FROBENIUS ALGEBRAS AND (2D) TQFTs

What is a TQFT?

a physical quantity that has a value at each point in space and time (Magnetic field)

TOPOLOGICAL QUANTUM FIELD THEORY

a description of how the field changes with respect to time and other variables (Maxwell's equations for magnetic field)

'QUANTIZE'

Classical electromagnetism field theory assume light travels as a wave. However photoelectric effect disagrees with this and shows that light has a particle nature → call this 'quantum phenomenon'

QFT is a field theory which takes into account various quantum phenomenon

TQFTs are QFTs where the theory does not depend on the metric of the spacetime, i.e., only topology matters.

As usual, mathematicians (in this case Atiyah) took various common features of TQFTs and turned them into axioms to define a TQFT

In this talk, two categories will be very important.

① The category of finite dimensional vector spaces over $\mathbb{C}$ (Vec)

Objects: f.d. vectors spaces

Morphisms: $\mathbb{C}$-linear maps

this category has more structure

given two vector spaces, we can form their $\otimes$-product

(i) $\otimes$: $V, W \in Vec \rightsquigarrow V \otimes W \in Vec$

(ii) $\otimes$ is 'associative'

$a_{U,V,W}: (U \otimes V) \otimes W \cong U \otimes (V \otimes W)$

(iii) We have a unit for tensor product : $\mathbb{C}$

$r_V: \mathbb{C} \otimes V \cong V$, $l_V: V \otimes \mathbb{C} \to V$

(iv) $a_{U,V,W}$, r_V, l_V satisfy certain coherence axioms.

(v) For $U, V \in Vec$,

$c_{U,V}: U \otimes V \cong V \otimes W$ (flip map)

and the maps $c_{U,V}$ are nicely compatible with the maps $a_{U,V,W}$, l_V, r_V.

categorification of a monoid

Categorification of Commutativity

(vi) $U \otimes V \xrightarrow{c_{U,V}} V \otimes U \xrightarrow{c_{V,U}} U \otimes V$ is identity map

(symmetric property)

Such a category (satisfying (i) – (vi)) is called a symmetric monoidal category.

Now lets discuss the second most important category of this talk.

② Cob_n (Cob ⇝ Cobordism) (all mflds will be smooth)

<u>Objects</u> : $(n-1)$ oriented closed manifolds Σ

<u>Morphisms</u> : $M: \Sigma_0 \to \Sigma_1$

is a n-mfld with boundary components Σ_0 and Σ_1.

<u>composition</u> : stacking pictures

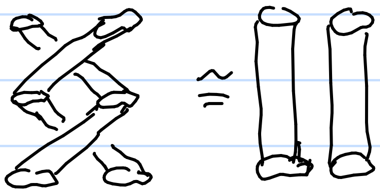
$\otimes$: $\Sigma_0 \otimes \Sigma_1 := \Sigma_0 \sqcup \Sigma_1$ (put next to each other

unit : The empty set consided as closed $(n-1)$ mfld

then $(\Sigma_0 \sqcup \Sigma_1) \sqcup \Sigma_2 = \Sigma_0 \sqcup (\Sigma_1 \sqcup \Sigma_2)$

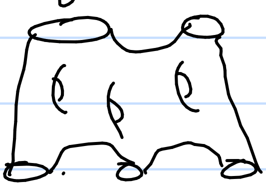
$$\Sigma_0 \sqcup \phi = \Sigma_0 = \phi \sqcup \Sigma_0$$

<u>commutativity</u> : $c_{0,1}: \Sigma_0 \sqcup \Sigma_1 \cong \Sigma_1 \sqcup \Sigma_0$ via cobordism

then c is symmetric : 

<u>Example</u> : Cob_2

<u>objects</u> : disjoint union of S^1

<u>morphisms</u> :  (read bottom to top)

<u>composition</u> : 

$O \otimes O = OO$

We have defined two symmetric monoidal. Functors between them which respect the extra symmetric monoidal structure are called symmetric monoidal categories

An n-dim'l TQFT is a symm. monoidal functor

$$Z: \mathrm{Cob}_n \longrightarrow \mathrm{Vec}_{\mathbb{C}}$$

being symmetric monoidal is categorification on being a monoid map.

