2category
  $\underline{\underline{\mathrm{Alg}_2}}$: Algebras, bimodules, bimodule maps.

  $\otimes_{\mathbb{k}}$

$\underline{\underline{\mathrm{Rex}}}$: · Finitely cocomplete $\mathbb{k}$-linear categories
(complete under colimits)
(eg. abelian categories)
· right exact sequences
(functors preserve finite colimits)
· natural transformations
· Deli -Kelly tensor product : $\boxtimes$
it is universal for right exact bifunctors

$$\begin{tikzcd} \mathcal{C} \times \mathcal{D} \arrow[r, "\text{r.e.}"] \arrow[d, dotted] & \mathcal{E} \\ \mathcal{C} \boxtimes \mathcal{D} \arrow[ur] & \end{tikzcd}$$

FACT : $(A\text{-mod}) \boxtimes (B\text{-mod}) = (A \otimes B)\text{-mod}$

There is a functor
$$\mathrm{Alg}_2 \longrightarrow \mathrm{Rex}$$
$$A \longmapsto A\text{-mod}$$

The fact tells us that this functor preserves monoidal structure.

2-Vec : finite semisimple $\Bbbk$-linear categories,

$\boxed{\text{Defn1}}$ functors, natural transformations,

$\boxtimes$

If simples in $\mathcal{C}$ are $X_1, \dots, X_n$ & in $\mathcal{D}$ are $Y_1, \dots, Y_n$
then simples in $\mathcal{C} \boxtimes \mathcal{D}$ are
$X_i \boxtimes Y_j$

(additive functors b/w semisimple cats. are automatically right exact)

(Recall there are 2 equivalent descriptions of Vec
(i) vector spaces, linear
(ii) $\mathbb{N}$, $n \times m$ matrices)

$\boxed{\text{Defn2}}$ $\mathbb{N}$, matrices of vec-spaces, matrices of linear maps

(requires $\Bbbk$ - algebraically closed)

$\boxed{\text{Defn3}}$ semisimple algebras, bimodules, bimodule maps

$$\begin{tikzcd} 2\mathrm{Vec} \arrow[r] \arrow[rr, bend right] & \mathrm{Alg}_2 \arrow[r] & \mathrm{Rex} \end{tikzcd}$$

Defn: An extended (once extended) TFT with values in a symmetric monoidal 2-cat $\mathcal{S}$ is a symm. monoidal 2-functor

$$\mathcal{F} : \mathrm{Bord}_{n,n-1,n-2} \longrightarrow \mathcal{S}$$

Given $\mathcal{F} : \mathrm{Bord}_{n,n-1,n-2} \longrightarrow \mathcal{S}$

we can get

$$\mathcal{F}' : \mathrm{Bord}_{n,n-1} \longrightarrow \mathrm{End}(\mathbb{1})$$

$$\mathrm{Bord}_{n,n-1} = \mathrm{End}(1) \text{ in } \mathrm{Bord}_{n,n-1,n-2}$$

we want it to be Vec

i.e. $\mathcal{S}$ is the "delooping of Vec"

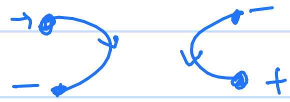
the 2-categories we discussed have this property.

Question: Classify 210 TFTs (following Chris Schommer-Pries' thesis)

We want to write generators and relations for $\mathrm{Bord}^{or}_{2,1,0}$

Recall for $\mathrm{Bord}^{or}_{1,0}$:

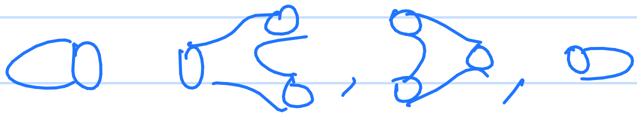
gen obj : $\bullet_+ \quad \bullet_-$

gen morphisms : 

relations : 

$\mathrm{Bord}^{or}_{2,1}$

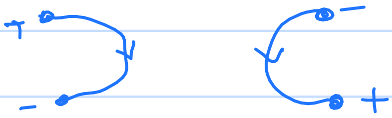
gen object : 

