3-manifold invariants & the fusion category

An extended TQFT is a symmetric monoidal 2-functor

$$Z : \mathrm{Bord}_1^3 \longrightarrow \mathrm{LinCat}$$

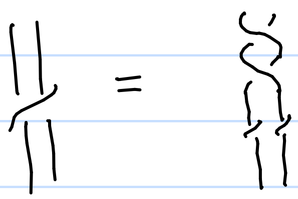
- We will construct an extended TQFT using a ribbon category $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r, c, \theta)$

Defn: Let $\mathcal{C}$ be a braided tensor category. $\mathcal{C}$ is said to be balanced if $\mathcal{C}$ is equipped with a twist, i.e. a natural isomorphism

$$\theta = \{ \theta_X : X \xrightarrow{\sim} X \} \quad \leftrightsquigarrow$$

satisfying the following conditions

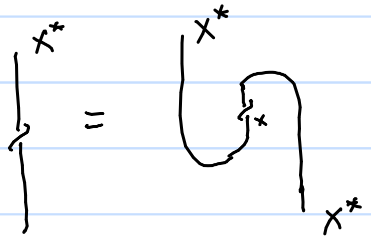
(1)



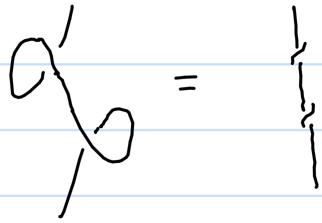
(2) $\theta_{\mathbb{1}} = id_{\mathbb{1}}$

Defn: A balanced tensor category is called ribbon if it satisfies

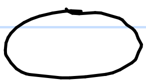
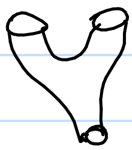
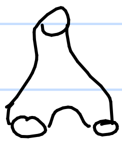
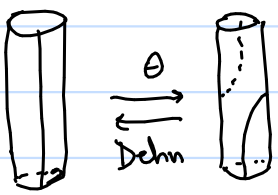
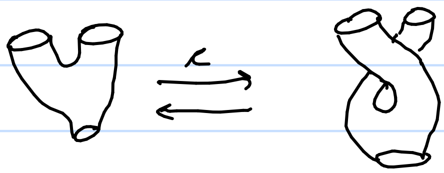
(i)

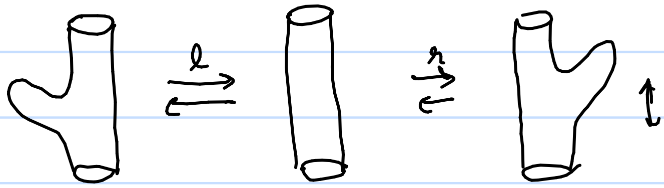


(ii)

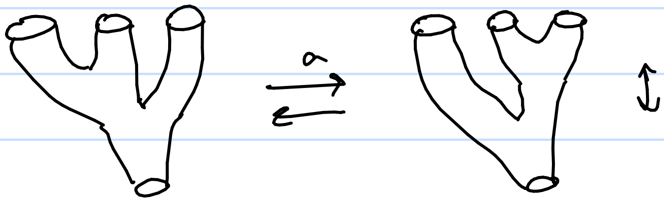


To construct an extended TQFT we first give a generators and relations description of Bord_1^3

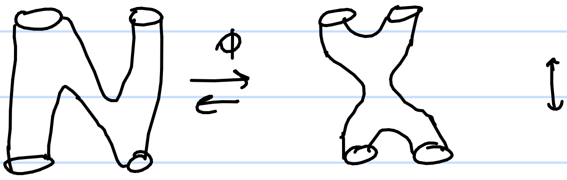
Bord_1^3	LinCat
<u>Objects</u>: • 	$\mathcal{C}$
<u>1-morphisms</u>: • 	$\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$
• 	$u : \mathrm{Vect} \to \mathcal{C}$, $\Bbbk \mapsto \mathbb{1}$
• 	adjoint of $\otimes$
• 	adjoint of u
<u>2-morphisms</u> • 	$\theta : \mathrm{Id}_\mathcal{C} \Longrightarrow \mathrm{Id}_\mathcal{C}$ twist
• 	$\otimes \xRightarrow{c} \otimes \circ \tau$ τ=twist