monoidal functor means
$$\begin{cases} Z(\Sigma_1 \sqcup \Sigma_2) \cong Z(\Sigma_1) \otimes Z(\Sigma_2) \\ Z(\phi) \cong \mathbb{C} \end{cases}$$

symmetric means

Σ_1 Σ_0

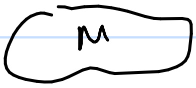
$Z\left(\right.$  $\left.\right)$: $Z(\Sigma_0) \otimes Z(\Sigma_1) \to Z(\Sigma_1) \otimes Z(\Sigma_0)$

Σ_0 Σ_1

is the flip map

Then closed closed n-mfld M

think of M

ϕ  ϕ

$$\left[\phi \xrightarrow{M} \phi \right] \xrightarrow{Z} \begin{matrix} (Z(\phi) \longrightarrow Z(\phi)) \\ \parallel \\ (\mathbb{C} \xrightarrow{Z(M)} \mathbb{C}) = \mathbb{C} \end{matrix}$$

using M, Z get $\mathbb{C}$ linear map $\mathbb{C} \to \mathbb{C}$
every such map is mult. by a scalar $Z(M)$

$M \underset{\text{diffeo}}{\cong} M'$ then $Z(M) = Z(M')$

thus we get invariants of smooth manifolds.

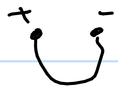
(ASK FOR QUESTIONS)

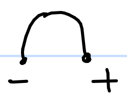
Examples 1-D TQFTs

Given $Z : \mathrm{Cob}_1 \longrightarrow \mathrm{Vec}$, have

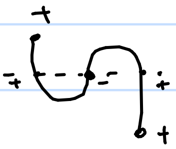
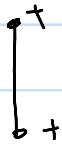
Objects $\begin{cases} \bullet^{+} \longmapsto V \\ \bullet^{-} \longmapsto W \end{cases}$

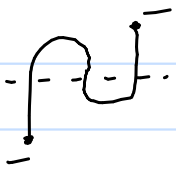
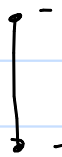
Morphisms

 $\xmapsto{\text{coev}}$ $\mathbb{C} \to V \otimes W$

 $\xmapsto{\text{ev}}$ $W \otimes V \to \mathbb{C}$

these satisfy

(*)  $=$  $\longmapsto \left(\mathbb{C} \otimes V = V \xrightarrow{\text{coev} \otimes id} V \otimes W \otimes V \xrightarrow{id \otimes \text{ev}} V \otimes \mathbb{C} = V \right) = id_V$

(**)  $=$  $\longmapsto \left(W \otimes \mathbb{C} \to W \otimes V \otimes W \to \mathbb{C} \otimes W \right) = id_V$

(Linear alg exercise) (*) + (**) $\Rightarrow$ V is a finite diml. and $W \cong \mathrm{Hom}(V, \mathbb{k}) = V^*$

with $\text{ev} : W \otimes V \to \mathbb{k}$, $f \otimes v \mapsto f(v)$ | $\text{coev} : \mathbb{k} \to V \otimes W$, $1 \mapsto \sum e_i \otimes e_i^*$

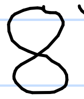
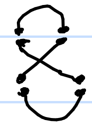
Thus Z determines a f.d. v.s. V

Converse is also true

$\{\text{1-D TQFTs}\} \longleftrightarrow \{\text{f.d. vector spaces}\}$

Invariants?

1 D TQFTs $\rightsquigarrow$ invariants of 1-D closed mflds

$Z($$) = Z($$) = Z($$) = \text{ev} \circ \text{flip} \circ \text{coev}(1)$

$= \text{ev} \circ \text{flip}\left(\sum_{i=0}^{\dim(V)} e_i \otimes e_i^* \right)$

$= \text{ev}\left(\sum e_i^* \otimes e_i \right) = \sum e_i^*(e_i) = \dim(V)$

2-D TQFTs

given $Z : Cob_2 \longrightarrow Vec_{\mathbb{C}}$

objects  $\longmapsto V$

morphisms

 $\longmapsto \left(V\otimes V \xrightarrow{m} V \right)$

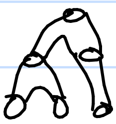
Disk  $\longmapsto \left(\mathbb{C} \xrightarrow{u} V \right)$

