SNAD March 2023

UNIMODULAR COMODULE ALGEBRAS

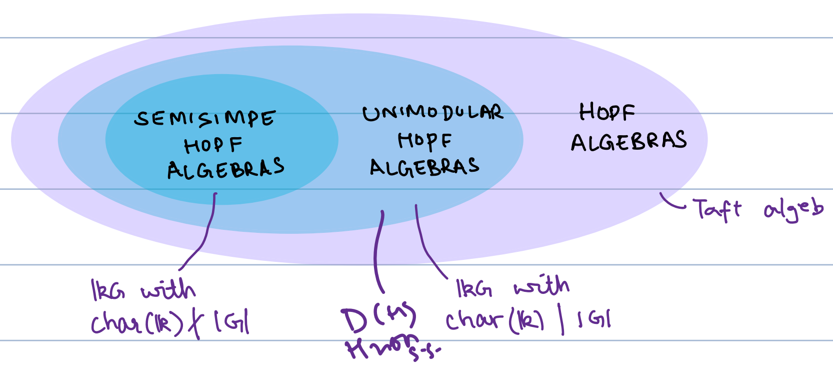
(based on 2302.06192)

- An integral matrix A is called unimodular if $\det(A) = \pm 1$.
- Using this idea, unimodularity of lattices, bilinear forms, topological groups, Poisson algebras, Hopf algebras, tensor categories is defined.
- Today's focus is on Hopf algebras and tensor categories.

$(H, m, u, \Delta, \varepsilon, S) =$ f.d. Hopf algebra over $\Bbbk$ (alg. closed)

Defn:
- A left integral is an element $\Lambda^{\ell} \in H$ satisfying $h\Lambda^{\ell} = \varepsilon(h)\Lambda^{\ell} \quad \forall\, h \in H$.
- A right integral is an element $\Lambda^{r} \in H$ satisfying $\Lambda^{r} h = \varepsilon(h)\Lambda^{r} \quad \forall\, h \in H$.
- Given a left integral Λ^{ℓ} $\exists$ $\alpha : H \to \Bbbk$ (distinguished character) such that $\Lambda^{\ell} h = \langle \alpha, h \rangle \Lambda^{\ell}$

We call H unimodular if $\alpha = \varepsilon$, i.e. Λ^{ℓ} is also a right integral.



• H f.d. Hopf algebra $\rightsquigarrow$	• $\mathcal{C} = \mathrm{Rep}(H)$ finite tensor category $(\mathcal{C}, \otimes, \mathbb{1} = \Bbbk_\varepsilon)$
• α distinguished char. $\rightsquigarrow$	• $D = \Bbbk_\alpha \in \mathrm{Rep}(H)$ $\hookrightarrow \begin{cases} = \Bbbk \text{ as vector spaces} \\ h \cdot c := \alpha(h)c \text{ for } c \in \Bbbk \end{cases}$ distinguished invertible object of $\mathcal{C}$.
• H unimodular $\rightsquigarrow$	• $\Bbbk_\alpha = \Bbbk_\varepsilon$

__Defn:__ A finite tensor category $\mathcal{C}$ is called unimodular if $D \cong \mathbb{1}$.

__Defn:__ A left H-comodule algebra is an algebra A with H-comodule structure $\rho : A \to H \otimes A$ so that ρ is an algebra map, i.e.

$\rho(a) = a_{-1} \otimes a_0$

$$\rho(a a') = a_{-1} a'_{-1} \otimes a_0 a'_0$$
$$\rho(1_A) = 1_H \otimes 1_A$$

__Continuing our table__

Left H-comodule algebras A $\rightsquigarrow$	yield functors $\triangleright : \mathrm{Rep}(H) \otimes \mathrm{Rep}(A) \to \mathrm{Rep}(A)$ $X \qquad M \mapsto X \otimes M$ $A \curvearrowright X \otimes M$ $a \cdot (x \otimes m) := a_{-1} \cdot x \otimes a_0 \cdot m$ this turns $(\mathrm{Rep}(A), \triangleright)$ into a left $\mathrm{Rep}(H)$-module category.
Left H-comodule alg. A is called exact if $\mathrm{Rep}(A)$ $\leftsquigarrow$	The module category $\mathrm{Rep}(A)$ is called exact if $\forall M \in \mathrm{Rep}(A)$

is exact $\mathrm{Rep}(H)$-module category.

& projective H-module X. $X \otimes M$ is projective A-module.

Tensor categories important. Can use module categories to get new ones

$\mathcal{C} = \mathrm{Rep}(H)$, $\mathcal{M} = \mathrm{Rep}(H)$

$\mathrm{End}_{\mathrm{Rep}(H)}(\mathrm{Rep}(A)) = {}^{H}_{A}\mathcal{M}_{A}$

category of A-bimodules with compatible H-coaction

$\mathcal{C}$ tensor cat, $\mathcal{M}$ exact module cat

$\mathrm{End}_{\mathcal{C}}(\mathcal{M}) =$ $\mathcal{C}$-module functors $\mathcal{M} \to \mathcal{M}$

This is a new tensor category.

