Recall: Had defined monoidal, braided monoidal, symmetric categories

- $X \in \mathcal{C}$ is invertible if both $ev: X^* \otimes X \xrightarrow{\sim} 1$
  $coev: 1 \xrightarrow{\sim} X \otimes X^*$
  satisfy snake equations

Example: ① Pointed (all simple objects are invertible)
$\hookrightarrow V_G^{\omega}$

② Pointed braided $\mathcal{C}(A, q)$
$\nwarrow$ quadratic form

③ $Rep(G)$, $Rep(H)$

Defn: Tensor category = $\mathbb{k}$-linear (locally finite) rigid monoidal, abelian, $\mathbb{1}$ is simple
(Hom space are finite dim.)

Defn: Fusion category = tensor category, semisimple finitely many iso. classes of simple objects

Goal for today: come up with many examples of above

Two ideas:

$\mathcal{C}$ – given category (fusion or tensor)

(I) We can do algebra in $\mathcal{C}$

algebra: (A, m, i) where $A \in \mathcal{C}$
$m: A \otimes A \to A$
$i: \mathbb{1} \to A$

$$\begin{tikzcd}
 & (A\otimes A)\otimes A \arrow[r, "m\otimes I"] & A\otimes A \arrow[d, "m"] \\
A\otimes(A\otimes A) \arrow[ur, "a"] \arrow[r, "I\otimes m"'] & A\otimes A \arrow[r, "m"'] \arrow[ur, phantom, "\circlearrowleft"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1}\otimes A \arrow[r, "m"] \arrow[dr, "\lambda"'] & A \arrow[d, equal] & A\otimes\mathbb{1} \arrow[l, "m"'] \arrow[dl, "\rho"] \\
 & A &
\end{tikzcd}$$

Example: ① $\in \mathrm{Vec}_G$ — G graded vector spaces

$$A = \bigoplus_{g \in G} A_g$$

$$m : A_g \otimes A_h \longrightarrow A_{gh}$$

Algebras in Vec_G are G-graded algebras.

Examples of G-graded algebras:

(i) $\Bbbk[G] = \bigoplus_{g \in G} \Bbbk g$ is a G-graded algebra

(ii) $\Bbbk[G]_\psi$ = $\Bbbk[G]$ twisted by a 2-cocycle ψ (it gives a skew graded algebra)

(iii) $H \leq G$

$\Bbbk[H]_\psi \rightsquigarrow$ still G-graded but in a slightly degenerate way

② Vec_G^ω

(i) For $H \leq G$,

$\Bbbk[H]_\psi$ where ψ is s.t. $\partial\psi = \omega|_H$

$\Bbbk[H]_\psi$ is an algebra in Vec_G^ω

<u>Exercise</u>: associativity corresponds to exactly the condition $\partial\psi = \omega|_H$

Given an algebra A in $\mathcal{C}$
we can form $\mathcal{C}_A$ (or A-mod$(\mathcal{C})$)
= category of (right) A-modules in $\mathcal{C}$

- $(M, \mu) \in \mathcal{C}$ is called right A-module if
$\mu : M \otimes A \to M$ satisfies

$$\begin{tikzcd}
(M\otimes A)\otimes A \arrow[rr, "\mu\otimes I"] \arrow[d, "a"'] & & M\otimes A \arrow[d, "\mu"] \\
M\otimes(A\otimes A) \arrow[r, "I\otimes\mu"'] & M\otimes A \arrow[r] \arrow[ur, phantom, "\cong"] & M
\end{tikzcd}$$

- A semisimple $\Longleftrightarrow$ $\mathcal{C}_A$ is (right) semisimple category

Exercise: Right semisimple $\Longleftrightarrow$ left semisimple
(Proof uses rigidity)

(Can also form category of Hopf modules, right?)
in $\mathcal{C}$

(II) $\mathcal{C}$ is itself a "ring"
modules categories over $\mathcal{C}$
are categories $\mathcal{M}$ with
bifunctor $\boxtimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$
$(X, M) \mapsto X \boxtimes M$

($\mathcal{M}$ is $\Bbbk$-linear, abelian)
(later want s-s as well)

need associativity constraint
$(X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M)$
+ need this constraint to satisfy a pentagon axiom

Example: $\mathcal{C} = \mathrm{Vec}$

any additive $\mathbb{k}$-linear category $\mathcal{M}$ is a module category of $\mathcal{C}$

(using just tensor product?)

