QA learing group

Sept 22, 2021

$\boxed{\text{TALK 3 : Section 2.3 of FFRS 2006}}$

(by H. Yadav)

<u>Last time</u> : Defn. of MTC
Idempotents and retracts, Karoubi completion of $\mathcal{C}$

<u>GOAL</u> : This section defines the different algebraic structures that will be used throughout the paper.
(basically we define the usual algebraic notions like algebra, modules in monoidal categories.

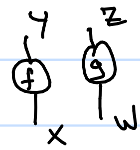
Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a strict monoidal category.

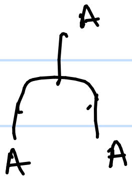
<u>Defn</u> : An algebra in $\mathcal{C}$ is a triple (A, m, u) where
- $A \in obj(\mathcal{C})$
- $m : A \otimes A \to A$, $u : \mathbb{1} \to A$ $\in Mor(\mathcal{C})$
- m is associative u is unital

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "id\otimes m"] \arrow[d, "m\otimes id"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "m"] \\ A\otimes A \arrow[r, "m"'] & A \end{tikzcd} \qquad \begin{tikzcd} A\otimes \mathbb{1} \arrow[r, "id\otimes u"] \arrow[dr, equal] & A\otimes A \arrow[d, "m"] & \mathbb{1}\otimes A \arrow[l, "u\otimes id"'] \arrow[dl, equal] \\ & A & \end{tikzcd}$$

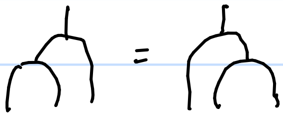
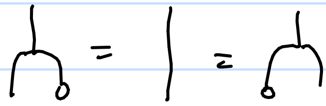
commute

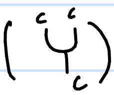
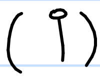
Writing commutative diagrams takes time, so we use graphical calculus (it is also very visual!)

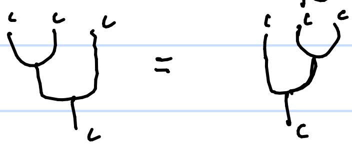
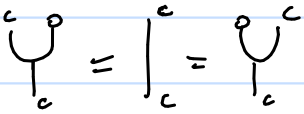
<u>Rules</u>: $X \xrightarrow{f} Y$  , $X \xrightarrow{f} Y$, $W \xrightarrow{g} Z$ then $f\otimes g =$ 

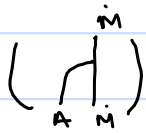
$Y \xrightarrow{h} Z$, then $h\circ f =$  , don't usually draw the unit object unit map of algebra $\mathbb{1} \xrightarrow{u} A$ , multiplication 

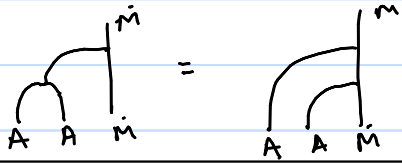
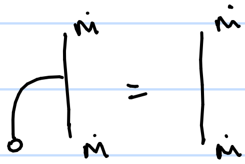
Using graphical notation,

associativity is , unitality is 

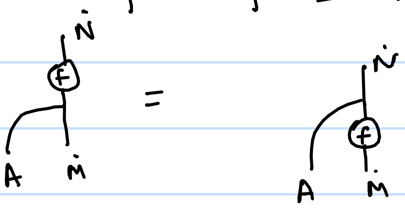
Defn: A coalgebra in $\mathcal{C}$ is a triple (C, Δ, ε) where
- $C \in Obj(\mathcal{C})$
- $\Delta : C \to C \otimes C$ (), $\varepsilon : A \to \mathbb{1}$ () $\in Mor(\mathcal{C})$
- Δ, ε satisfy

, 

Defn: A left module over an algebra A in $\mathcal{C}$ is a pair $M = (\dot{M}, \rho)$ where
- $\dot{M} \in Ob(\mathcal{C})$
- $\rho : A \otimes \dot{M} \to \dot{M}$ () is a morphism satisfying