 $\longmapsto \left(V \xrightarrow{\Delta} V\otimes V \right)$

 $\longmapsto \left(V \xrightarrow{\varepsilon} \mathbb{C} \right)$

(Fact: Using Morse theory one can show that every morphism in Cob_2 is made up by compositions of these 4 morphisms & )

In Cob_2 there are diffeomorphic morphisms

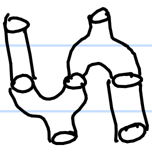
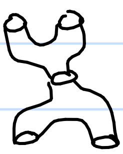
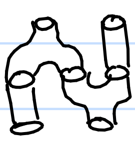
 $\cong$  $\longmapsto$

$$\begin{tikzcd} & V & \\ V\otimes V \arrow[ur, "m"] & & V\otimes V \arrow[ul] \\ & V\otimes V\otimes V \arrow[ul, "id\otimes m"] \arrow[ur, "m\otimes id"'] & \end{tikzcd}$$

 $\cong$  $\cong$  $\longmapsto$ m is unital with unit u

$\rightarrow$ (Δ, ε) are coassociative counit

 $\cong$  $\rightarrow$ m is commutative

 $=$  $=$  $\rightarrow$ Frobenius conditions

$(V, m, u, \Delta, \varepsilon)$ is a commutative Frobenius algebra.

In fact

$$\{\text{2D TQFTs}\} \longleftrightarrow \{\text{commutative Frobenius algebras}\}$$

Invariants? (for closed 2-mflds)

<u>Example</u>: Take $V = \dfrac{\mathbb{k}[t]}{(t^2-1)}$ (Group algebra of cyclic group with two elements)

$m, u = 1$

$\Delta : V \longrightarrow V \otimes V$

$t \longmapsto t \otimes t \quad , \quad 1 \mapsto 1 \otimes 1 + t \otimes t$

$\varepsilon : V \longmapsto \mathbb{C}$

$t \longmapsto 0 \quad , \quad 1 \longmapsto 1$

Then $(V, m, u, \Delta, \varepsilon)$ is a comm. Frob. algebra

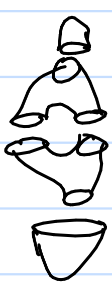
$Z\big($  $\big) = \quad \underset{Z(\emptyset)}{\overset{\mathbb{k}}{\shortparallel}} \xrightarrow{Z(\text{cup}) = u} \underset{Z(\bigcirc)}{\overset{A}{\shortparallel}} \xrightarrow{Z(\text{cap}) = \varepsilon} \mathbb{k}$

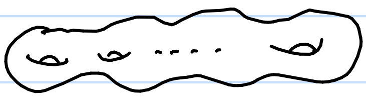
$1 \longmapsto 1 \overset{\varepsilon}{\longmapsto} 1$

$\therefore Z(\text{sphere}) = 1$

$Z\big($  $\big) = \mathbb{k} \xrightarrow{u} A \xrightarrow{\Delta} A \otimes A \xrightarrow{m} A \xrightarrow{\varepsilon} \mathbb{k}$

$1 \longmapsto 1 \longmapsto 1 \otimes 1 + t \otimes t \longmapsto 1 + 1 = 2 \longmapsto 2$



In fact, $Z\big($  $\big) = 2^g$

genus g

3-D TQFTs

~1985 Jones polynomial

takes knot $\longrightarrow$ assigns polynomial via skein relations

$$\times = q \asymp + q^{-1})($$

gives invariant of knots

'VERY BIG DEAL'

~1989 Atiyah asks if there is intrinsic 3-dim'l defn of Jones poly.

~1991 Reshetikhin – Turaev define a 3D TQFT

$$Z^{H}_{RT} : Cob_3 \longrightarrow Vec.$$

takes as input a modular Hopf algebra H.

In particular get invariants of 3-mflds

When $H = \mathcal{U}_q(sl_2)$

then $Z^{\mathcal{U}_q(sl_2)}_{RT}(S^3 \setminus K)$ = Jones poly. of K.