Eg: $\mathcal{C} = \mathrm{Rep}(G)$

$H \leq G$, take $\mathcal{M} = \mathrm{Rep}(H)$

want bifunctor $\mathrm{Rep}(G) \times \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(H)$

$$X, M \mapsto X \boxtimes M$$

where $X \boxtimes M := (\mathrm{Res}^G_H X) \otimes M \in \mathrm{Rep}(H)$

Eg: ψ: 2-cocycle on H

$\mathrm{Rep}(\mathbb{k}[H]_\psi)$ is a module category over $\mathrm{Rep}(G)$

$$X \boxtimes M := (\mathrm{Res}^G_H X) \otimes H$$

Eg: for algebra $A \in \mathcal{C}$

$\mathcal{C}_A$ is a (left) module category over $\mathcal{C}$

$$\mathcal{C} \times \mathcal{C}_A \longrightarrow \mathcal{C}_A$$

$$X \boxtimes M := X \otimes M$$

· Ex: check that $X \otimes M \in \mathcal{C}_A$ and associativity is true

A semisimple $\iff$ $\mathcal{C}_A$ semisimple

(today all module cats. are semisimple)

$\boxed{\text{Theorem}}$ If $\mathcal{C}$ is fusion cat. & $\mathcal{M}$ is a semisimple (left) module category over $\mathcal{C}$ then $\exists\ A \in \mathcal{C}$ s.t.

$$\mathcal{M} \simeq \mathcal{C}_A$$

as $\mathcal{C}$-modules (make this precise categorically)

WARNING: This thm says that A exists, but A is not unique.

Defn: A_1, A_2 algebras in $\mathcal{C}$ are called Morita equivalent if

$$\mathcal{C}_{A_1} \simeq \mathcal{C}_{A_2}$$

as $\mathcal{C}$-module categories

- We can restrict to indecomposable algs. and simple $\mathcal{C}$-module categories

Eg: $\mathcal{C} = \mathrm{Rep}(G)$

What are semisimple indec. algebras in $\mathcal{C}$?

Algebras in $\mathrm{Rep}(G)$ are algebras in Vec with action of G s.t. m, i are G-module maps.

(i) A has to be semisimple in Vec

(ii) By A.W., $A = \bigoplus_i \mathrm{Mat}_{n_i}(\mathbb{k})$

$\{e_i\}$ – primitive central idempotents

↳ one for each summand

G acts on $\{e_i\}$.

- for the algebra to be indecomposable, G action on $\{e_i\}$ should be transitive

- this action should be same as $G \curvearrowright G/H$ for some $H \leq G$ (H = stabilizer of action)

- H acts on one particular summand $Mat_n(\mathbb{k})$

$$Mat_n(\mathbb{k}) = V \otimes V^*$$

& H acts projectively on V

$\Rightarrow$ $\exists$ 2-cocycle on H &

V = $\mathbb{k}[H]_\psi$ - module

To summarize:

we need $H \subset G$, ψ = 2-cocycle on H

V = module over $\mathbb{k}[\psi]_H$

then, $V \otimes V^* \in Rep(H)$

$\Rightarrow$ $Ind_H^G (V \otimes V^*)$

$V \otimes V^* = Mat_n(\mathbb{k})$ is algebra

then $Ind_H^G (V \otimes V^*)$ is also an algebra

(with H, ψ fixed if we change V, we get something Morita eq)

(we can conjugate H, that too gives something morita eq.)

with $A = Ind_H^G (V \otimes V^*)$, one gets

$$\underset{\substack{\uparrow \\ \mathcal{C}\text{-module}}}{\mathcal{C}_A \simeq} \; Rep(\mathbb{k}[H]_\psi)$$

turns out, $\mathrm{Rep}(\Bbbk[H]_\psi)$ are all $\mathcal{C}$-modules !!

given (H,ψ)

(H,ψ) & (H',ψ') are M-E. if they are "conjugates"

Eg: If $\mathcal{C} \underset{\text{monoidal}}{\simeq} \mathcal{D}$, then module categories are the same.

