On non-counital Frobenius algebras

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(joint work with Amanda Hernandez and Chelsea Walton)

Motivation

History

- An *algebra* is a triple $(A, m : A \otimes A \to A, u : \Bbbk \to A)$ such that m is associative and unital.
- A is a right A-module under multiplication

  $\Rightarrow A^*$ is a left A module.
- Frobenius 1903 studied (using the language of linear algebra) when $A \cong A^*$ as a left A module.
- Brauer and Nesbitt 1937 named such algebras as *Frobenius algebras*.
- Nakayama 1939 analyzed the rich duality structure and gave new equivalent definitions.
- Later Eilenberg, Thrall, Dieudonné made contributions.

Examples: Group algebras, exterior algebras, Hopf algebras

History

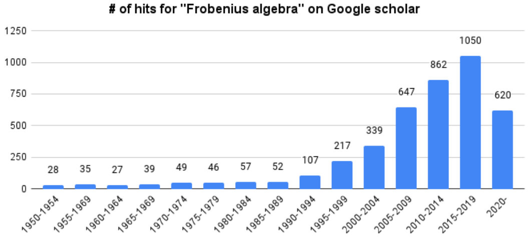
The same group of mathematicians parallelly also developed the theory of quasi-Frobenius algebras.

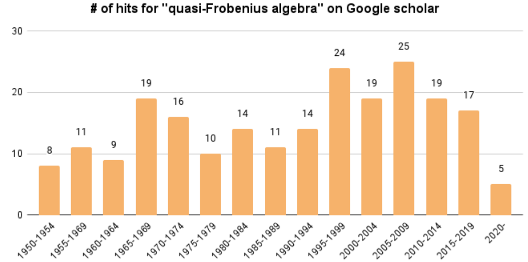
- An algebra A is called *quasi-Frobenius* is A is injective as a left A-module.
- Equivalently, A and A^* have the same indecomposable summands.
- Thus Frobenius algebras are quasi-Frobenius.

<u>Example</u>:

- Groupoid algebras ($\Bbbk\mathcal{G}$)
    - $\mathcal{G}$: Obj($\mathcal{G}$) $= \{1, \ldots, n\}$, all morphisms are invertible
    - $\Bbbk\mathcal{G}$:=vector space spanned by Mor($\mathcal{G}$)
    - Product: given by composition
    - Unit: $1_{\Bbbk G} = \sum_{i=1}^{n} \mathrm{id}_i$.
- Weak Hopf algebras

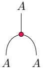
Frobenius vs Quasi-Frobenius



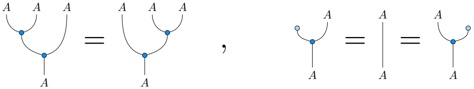


What happened in the 1990s?

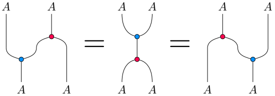
Quinn 1995, Abrams 1996 gave a new defintion of Frobenius algebras.
A Frobenius algebra is a 5-tuple $(A, m, u, \Delta : A \to A \otimes A, \varepsilon : A \to \mathbb{k})$ such that

- $(A,\ m =$  $, u =$  $)$ is an algebra

- $(A,\ \Delta =$  $\varepsilon =$  $)$ is a coalgebra



- Frobenius axiom is satisfied

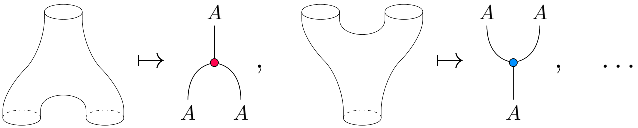


Connection to TQFTs

Commutative Frobenius algebras allow us to construct functors

$$\mathcal{Z} : \text{Category of } 2\text{D shapes} \longrightarrow \text{Vector spaces}$$

Given a commutative Frobenius algebra A, we can define a functor $\mathcal{Z}_A$ which sends the shapes



In this way, closed 2D manifolds map to endomorphism of $\Bbbk$. Thus, we get invariants of manifolds.

Quasi-Frobenius algebras

Question: Do quasi-Frobenius algebras admit a comultiplication which satisfies the Frobenius law?

Definition

An algebra (A, m, u) equipped with a map $\Delta : A \to A \otimes A$ is called *non-counital Frobenius* if:

- Δ is coassociative
- Δ satisfies the Frobenius axiom

We investigate whether Quasi-Frobenius algebras are non-counital Frobenius. In particular we get that

Theorem (Hernandez-Walton-Y.)

