Unimodularity

- classical concept
- square integer matrix A is called unimodular if $\det(A) = \pm 1$.
- building upon this, unimodularity is defined for lattices, bilinear form, topological groups, Poisson algebras, Hopf algebras.
- Unimodular lattices (E_8 & Leech lattices) were used to obtain efficient sphere packings in dim 8 and 24 in recent (fields medal winning) work of Maryna Viazowska and coauthours.

For today, I am primarily concerned with unimodular Hopf algebras.

The story in this direction began with locally compact topological groups.

Take such a group G (eg. $G = \mathbb{R}^n$)
→ come equipped with left (and right) invariant Haar measures. We can integrate wrt these measures.

$$\int^{\ell} f \, d\mu \qquad\qquad \int^{r} f \, d\mu$$

left invariant $\qquad f : G \to \mathbb{k}$ $\qquad$ right invariant

If left invariant Haar is also right invariant,

we call the group unimodular.
(this tells us that we should take our analysis classes more seriously)

Larson & (big deal in Hopf algebras)
→ Generalizing this, Sweedler defined left and right integrals for Hopf algebras.

Let $(H, m, u, \Delta, \varepsilon, S)$ be a f.d. Hopf algebra

Defn: • A left integral is an element $\Lambda^{\ell} \in H$ satisfying $h\Lambda^{\ell} = \varepsilon(h)\Lambda^{\ell} \quad \forall\, h \in H$.
• A right integral is an element $\Lambda^{r} \in H$ satisfying $\Lambda^{r} h = \varepsilon(h)\Lambda^{r} \quad \forall\, h \in H$.

Space of left (right) integrals is denoted as $\int_{\ell}$ ($\int_{r}$).

Example ① G = finite group
$H = \Bbbk G$ then
$\Lambda = \frac{1}{|G|} \sum_{g \in G} g$ is a left and right integral.

② H = Sweedler's Hopf algebra $H_{2,1}$
$$H = \frac{\Bbbk\langle x, g\rangle}{(x^2 = 0,\ g^2 = 1,\ gx = i\,xg)} \qquad i = \sqrt{-1}$$
$\Delta(g) = g \otimes g \qquad \Delta(x) = x \otimes 1 + g \otimes x$
$S(g) = g^{-1} \qquad S(x) = -gx^{-1}$
$\varepsilon(g) = 1 \qquad \varepsilon(x) = 0$

$\Lambda^{\ell} = x + gx$ is a left integral (not a right integral)

$\Lambda^{r} = x + xg$ is a right integral

e.g. • $g(x+gx) = gx + g^2 x = x + gx$
$= \varepsilon(g)[x+gx]$

• $x(x+gx) = x^2 + xgx = xgx$
$= i x^2 g$
$= 0$

Integrals are very important for studying Hopf algebras.

<u>Theorem</u> (Larson-Sweedler): Let H be a f.d. Hopf algebra. Then

H is semisimple $\iff$ $\exists$ left (right) integral Λ^{ℓ} satisfying $\varepsilon(\Lambda) \neq 0$

• Using this check that • $\Bbbk G$ is semisimple.
• $H_{2,1}$ is not semisimple

• Integrals for infinite diml. Hopf algebras have been defined by James and coauthors.

• Using them, they obtained a (nice) generalization of Larson-Sweedler's result.

Defn: A f.d. Hopf algebra is called unimodular if admits an element $\Lambda \in H$ that is both a left integral and a right integral.

Eg: • Group algebras are unimodular
• In fact, all semisimple Hopf algebras are unimodular
• Taft algebras are not unimodular

Why care about unimodular Hopf algebras?

↳ Main motivation is topological
$\exists$ various constructions of invariants of 3-manifolds using Hopf algebras as input. Here unimodularity plays an important role.

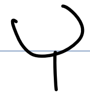
Distinguished character
Take a left integral Λ
. Then $\mathbb{k}\Lambda$ is a 1-diml left ideal of H
. $\int^{\ell}$ is ideal + 1-diml
$\Rightarrow \exists \ \alpha : H \to \mathbb{k}$ s.t. $\Lambda h = \langle \alpha, h \rangle \Lambda \ \forall h \in H$

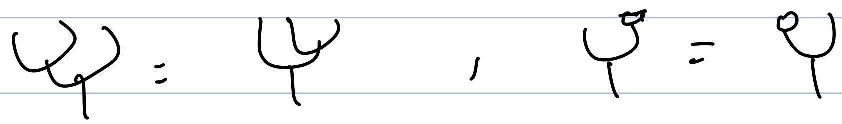
α is a character

. H unimodular $\iff \int_{\ell} = \int_{r} \iff \alpha = \varepsilon$

Left H-comodule algebras

- it is an associative, unital algebra (A, m, u)

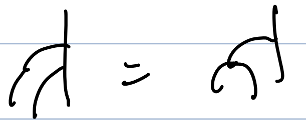
$m =$  $u =$ 

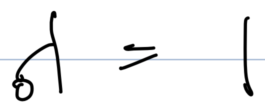


- it is a left H-comodule $(A, \delta : A \to H \otimes A)$

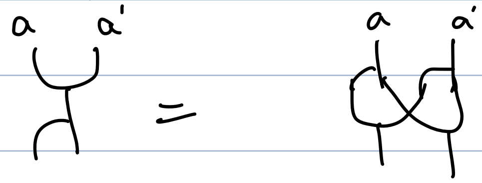
$\delta =$  $: A \to H \otimes A$

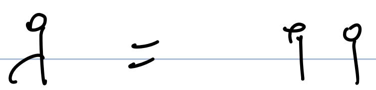
$a \mapsto a_{(-1)} \otimes a_{(0)}$

want  : $(a_{(-1)})_{(1)} \otimes (a_{-1})_{(2)} \otimes a_{(0)}$
$= a_{-1} \otimes a_{0,-1} \otimes a_{0,0}$

 : $\varepsilon(a_{(-1)})\, a_0 = a$

- m, u are maps of left H-comodules

$\to$  $(aa')_{(-1)} \otimes (aa')_{(0)} = a_{(-1)} a'_{(-1)} \otimes a_{(0)} a'_{(0)}$

$\to$  $\delta(1_H) = 1_A \otimes 1_H$

Example:

$$H = H_{2,1} = \frac{\Bbbk\langle x, g\rangle}{\langle x^2 = 0,\ g^2 = 1,\ gx = i\,xg\rangle}$$

- $A(2) = \Bbbk\langle G \mid G^2 = 1\rangle$

$\delta(A) = 1 \otimes G$