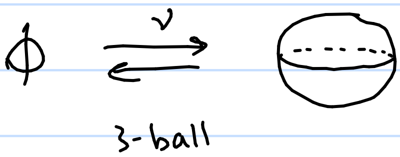
• 

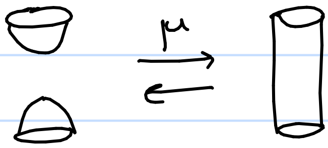
l, r (left, right unitor

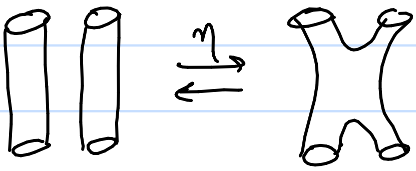
• 

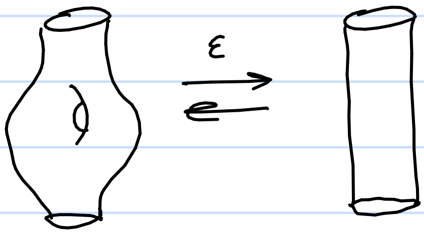
a (associator)

• 

• 

• 

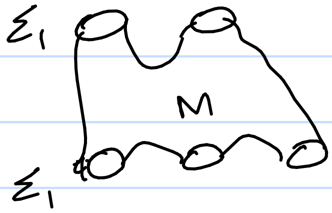
• 

• 

+ relations between these

INTERNAL STRING DIAGRAMS

Say we have a surface $\Sigma_1 \xrightarrow{M} \Sigma_2$



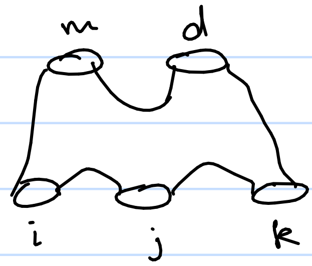
then applying Z we get a functor

$$Z(\Sigma_1) \xrightarrow{\Sigma(M)} Z(\Sigma_2)$$

FACT The functor $\Sigma(M)$ is encoded by the hom spaces

$$\mathrm{Hom}\,(Z(\Sigma)(m \boxtimes d),\ i \boxtimes j \boxtimes k)$$

for $m, d, i, j, k \in \mathrm{Ob}(\mathcal{C})$

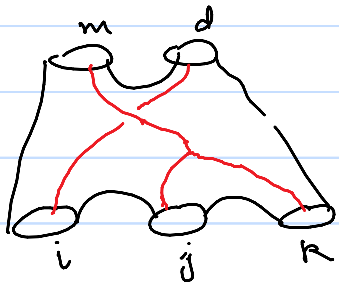


Prop/Thm $\mathrm{Hom}\,(Z(\Sigma)(m \boxtimes d),\ i \boxtimes j \boxtimes k)$

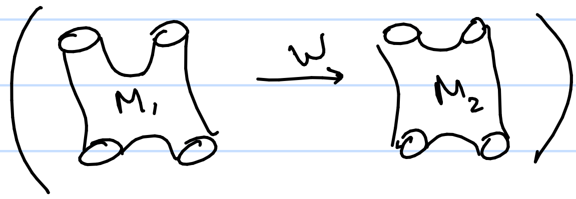
$\simeq$ internal string diagrams $/ \sim$

→ Internal string diagrams are string diagrams as usual drawn in the interior of the surface M.

<u>eg</u>



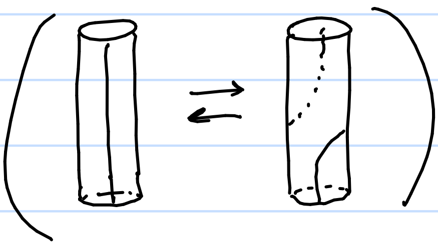
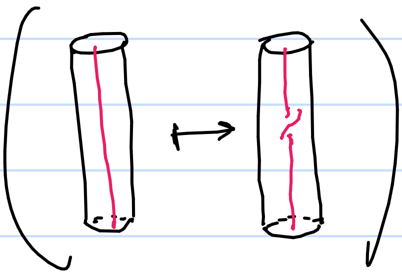
Thus, to describe a natural transformation

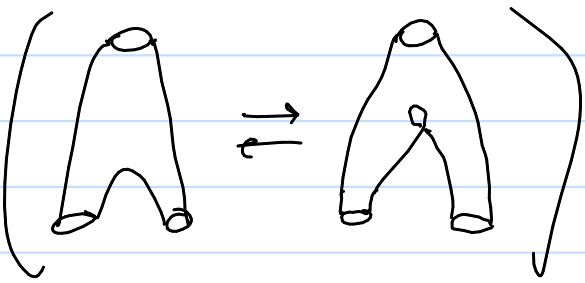
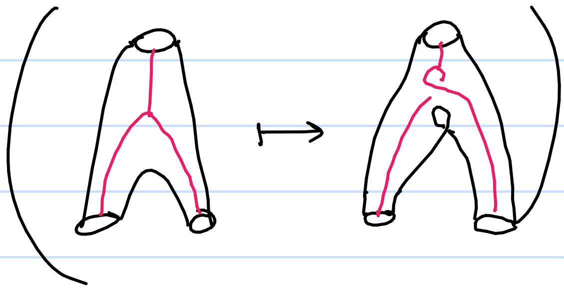
Z  $\quad Z(M_1) \xRightarrow{Z(W)} Z(M_2)$

it suffices to describe what $Z(W)$ does on to internal string diagram for $Z(M_1)$