Call an exact H-comodule algebra A unimodular if the tensor category ${}^{H}_{A}\mathcal{M}_{A}$ is unimodular

Call $\mathcal{M}$ unimodular if $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ is a unimodular tensor category.

$\boxed{\text{E.g. } {}^{H}_{H}\mathcal{M}_{H} = \mathrm{Rep}(H^{\mathrm{cop}}), \quad {}^{H}_{\Bbbk}\mathcal{M}_{\Bbbk} = \mathrm{Rep}(H^{*}) \quad + \text{Drinfeld twists}}$

Why you might care about unimodularity

Let $A = \Bbbk$ be the H-comodule alg.

Then ${}^{H}_{A}\mathcal{M}_{A} \cong \mathrm{Rep}(H^{*})$

$\therefore$ $\underset{= \mathrm{Vect}_{\Bbbk}}{\mathrm{Rep}(\Bbbk)}$ is unimodular $\iff$ $\Bbbk$ is unimodular H-com. alg

$\iff$ ${}^{H}_{\Bbbk}\mathcal{M}_{\Bbbk} \cong \mathrm{Rep}(H^{*})$ is unimodular

$\iff$ H^{*} is unimodular Hopf alg.

Main result ① : An exact H-comodule alg A is unimodular element $\iff$ A admits a unimodular element.

Rmk: By Skryabin's result, exact H-comodule algebras are Frobenius. Let ν denote the Nakayama aut & $\{a^i\}\{b_i\}$ two bases s.t. $\langle \lambda_A, a^i b_j\rangle = \delta_{ij}$
λ_A Frob form

Defn: A unimodular element of A is an invertible element $\tilde{g} \in A$ such that

① $\tilde{g}\, a\, \tilde{g}^{-1} = \langle \alpha_H, S(a_{-1})\rangle\, \nu^2(a_0) \quad \forall a \in A$

② $1_H \otimes \tilde{g} = \tau \cdot \rho(\tilde{g})$

where

$$\tau = \langle \lambda_A . a^i_0\rangle \langle \lambda_A . a^j_0\rangle\, g_H\, S^{-3}(a^j_{-1})\, S^{-1}(a^i_{-1}) \otimes \nu(b_j b_i)$$
$\in H \otimes L$

Special cases

i) $A = \mathbb{k}$, then

① becomes redundant

② becomes $1_H \otimes g_H = \rho(\tilde{g}) = 1 \otimes \tilde{g}$ but $\tilde{g} = 1$

$\Rightarrow \boxed{g_H = 1}$

Question: Is there a way to define integrals for H comodule algebras and use them to characterize unimodularity?

Main result ②: Established multiple characterizations of unimodular module categories.

Thm: Let $\mathcal{C}$ be a finite tensor category. Let $\mathcal{M}$ be an indecomposable, exact left $\mathcal{C}$-module category. Then TFAE

(i) $\mathcal{M}$ is a unimodular $\mathcal{C}$-module category

(ii) $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$ is a unimodular finite tensor cat.

(iii) $\mathbb{S}_{\mathcal{M}} \circ \mathbb{N}_{\mathcal{M}} \cong \mathrm{id}_{\mathcal{M}}$ as a $\mathcal{C}$-module functor.

(iv) Consider functor $\varphi : \mathcal{Z}(\mathcal{C}) \to \mathrm{End}_{\mathcal{C}}(\mathcal{M})$
$\varphi^{ra}(\mathrm{id}_{\mathcal{M}})$ is a Frobenius algebra in $\mathcal{Z}(\mathcal{C})$.

(v) φ^{ra} is a Frobenius monoidal functor.

→ For $\mathcal{C} = \mathrm{Rep}(H)$, $\mathcal{M} = \mathrm{Rep}(A)$ case

$$\varphi : {}^{H}_{H}\mathcal{YD} \longrightarrow {}^{H}_{A}\mathcal{M}_{A}$$

We also know φ^{ra} and can use it to calculate the algebra $\varphi^{ra}(A)$ in ${}^{H}_{H}\mathcal{YD}$.

→ This can be used to attack a weak version of Kaplansky's 5th conjecture (H s.s. or coss $\Rightarrow S^2 = \mathrm{id}_H$)

H f.d. Hopf cosemisimple $\Rightarrow$ H unimodular

this is equivalent to

H f.d. semisimple Hopf $\Rightarrow$ H^* unimodular

To show this, suffices to show that
$\mathrm{Vect}_{\mathbb{k}}$ is unimodular $\mathrm{Rep}(H)$-module category
(thus need ①, ②)