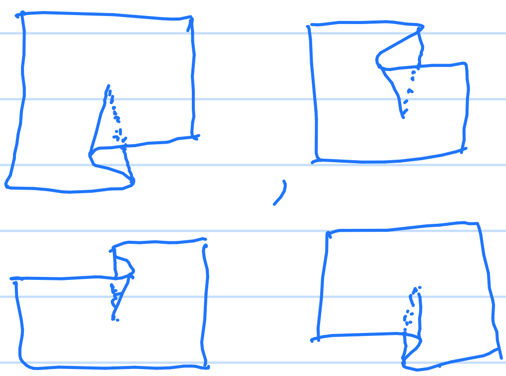
gen morphisms : 

relations : axioms for Frobenius algebra

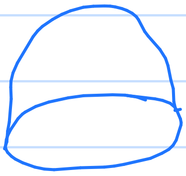
$\mathrm{Bord}^{or}_{2,1,0}$: the thesis of Chris gives a description as above

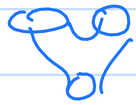
Gen. objects : $\bullet^+ \quad \bullet^-$

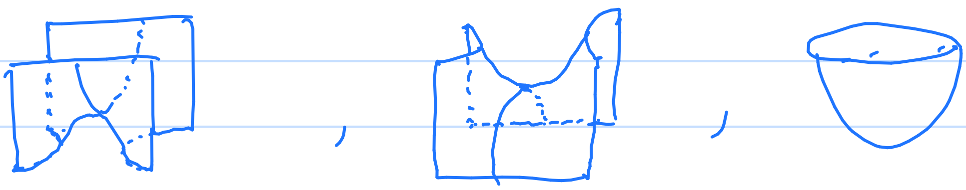
Gen. 1-mor : 

Gen 2-morph. : 

(these come from relations in $\mathrm{Bord}^{or}_{1,0}$)



One would think we need  next but we can break ~~it~~ into simpler mflds. with corners, so we get

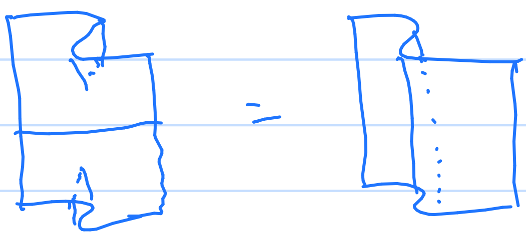


By morse theory arguments, one can conclude that these are all the generating 2-morphisms.

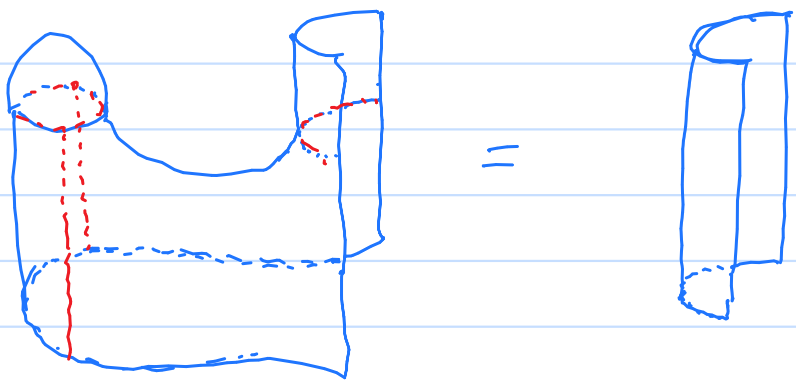
(boundaries of boundaries give corners)

Relations: To many to write (1-page in CSP's thesis)