, 

Defn: Take two left A-modules $(\dot{M}, \rho_M)$, $(\dot{N}, \rho_N)$. Then $f : \dot{M} \to \dot{N}$ is called a map of left A-modules if f satisfies



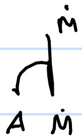
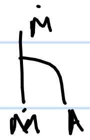
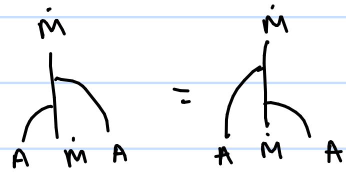
Left A-modules and A-module maps form a category which we denote as $\mathcal{C}_A$.

→ Can similarly define right A-modules

- $M \in \mathcal{C}_A$ is called simple if it simple as an object of $\mathcal{C}_A$

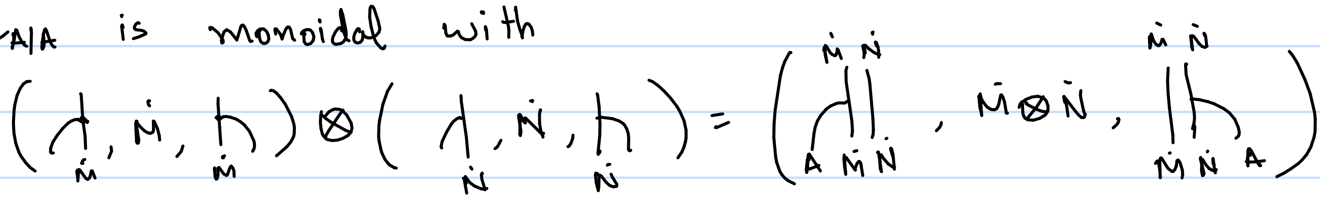
- For $U \in Ob(\mathcal{C})$, denote $Ind_A(U) = A \otimes U$. Then $Ind_A(U) \in \mathcal{C}_A$ via the multiplication in A. Denote by $\mathcal{C}_A^{ind} \subset \mathcal{C}_A$, the full subcategory of A whose objects are $Ind_A(U)$ for some $U \in Ob(\mathcal{C})$

<u>Defn</u>: An A-bimodule is a triple $M = (\dot{M}, \rho_l, \rho_r)$ such that $(\dot{M}, \rho_l)$ is a left A-module and $(\dot{M}, \rho_r)$ is a right A-module, and left and right actions commute.

$\rho_l =$  , $\rho_r =$  then 

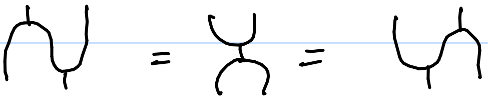
→ A-bimodule and bimodule maps (i.e. left & right module maps) between them form a category that we will denote as $\mathcal{C}_{A|A}$.

→ $\mathcal{C}_{A|A}$ is monoidal with

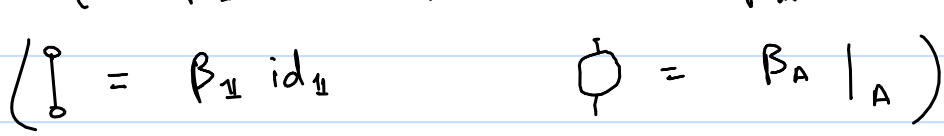


<u>Defn</u> A Frobenius algebra in $\mathcal{C}$ is a 5-tuple $(A, m, u, \Delta, \varepsilon)$ such that

- (A, m, u) is an algebra in $\mathcal{C}$
- (A, Δ, ε) is a coalgebra in $\mathcal{C}$
- Δ - m satisfy the Frobenius law



- A Frobenius algebra is called special if
  $\varepsilon \circ \eta = \beta_{\mathbb{1}}\, \mathrm{id}_{\mathbb{1}}$, $m \circ \Delta = \beta_A\, \mathrm{id}_A$