Question: When does $\mathrm{Rep}(\Bbbk[H]_\psi)$ have only 1-simple object isomorphism?

Ans: This is classical alg question. When does $\Bbbk[H]_\psi$ have 1-irr representation?

want $\Bbbk[H]_\psi \simeq \mathrm{Mat}_n(\Bbbk)$

$\Rightarrow$ $|H|$ has to be a square.

Eg: (i) $H = C_2 \times C_2$ $\qquad C_2 = \{g \mid g^2 = 1\}$

can be twisted to get $\mathrm{Mat}_2(\Bbbk)$

(ii) $H = C_4$ second cohomology is trivial

$\therefore$ Can't be twisted to $\mathrm{Mat}_2(\Bbbk)$.

(iii) $\mathcal{C} = \mathrm{Rep}(D_8)$ $\qquad \mathcal{D} = \mathrm{Rep}(Q_8)$

Q What are modules with exactly 1 object?

D_8: $\{e\}$

subgroups of order 4 (want those of type $C_2 \times C_2$ not C_4)

Q_8 has only 1 element of order 2

$\Rightarrow$ has no subgroup of type $C_2 \times C_2$

but there are some of this type in D_8

$\therefore$ $\mathrm{Rep}(D_8)$ and $\mathrm{Rep}(Q_8)$ are not equivalent.

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endofunctors form a monoidal category

Given $\mathcal{C}$, $\mathcal{M}$, one can form

$$\mathcal{C}^*_{\mathcal{M}} := \mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})^{op} \to \text{lk-linear functors}$$

monoidal!

lk-linear

Thm (Etingof - Nikshych - O.) If char $\Bbbk = 0$, $\mathcal{C}$ fusion, $\mathcal{M}$ semisimple, indecomposable then $\mathcal{C}^*_{\mathcal{M}}$ is fusion.

(this is good way of producing new monoidal categories)

Example: Take $\mathcal{C} = \mathrm{Rep}(G)$

$\mathcal{M} = \mathrm{Rep}(\Bbbk[H]_\psi)$

take $H = \{e\}$

then $\mathcal{C}^*_{\mathcal{M}} = \mathrm{Vec}_G$!!

$\mathcal{M} = \mathcal{C}_A$, then $\mathcal{C}^*_{\mathcal{M}} = {}_A\mathcal{C}_A$ - A-bimodules in $\mathcal{C}$

Eg: Take $\mathcal{C} = Vec_G$

$A = \Bbbk[H]_\psi$ $\quad H \le G$, ψ - 2 cocycle

Take $\psi = Id$

then $\mathcal{C}^*_{\mathcal{e}_A} = {}_A\mathcal{C}_A$

There is a geometric way to think about this

$\mathcal{C}_{\mathcal{e}_A} = {}_A\mathcal{C}_A$ = H- equivariant sheaves on G/H

This gives new understanding

eg: $G = S_n$ $\quad H = S_{n-1}$

then $G/H = \{ \underbrace{1, 2, \dots, n-1}, n \}$

2 orbits

H- equivariant sheaves

$= Rep(S_{n-1}) \oplus Rep(S_{n-2})$

(there is a way to impose fusion structure on it)

But $Rep(S_{n-1})$ has a structure of fusion category.

$\mathcal{M}$ is module category for $\mathcal{C}^*_{\mathcal{M}}$

and it turns out

$Fun_{\mathcal{C}^*_{\mathcal{M}}}(\mathcal{M}, \mathcal{M}) \cong \mathcal{C}$

Thm: Module categories over $\mathcal{C}$

$\updownarrow$

Module categories over $\mathcal{C}^*_{\mathcal{M}}$ ($\mathcal{M}$ is s.s. indecom. module)

[$\therefore$ Vec_G module categories $\longleftrightarrow$ $Rep(G)$-module categories]

Defn: $\mathcal{C}$ is (weak) Morita equivalent to $\mathcal{C}^*_{\mathcal{M}}$

want this to be

- reflexive → exercise
- symmetric → we discussed
- transitive → requires work

so that it becomes a relation.