Defn: Category $\mathcal{C}$ is premodular if it is spherical, braided and fusion.

Example: $\mathcal{C}(A,q)$ A-abelian grp, q- quadratic form

$\mathrm{Rep}(G)$ G-finite

Conj: Every fusion cat. has a spherial structure?

$\mathcal{C}(A,q)$ motivating example for defind (modular $\otimes$ cat)

(sometimes q is not degenerate)

Recall: using q we can produce a bicharactor

$$q \rightsquigarrow \frac{q(xy)}{q(x)\,q(y)} : A \times A \to \mathbb{k}^*$$

(for fixed x or y, we get a character of A)

$\mathcal{C}$ premodular $\rightsquigarrow S_{ij}$

simple objects $\{X_i\}_{i \in I}$

we get maps $X_i \otimes X_j \xrightarrow{c} X_j \otimes X_i \xrightarrow{c} X_i \otimes X_j$

with spherical structure, we can talk about trace.

Set $S_{ij} = \mathrm{Tr}\left(c_{X_j, X_i} \circ c_{X_i, X_j}\right) \in \mathbb{k}$

Defn: A premodular $\mathcal{C}$ is modular tensor cat (MTC) iff S is (invertible) non-degenerate.

- $\mathrm{Rep}(G)$ is MTC $\iff G = \{e\}$

- $\mathcal{C}(A, q)$ is MTC $\Longleftrightarrow$ q is non-degenerate

MTC $\rightsquigarrow$ model Topological Phases of Matter

- $TLJ^{\mu}(A)$ are also MTC

$\mathcal{C}$ - fusion cat , $\mathcal{M}$ - module category

$\rightsquigarrow$ $\mathcal{C}^*_{\mathcal{M}} = Fun_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$

Special case: $\mathcal{C}$ is a $\mathcal{C}$-bimodule (or $\mathcal{C} \boxtimes \mathcal{C}^{op}$-module)

then $(\mathcal{C} \boxtimes \mathcal{C}^{op})^*_{\mathcal{C}} = Fun_{\mathcal{C} \boxtimes \mathcal{C}^{op}}(\mathcal{C}, \mathcal{C}) := Z(\mathcal{C})$

- What is the classical anology?

R - ring (algebra over $\mathbb{k}$)

can think of R as R-R bimodule or $R \otimes R^{op}$ bimodule

then

$$E_{R \otimes R^{op}}(R) = Z(R) \quad \text{center of } R$$

$(\mathcal{C} \boxtimes \mathcal{C}^{op})^*_{\mathcal{C}} = Fun_{\mathcal{C} \boxtimes \mathcal{C}^{op}}(\mathcal{C}, \mathcal{C}) := Z(\mathcal{C})$

= Drinfeld center of $\mathcal{C}$

Objects $(Z(\mathcal{C})) = (X, \varphi)$

X: object of $\mathcal{C}$; φ: central structure

$$\varphi : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

isomorphisms of functors of variable Y

$$\begin{tikzcd}
X\otimes(Y_1\otimes Y_2) \arrow[rrr, "\varphi"] \arrow[d, "a"'] & & & (Y_1\otimes Y_2)\otimes X \\
(X\otimes Y_1)\otimes Y_2 \arrow[dr, "\varphi"'] & & & Y_1\otimes(Y_2\otimes X) \arrow[u, "a"'] \\
& (Y_1\otimes X)\otimes Y_2 \arrow[r, "\varphi"'] & Y_1\otimes(X\otimes Y_2) \arrow[ur] &
\end{tikzcd}$$

(This is one of the hexagon axioms of B.T.C.)

There are some axioms of unit objects too

<u>Morphisms</u>: $X \to Y$ respecting φ

<u>$\otimes$ structure</u>: $(X_1, \varphi_1)\otimes(X_2, \varphi_2)$

$$= (X_1\otimes X_2, \psi)$$

$$\psi: \begin{tikzcd}
(X_1\otimes X_2)\otimes Y \arrow[r, "a"] \arrow[d, dotted] & X_1\otimes(X_2\otimes Y) \arrow[r, "\varphi_2"] & X_1\otimes(Y\otimes X_2) \arrow[d, "a"] \\
Y\otimes(X_1\otimes X_2) & (Y\otimes X_1)\otimes X_2 \arrow[l, "a"'] & (X_1\otimes Y)\otimes X_2 \arrow[l]
\end{tikzcd}$$

The φ makes $Z(\mathcal{C})$ a braided monoidal category.