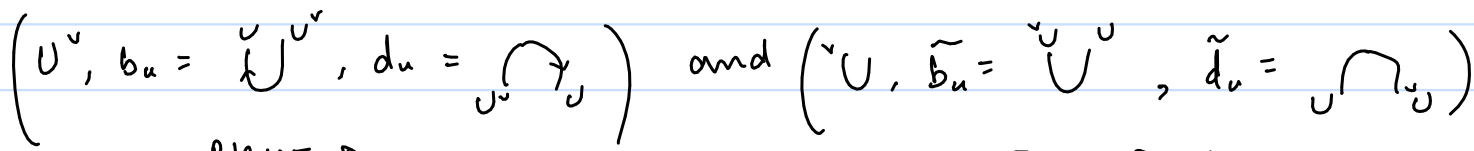


($= \beta_{\mathbb{1}}\, \mathrm{id}_{\mathbb{1}}$, $= \beta_A\, |_A$)

for some non-zero numbers $\beta_{\mathbb{1}}$ and β_A.

Updating graphical calculus for $\mathcal{C}$-rigid

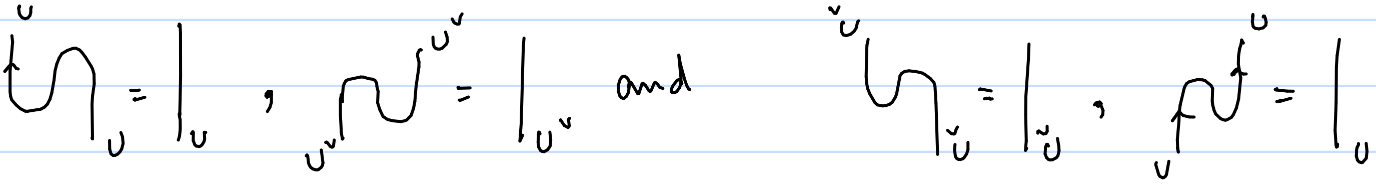
Recall that if $\mathcal{C}$ is rigid, then $\forall\ U \in \mathcal{C}$, we have



$(U^\vee,\ b_u = \ ,\ d_u = \ )$ and $({}^\vee U,\ \tilde{b}_u = \ ,\ \tilde{d}_u = \ )$

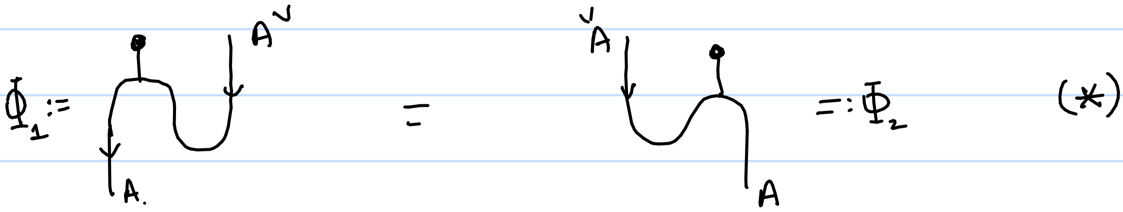
RIGHT DUAL   LEFT DUAL

satisfying



and

Defn: (i) An algebra A in rigid monoidal category $\mathcal{C}$ together with a morphism $\varepsilon : A \to \mathbb{1}$ (denoted) is called a symmetric algebra if

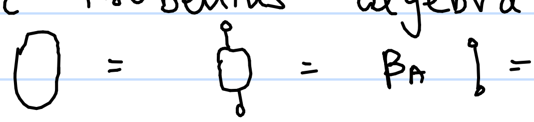


$\Phi_1 := \quad = \quad =: \Phi_2$ $\qquad (*)$

(ii) A special symmetric Frob. algebra is a special Frobenius algebra A that is symmetric with $\varepsilon =$ counit of A.

→ For a special symmetric Frobenius algebra A, we have

$\dim(A) = \quad = \quad = \beta_A \quad = \beta_A \beta_{\mathbb{1}}$



→ We normalize ε, Δ so that $\beta_{\mathbb{1}} = \dim(A)$, $\beta_A = 1$
This convention is followed unless mentioned otherwise.