Example:

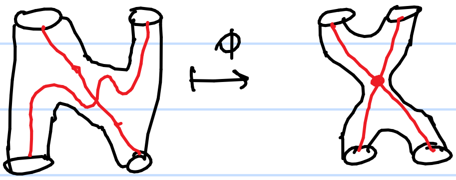
To describe the what Z does to the Dehn Twist

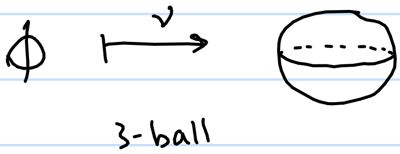
①  it suffices to desribe how internal string diagrams get changed 

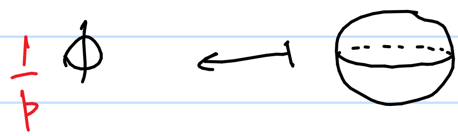
②  $\rightsquigarrow$ 

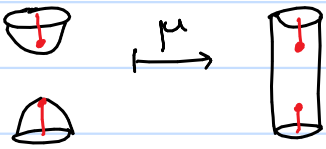
and so on

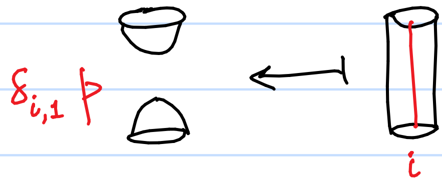
Here is the description of remaining 2-morphisms in terms of string diagrams.

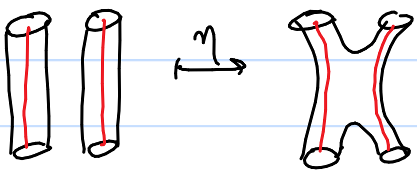
• 

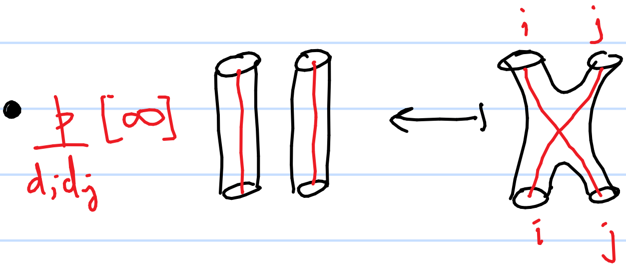
• 

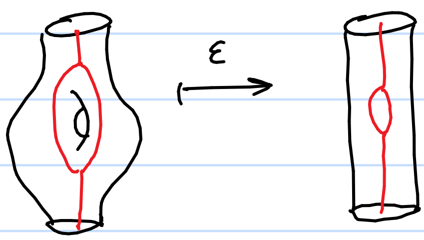
• 

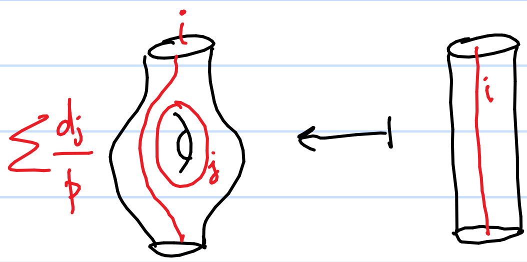
• 

• 

• 

• 

• 

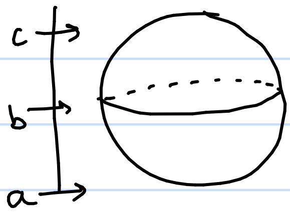
• 

$$p = \sqrt{\sum_{i=1}^{n} d_i^2}$$

Now we can construct invariants of 3-manifolds using the extended TQFT we just constructed

Example ① $S^3 \to \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2=1\}$

First recall S^2 the unit ball



if we stay only in 2D world and we need to describe S^2, we would write it as

$$\phi \xrightarrow[\text{disk}]{\text{add}} S^1 \xrightarrow[\text{disk}]{\text{add}} \phi$$

point a   point b   point c

Similarly if we decompose S^3, we get

$\phi \xrightarrow[\text{3-ball}]{\text{add}}$  $\xrightarrow[\text{2-ball}]{\text{add}} \phi$

S^2

we know what $\phi \longrightarrow$  and  $\to \phi$

are in terms of string diagrams and can use that to get the invariant

$\Big(\phi \longrightarrow$  $\longrightarrow \frac{1}{p}\,\phi \Big) = \frac{1}{p}$