<u>Thm</u>: If $\mathcal{C}$ is spherical, fusion category, then $Z(\mathcal{C})$ is Modular Tensor category.

(for this to be s.s., need char $k = 0$)

We always have a functor
$F : Z(\mathcal{C}) \longrightarrow \mathcal{C}$ tensor functor

this F is surjective, i.e. every $X \in \mathcal{C}$ is direct summand of $F(Y)$ for some $Y \in Z(\mathcal{C})$

- To get a handle on $\mathcal{C}$, it is helpful to look at $Z(\mathcal{C})$.

Ex: $\mathcal{C} = \mathrm{Vec}_G$

$\underbrace{\mathrm{Vec}_G}$ as module over $\mathrm{Vec}_G \times \mathrm{Vec}_G^{op} = \mathrm{Vec}_{G\times G}$

↳ should be described by some subgroup H of $G \times G$.

There is a canonical subgroup
$H = \Delta G = \{(g,g) \mid g \in G\}$

Take 2-cocycle ψ to be trivial

$(\mathcal{C} \boxtimes \mathcal{C}^{op})^*_{\mathcal{C}}$ = $\underset{\underset{G}{\shortparallel\wr}}{\Delta G}$ equivariant sheaves on $G \times G / \Delta G = G$

= G equivariant sheaves on G

What is the action of ΔG on $G \times G / \Delta G$
(it is the adjoint action
$x \cdot g = xgx^{-1}$)

$\mathcal{C} = \mathrm{Vec}_G \rightsquigarrow Z(\mathrm{Vec}_G)$

$\cong$

$\mathrm{Rep}_{D(G)}$

category of reps. of Drinfeld double of G.

Stabilizer of g = things commuting with G

Simple objects $\longleftrightarrow \left\{ (X, p) \quad \begin{array}{l} X \in G \\ p \in \underset{\mathrm{Irr}}{\cancel{\mathrm{Rep}}} (C_G(X)) \end{array} \right\} \Big/ G$

Example: $\mathcal{C} = \mathrm{Vec}_G^{\omega} \rightsquigarrow Z(\mathrm{Vec}_G^{\omega})$ = twisted Drinfeld double

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1) If $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent (i.e. $\mathcal{D} = \mathcal{C}_M^*$)
then $Z(\mathcal{C}) \xrightarrow{\sim} Z(\mathcal{D})$
braided equivalence

2) If $\mathcal{C}, \mathcal{D}$ are fusion categories, then
$Z(\mathcal{C}) \underset{\text{braided}}{\xrightarrow{\sim}} Z(\mathcal{D}) \Rightarrow \mathcal{C}$ and $\mathcal{D}$ are morita equivalent

Q How to differentiate $Z(\mathrm{Vec}_G^\omega)$?

Q When is some MTC $\mathcal{C}$ of the form $Z(\mathrm{Vec}_G^\omega)$?

Both $Z(\mathrm{Vec}_G)$ & $Z(\mathrm{Vec}_G^\omega)$ contain $\mathrm{Rep}(G)$ as a braided tensor subcategory.