Various remarks about Frobenius algebras :

- $\mathcal{C}$ semisimple, A special Frobenius $\Rightarrow$ $\mathcal{C}_A$ is semisimple

- We have the induction & restriction functors

  $\mathrm{Ind}: \mathcal{C} \longrightarrow \mathcal{C}_A$ $\qquad\qquad$ $\mathrm{Res}: \mathcal{C}_A \longrightarrow \mathcal{C}$

  $U \longmapsto (A \otimes U, \, \curlywedge|)$ $\qquad\qquad$ $M = (\dot{M}, \rho_e) \longmapsto \dot{M}$

  For any algebra A, we have reciprocity relation that is, a canonical natural bijection

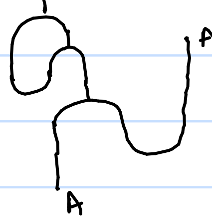
  $$\mathrm{Hom}_A(\mathrm{Ind}_A(U), M) \xrightarrow{\cong} \mathrm{Hom}(U, \dot{M})$$

  Categorically, this amounts to saying that

  $\mathrm{Ind}_A \dashv \mathrm{Res}$ $\qquad$ (left adjoint)

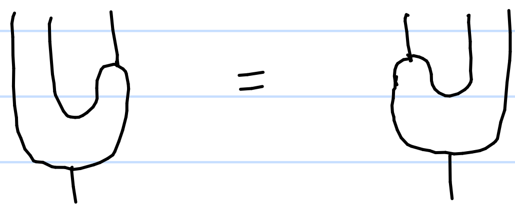
  If A is Frobenius, $\mathrm{Res} \dashv \mathrm{Ind}$ as well.

- Given an algebra (A, m, u) in a pivotal category,

  $\exists\ \Delta, \varepsilon$ making $(A, m, u, \Delta, \varepsilon)$ symmetric special Frobenius $\Leftrightarrow$ 

is invertible

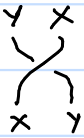
- In terms of graphical calculus, an algebra being symmetric Frobenius means that multiplication and comultiplication can be moved past each other in all possible arrangements.

example 

Now $\mathcal{C}$ is braided as well.

Then we have a natural isomorphism

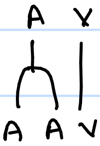
$$\{c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$$

$c_{X,Y} \leftrightarrow$ 

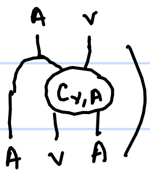
Defn: For A an algebra in a braided tensor category $\mathcal{C}$, we have the functors

$$\alpha_A^{\pm} : \mathcal{C} \longrightarrow \mathcal{C}_{A|A}$$

of α-induction.

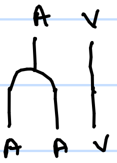
$\alpha_A^+(V) := \Big(A \otimes V,$ 

left action

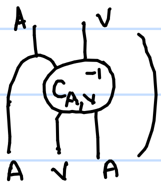
, 

right action

$\Big)$; $\alpha_A^+(f) = \mathrm{id}_A \otimes f$

$\alpha_A^-(V) := \Big(A \otimes V,$ 

left action

, 

right action

$\Big)$; $\alpha_A^-(f) = \mathrm{id}_A \otimes f$

Remark: $\alpha^{\pm} : \mathcal{C} \to \mathcal{C}_{A|A}$ are monoidal functors.

Defn: An algebra A in a braided monoidal category is said to be commutative if

$$m \circ c_{A,A} = m .$$

Remark: → For general A, $\mathcal{C}_A$ is not monoidal.
→ But for A commutative, $\mathcal{C}_A$ is monoidal. There are two choices for monoidal structure.
→ Later study $\mathcal{C}_A^{\mathrm{loc}}$, full subcategory of $\mathcal{C}_A$. It has a canonical tensor structure.

Now say $\mathcal{C}$ is a braided pivotal category. Then the twist θ satisfies

$\theta =$ 

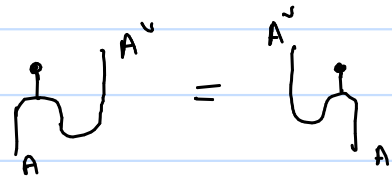
PROP 2.25 Let $\mathcal{C}$ be a braided pivotal category

(i) A commutative symmetric Frobenius algebra has trivial twist, i.e., $\theta_A = id_A$.

