REPRESENTATION OF THE MODULAR GROUP

- → What is the 'modular group'
- → Modular categories give representation of modular groups
- → Congruence subgroup problem for $SL_2(\mathbb{Z})$

§ 1 MODULAR GROUP

- It is the group
$$\Gamma := SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; \begin{array}{c} a,b,c,d \in \mathbb{Z} \\ ad-bc=1 \end{array} \right\}$$

- Center of $SL_2(\mathbb{Z}) = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$

- The quotient $\dfrac{\Gamma}{Z(\Gamma)} = PSL_2(\mathbb{Z})$

- $PSL_2(\mathbb{Z})$ is the group of fractional linear transformation of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az+b}{cz+d}, \quad z \in \hat{\mathbb{C}}$$

- Given two complex number ω_1 and ω_2, we can form the 2D lattice

$$\Lambda(\omega_1,\omega_2) = \{ m\omega_1 + n\omega_2 \mid m,n \in \mathbb{Z} \}$$

$PSL_2(\mathbb{Z})$ is the symmetry group of $\Lambda(\omega_1,\omega_2)$, i.e. another pair of vectors α_1, α_2 generate the same lattice iff

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \quad \text{for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z})$$

$\longrightarrow$ leads to connection with modular forms and number theory.

- $PSL_2(\mathbb{Z})$ is the subgroup of the group of isometries of the Hyperbolic plane.

$\longrightarrow$ leads to connection with hyperbolic geometry

- $SL_2(\mathbb{Z})$ is the mapping class group of the torus.

- Γ is generated by 2 matrices

$$s := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad t := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- Γ can be described abstractly in terms of generators and relations as

$$\Gamma = \langle s, t \mid (st)^3 = s^2, \ s^4 = 1 \rangle$$

and

$$\Gamma / \langle \pm 1 \rangle = \langle s, t \mid (st)^3 = s^2 = 1 \rangle$$

§ 2 MODULAR CATEGORIES GIVE RISE TO REPRESENTATIONS OF MODULAR GROUPS

Recall the S and T matrices

$$S = (s_{XY})_{X,Y \in \mathcal{O}(\mathcal{C})} \qquad T = (t_{XY})_{X,Y \in \mathcal{O}(\mathcal{C})}$$

$$s_{XY} = \mathrm{tr}(c_{Y,X} \circ c_{X,Y}) \qquad t_{XY} = \delta_{X,Y}\, \theta_X^{-1}$$

[Thm 8.16.1, EGNO]
Let $\mathcal{C}$ be a modular category. We have $(ST)^3 = \tau^-(\mathcal{C})\, S^2$ and $S^4 = \dim(\mathcal{C})^2\, \mathrm{id}$, where id is the identity matrix.
Hence, the assignments

$$\mathfrak{s} \longmapsto S \quad \text{and} \quad \mathfrak{t} \longmapsto T$$

define a projective representation $\phi_{\mathcal{C}}$ of Γ.

(Note: after normalizing the T-matrix the representation can be made linear)

Proof: In Prop 8.14.2, we saw that

$$S^2 = \dim(\mathcal{C})\, E$$

where $E = (E_{xy})_{x,y \in \mathcal{O}(\mathcal{C})}$ and $E_{xy} = \begin{cases} 1 & x = y^* \\ 0 & \text{else} \end{cases}$

$$\therefore \quad S^4 = \dim(\mathcal{C})^2\, E^2$$

and $(E^2)_{x,y} = \sum_{z \in \mathcal{O}(\mathcal{C})} E_{x,z} E_{z,y} = \begin{cases} 1 & x = y \\ 0 & \text{else} \end{cases}$

$$\therefore E^2 = \mathrm{id}$$

$$\Rightarrow \boxed{S^4 = \dim(\mathcal{C})^2\, \mathrm{id}}$$

Let's prove the second relation

→ Consider the matrix $T^{-1}ST^{-1}$

$$(T^{-1}ST^{-1})_{XY} = \theta_X \, s_{XY} \, \theta_Y$$

→ Now consider the matrix STS

$$(STS)_{XY} = \sum_{V \in O(\mathcal{C})} s_{XV} \, \theta_V^{-1} \, s_{VY}$$

$$(ST)_{XY} = s_{XY} \, \theta_Y^{-1}$$

$\Bigg(s_{XV} = s_{VX} \quad \therefore \quad$ by Prop 8.13.10

$$s_{VX} s_{VY} = \dim(V) \sum_{W \in O(\mathcal{C})} N_{XY}^{W} \, s_{VW} \Bigg)$$

$$= \sum_{W \in O(\mathcal{C})} \left(\sum_{V \in O(\mathcal{C})} \theta_V^{-1} \dim(V) \, s_{VW} \right) N_{XY}^{W}$$