Answer: Assume $\mathcal{C}$ contains a copy of $\mathrm{Rep}(G)$ of "right size". Then
$\mathcal{C} \simeq Z(\mathrm{Vec}_G^\omega)$

Example:
① $G = C_4 \rightsquigarrow Z(\mathrm{Vec}_G)$

abelian $\Rightarrow$ $\begin{cases} \text{conjugacy classes} = \text{single elts} \\ \text{stabilizer} = G \end{cases}$

$Z(\mathrm{Vec}_{C_4})$ contains $\mathrm{Rep}(C_4)$ & $\mathrm{Rep}(C_2 \times C_2)$
pointed

Corollary: $\exists\ \omega \in H^3(C_2 \times C_2, \Bbbk^\times)$ s.t.
$Z(\mathrm{Vec}_{C_4}) \simeq Z(\mathrm{Vec}_{C_2\times C_2}^\omega)$
called 'electric-magnetic duality' in Physics

② $Z(\mathrm{Vec}_{D_8}) \supset \mathrm{Rep}(C_2 \times C_2 \times C_2)$
this tells
$Z(\mathrm{Vec}_{D_8}) \simeq Z(\mathrm{Vec}_{C_2\times C_2\times C_2}^\omega)$

replace D_8 by Q_8, it still works

$\mathrm{Vec}_{D_8} \underset{\text{Morita}}{\sim} \mathrm{Vec}_{C_2\times C_2\times C_2}^\omega$

Non-semisimple tensor categories:

Finite tensor categories

= tensor categories $\mathcal{C}$ s.t.

$\mathcal{C} \simeq \mathrm{Rep}(A)$

equivalent as Abelian categories

finite dim. algebra

Ex: (i) $\mathrm{Rep}(H)$ $\quad H$ = fin. dim (quasi) Hopf algebra

eg H = small quantum group

(ii) $\mathrm{Rep}(G)$ when char $\Bbbk \mid |G|$

$\mathrm{Rep}(G)$ $\quad G$ = "finite supergroup"

Q: Find more examples of these.

Defn of Finite tensor cats continued –

- all objects have finite length
- there are enough projectives
- # of simples is finite

Lemma: 1) P projective, X $\Rightarrow$ $P \otimes X$ projective

2) P is projective $\Rightarrow$ P^* is projective

3) $P(\mathbb{1})$ projective cover of $\mathbb{1}$ $\rightsquigarrow$ $P(\mathbb{1})^* = P(L)$ for L simple in $\mathcal{C}$

L is invertible

$X^{****} \simeq L \otimes X \otimes L^{-1}$

analogue of Radford's power of 4 formula.

4) $F : \mathcal{C} \to \mathcal{D}$ tensor surjective
then $F(\mathcal{D})$ projective
(categorical version of Nicolls-Zoeller thm)

MODULE CATEGORIES

Study "exact" module categories
$\mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$
$P \otimes M \to$ projective in $\mathcal{M}$

Ex $\mathcal{C} =$ fusion then 1 projective
$1 \otimes M = M$
$\Rightarrow M$ projective $\forall M \in \mathcal{M}$
all projective $\Rightarrow$ semisimple
Thus we are asking $\mathcal{C}$ to be semisimple.

Q Find more examples of these.

Tensor categories by generators & relations:

Generators : some objects + some morphisms

Relations : some relations between morphisms.

Eg: $\mathrm{Vec}_{C_n} \ni X$ $\qquad C_n = \langle g \mid g^n = 1 \rangle$

Generators: X, $a : X^{\otimes n} \longrightarrow 1$, $b : 1 \longrightarrow X^{\otimes n}$

Relations: $ab = \mathrm{Id}_1$, $ba = \mathrm{Id}_{X^{\otimes n}}$
$a \otimes \mathrm{Id} = \mathrm{Id} \otimes a : X^{n+1} \longrightarrow X$

Thm: The category obtained by these generators & relations is the pointed category Vec_{C_n}.

Universal property: $\mathcal{C}$ – tensor category

$$\left\{ \begin{array}{c} \text{tensor functor} \\ \mathrm{Vec}_{C_n} \xrightarrow{F} \mathcal{C} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{objects } X \in \mathcal{C} \\ \text{together with } a, b \\ \text{as before} \end{array} \right\}$$

this is a category

this too is category

So, what we have above is an equivalence of categories

$$F \longmapsto F(\delta_g)$$

(where $\delta_g \in \mathrm{Vec}_{C_n}$ is C_n graded vector space with $\mathbb{C}$ in one position g.)

Q State generators & relations & state universal property for TLJ

- Think about ideals in tensor categories