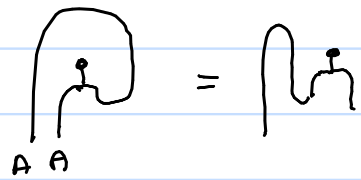
(ii) Every commutative Frobenius algebra with trivial twist is symmetric.

(iii) A commutative symmetric Frobenius algebra is also cocommutative.

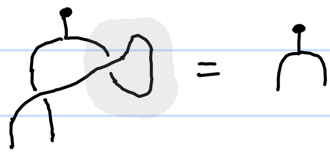
<u>Proof of (i)</u>: A is symmetric

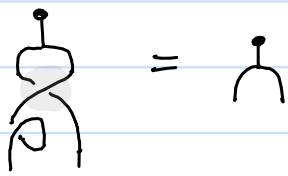
$\Rightarrow$ 

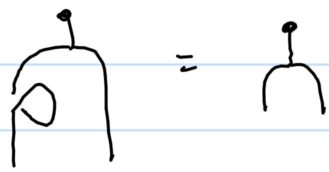
apply $d_A = \underset{A\;\;A^\vee}{\cap}$ $\Longrightarrow$ to both sides

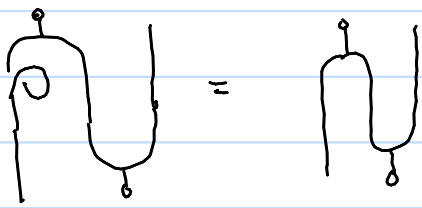


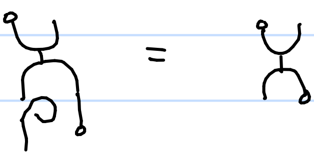
(using snake relation on RHS) $\Longrightarrow$ 

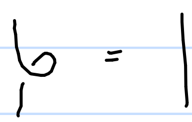
(moving the left thread over mult. map) $\Rightarrow$ 

(move the shaded part to left side) $\Rightarrow$ 

(commutativity of A) $\Rightarrow$ 

(adding extra maps to both sides) $\Rightarrow$ 

(Frobenius law) $\Rightarrow$ 

(by Unit /counit axioms) $\Rightarrow$ 

See paper for (ii), (iii) ∎

Defn: • An algebra is called simple if all bimodule endomorphisms of A as a bimodule over itself are multiples of the identity, i.e.

$$\mathrm{Hom}_{A|A}(A,A) = \Bbbk \, \mathrm{id}_A .$$

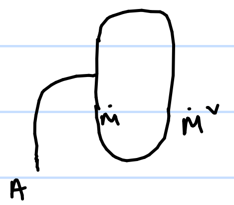
• An algebra A is called haploid if

$$\mathrm{Hom}_A(A,A) = \Bbbk \, \mathrm{id}_A \iff \dim(\mathbb{1}, A) = 1$$

→ In general haploid ⇒ simple, but the converse is not true

→ When A is commutative, haploid ⇔ simple.

Lemma 2.27: Let A be a simple symmetric special Frobenius algebra. Then $\forall \; M \in \mathcal{C}_A$ we have



$$= \frac{\dim(\dot{M})}{\dim(A)} \, \varepsilon$$

■

Defn: For $U, V \in \mathcal{C}$, set

$$\tilde{Z}(A)_{U,V} := \dim_{\Bbbk} \left[\mathrm{Hom}_{A|A} \left(\alpha_A^-(V), \alpha_A^+(V) \right) \right]$$

→ Since $A = \alpha_A^{\pm}(\mathbb{1})$, simplicity of A is equivalent to $\tilde{Z}(A)_{\mathbb{1},\mathbb{1}} = 1$.

(This covers everything in § 2.3 except a few remarks about retracts and interaction between module categories & Karoubi completion)