Cusp generators are inverse to each other

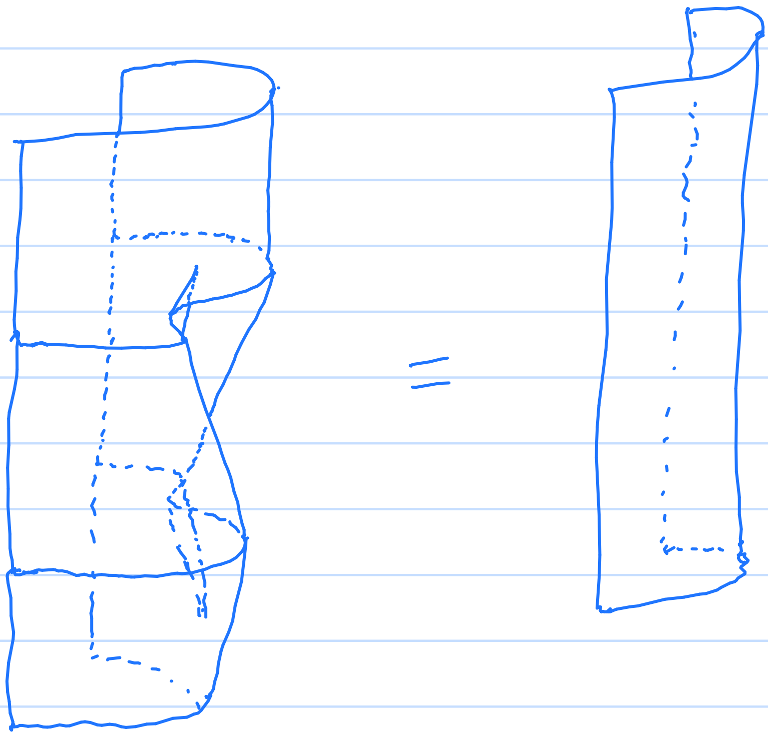


Most Important: Handle cancellations



· Cusps + handles

· Swallowtail

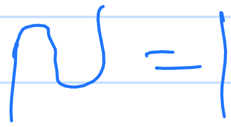


Uses Cerf theory

Morse theory has 1 direction
Cerf theory has 2 directions.

What do handle cancellations tell you?

Handle cancellation is similar to zigzag equations.



Handle cancellation gives duality b/w (and)

$\Rightarrow$ Some 1-morphisms are "~~dual~~ adjoint" to each other

In category terms, we have

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[rr, bend right=50, "id"', ""{name=A}] & Y \arrow[r, "G"] \arrow[d, phantom, "\Uparrow"] & X \arrow[r, "F"] \arrow[u, phantom, "\Uparrow"] & Y \arrow[ll, bend right=50, "id"'] \end{tikzcd}$$

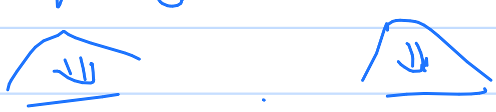
So, F and G are adjoint morphisms in a 2-category

The unit/counit defn. of ~~ad~~ adj. functors is purely 2-categorical
So we can use the same defn. in any 2-category.

a 1-mor F is adjoint L/R to 1-mor G if $\exists$ unit ... counit ..
(2-mor) 2-mor

$id \to F \circ G$ $\quad$ $G \circ F \to Id$

satisfying



Handle cancellations tell us that $\supset$ and $\subset$ are both left & right adjoint.

TRICKY POINT:

$$\supset = ev$$

$$ev^L = ev^R = \text{swap} \circ \text{coev} = \subset$$

This will make our life harder

"Fundamental problem with oriented bordisms"

$$\begin{array}{ccccc}
\mathrm{Bord}_{2,1,0} & \longrightarrow & \mathrm{Alg}_2 & \longrightarrow & \mathrm{Rex} \\
\bullet^+ & \longmapsto & A & \longmapsto & A\text{-mod} = \mathcal{C} \\
\bullet^- & \longmapsto & A^{op} & \longmapsto & \mathrm{Rex}(\mathcal{C}, \mathrm{Vec})
\end{array}$$

A-mod $\qquad$ $\mathbb{k}$-mod

right exact functors b/w them are given by $\mathbb{k}$–A bimodules or right A-modules

For general $\mathcal{C}$, we need to know when

$$\mathcal{C} \boxtimes \mathrm{Rex}(\mathcal{C}, \mathrm{Vec}) \xrightarrow{\sim} \mathrm{Rex}(\mathcal{C}, \mathcal{C})$$

for that we want to know when this maps hits $\mathrm{Id}_{\mathcal{C}} \in \mathrm{Rex}(\mathcal{C}, \mathcal{C})$

See Brandenburg--Chirvasitu--Johnson-Freyd

For $\mathcal{C} = A\text{-mod}$ answer is easier

$\lambda = ev \rightsquigarrow {}_{A\otimes A^{op}}A_{\Bbbk}$, $\varepsilon = coev \rightsquigarrow {}_{\Bbbk}A_{A\otimes A^{op}}$

they satisfy zigzag equations.

Q When does ${}_A M_B$ have a left and right adjoints?

these have to be

${}_B\mathrm{Hom}_A(M, A)_A$ ${}_B\mathrm{Hom}_B(M, B)_A$

we need to check when does

$$\mathrm{Hom}_A(M, A) \otimes_A M \longrightarrow \mathrm{Hom}_A(M, M)$$

hit the identity.

Thm: The above happens iff ${}_A M$ is finitely generated projective.

$\therefore$ $A_{\Bbbk}$ is f.g. projective
$\Rightarrow$ A is finite dimensional

${}_{A\otimes A^{op}}A$ is f.g. projective
$\Longleftrightarrow$ A is separable
($\simeq$ semisimple)

So, A has to be f.g. separable algebra.