(using Cor 8.15.5)

$$= \sum_{W \in O(\mathcal{C})} \left(\tau^{-1}(\mathcal{C}) \dim(W) \, \theta_W \right) N_{XY}^{W}$$

$$= \tau^{-1}(\mathcal{C}) \sum_{W \in O(\mathcal{C})} \dim(W) \, \theta_W \, N_{XY}^{W}$$

$$\sum_{W \in O(\mathcal{C})} \dim(W)\, \theta_W\, N_{XY}^W = \sum_{W \in O(\mathcal{C})} \mathrm{Tr}(\theta_W \mathrm{id}_W)\, N_{XY}^W$$

$$= \mathrm{Tr}\Big(\bigoplus_{W \in O(\mathcal{C})} \theta_W^{\oplus N_{XY}^W} \Big)$$

$$= \mathrm{Tr}(\theta_{X \otimes Y})$$

$$= \mathrm{Tr}\big((\theta_X \otimes \theta_Y) \circ (c_{Y,X} \circ c_{X,Y})\big)$$

$$= \theta_X \theta_Y \; \mathrm{Tr}(c_{Y,X} \circ c_{X,Y})$$

$$= \theta_X \theta_Y \, s_{XY}$$

$$\therefore \quad (STS)_{XY} = \tau^{-1}(\mathcal{C})\, \theta_X \theta_Y\, s_{XY} = \tau^{-1}(\mathcal{C})\, (T^{-1} S T^{-1})_{XY}$$

$$\Rightarrow \quad STS = \tau^{-1}(\mathcal{C})\, T^{-1} S T^{-1}$$

$$\Rightarrow \quad \boxed{(ST)^3 = \tau^{-1}(\mathcal{C})\, S^2}$$

§ 3 CONGRUENCE SUBGROUP PROBLEM FOR $SL_2(\mathbb{Z})$

→ In general, a congruence subgroup of a matrix group with integer entries is a subgroup defined by congruence conditions on the entries.
(e.g. non diagonal entries should be $0 \pmod 3$)

→ The simplest interesting case is that of $SL_2(\mathbb{Z})$.

- using the morphism
$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathbb{Z}/n\mathbb{Z} \\ a &\longmapsto a \bmod n \end{aligned}$$

we get a morphism
$$\pi_n : SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/n\mathbb{Z})$$

$\Gamma(n) := \ker(\pi_n)$ are called the principal congruence subgroups of $SL_2(\mathbb{Z})$.

More precisely,

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{matrix} a, d \equiv 1 \pmod{n} \\ c, b \equiv 0 \pmod{n} \end{matrix} \right\}$$

→ A subgroup $H < SL_2(\mathbb{Z})$ is called a congruence subgroup of $SL_2(\mathbb{Z})$ if it contains some principal congruence subgroup $\Gamma(n)$.

→ It was conjectured that
The kernel of the representation ϕ_e of $SL_2(\mathbb{Z})$ coming from a MTC $\mathcal{C}$, i.e. $\ker(\phi_e)$ is a congruence subgroup of $SL_2(\mathbb{Z})$.

Some history:

(see [8.27.10, EGNO] for references)

→ There is a (precise?) connection between modular tensor categories and 2D Rational conformal field theories.

2003 Bantay first gave a physical argument in favor of the conjecture in the context of conformal field theory.

2006 Xu showed that its possible to make Bantay's arguments rigorous using the operator algebra approach to CFT.

However it is not known if an arbitrary MTC can be realized via a 2D CFT.

2007 Sommerhauser-Zhu introduced generalizations of Frobenius-Schur indicators and used these to prove the conjecture for the case $\mathcal{C} = \mathrm{Rep}(D(H))$ for H a semisimple Hopf algebra.

2010 Ng-Schauenberg extend the theory of Frobenius-Schur indicators to pivotal categories and use these to settle the conjecture in full generality.