The following classes of quasi-Frobenius algebras are non-counital Frobenius:

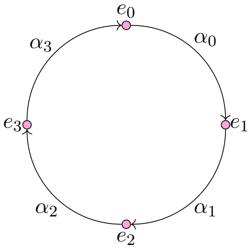
1. *Nakayama-Skowroński-Yamagata (NSY) algebras*
2. *Weak Hopf algebras*

Nakayama-Skowroński-Yamagata algebras

NSY algebras

- Take $n \in \mathbb{Z}_+$ and $1 \leq \ell \leq n-1$.
- $Q_{(n)}$: n-cycle quiver with
  - vertex set $Q_0 = \{e_0, e_1, \ldots, e_{n-1}\}$
  - arrow set $Q_1 = \{\alpha_i : e_i \to e_{i+1}\}_{i=0,\ldots,n-1}$
- R: arrow ideal of the path algebra $\Bbbk Q$
- $\mathcal{I}_\ell$: be the admissible ideal R^ℓ of $\Bbbk Q$
- $B_{n,\ell} := \Bbbk Q_{(n)} / \mathcal{I}_\ell$
- Basis: $\{\alpha_{i,k} := \alpha_i \; \alpha_{i+1} \; \cdots \; \alpha_{i+k}\}_{i,k}$
- Indec. right $B_{n,\ell}$ modules: $\{P_i = e_i B_{n,\ell}\}_i$

Example: $n = 4, \; l = 1$



$I^1 = R = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$
Basis $= \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}$

Definition (NSY algebras)

$B_{n,l}(m_0, \ldots, m_{n-1}) := \mathrm{End}_{B_{n,l}}(P_0^{\oplus m_0} \oplus \ldots \oplus P_{n-1}^{m_{n-1}})$

NSY algebras

We obtain the following about the NSY algebras:

- Nakayama permutation: $\nu(i) = (i + l - 1) \bmod n$
- Basis: $\{X_{i,j}^{r_i,s_j}\}_{i,j,r_i,s_{i+j}}$
  $X_{i,j}^{r_i,s_{i+j}} : P_{i+j}^{s_{i+j}} \longrightarrow P_i^{r_i}, \quad \alpha_{i+j,k}^{s_{i+j}} \mapsto (\alpha_{i,j} \cdot \alpha_{i+j,k})^{r_i} = \alpha_{i,j+k}^{r_i}$
- Dimension: $\sum_{i=0}^{n-1} \sum_{j=0}^{\ell-1} m_i m_{i+j}$
- Product: $X_{i,j}^{r_i,s_{i+j}} \cdot X_{a,b}^{r_a,s_{a+b}} = \begin{cases} \delta_{a,i+j}\ \delta_{r_a,s_{i+j}}\ X_{i,j+b}^{r_i,s_{a+b}}, & \text{for } j+b<\ell, \\ 0, & \text{else;} \end{cases}$
- Unit: $1_A = \sum_{i=0}^{n-1} \sum_{r_i=0}^{m_i-1} X_{i,0}^{r_i,r_i}$
- $B_{n,l}(m_0, \ldots, m_{n-1})$ is quasi-Frobenius (Skowroński and Yamagata 2006).
- $B_{n,l}(m_0, \ldots, m_{n-1})$ is Frobenius $\iff$ $m_i = m_{\nu(i)}$ for all $1 \leq i \leq n$
  - $B_{2,2}(2,1)$ is not Frobenius ($\nu(i) = i + 1$)
  - $B_{4,3}(1,2,1,2)$ is Frobenius ($\nu(i) = i + 2$)
  - $B_{4,3}(1,2,1,1)$ is not Frobenius ($\nu(i) = i + 2$)

Non-counital comultiplication

Theorem (Hernandez-Walton-Y.)

- $B_{n,l}(m_0, \dots, m_{n-1})$ *is non-counital Frobenius with*

$$\Delta(X_{i,j}^{r_i, s_{i+j}}) =$$

$$\sum_{k=0}^{\ell-1-j} \sum_{t_{i+j+k}=0}^{m_{i+j+k}-1} \sum_{t_{i+j+k-\ell+1}=0}^{m_{i+j+k-\ell+1}-1} \left(1 - \delta_{m_{i+j+k}, m_{i+j+k-\ell+1}}(1 - \delta_{t_{i+j+k}, t_{i+j+k-\ell+1}})\right)$$

$$\cdot X_{i,j+k}^{r_i, t_{i+j+k}} \otimes X_{i+j+k-\ell+1, \ell-1-k}^{t_{i+j+k-\ell+1}, s_{i+j}}.$$

- (A, Δ) *is Frobenius precisely when* $m_i = m_{i-\ell+1}$ *for all* $i = 0, \dots n-1$*; in which case,*

$$\varepsilon(X_{i,j}^{r_i, s_{i+j}}) = \delta_{j, \ell-1}\, \delta_{r_i, s_{i+j}}\, 1_{\Bbbk}$$

is the counit of Δ*.*

Weak Hopf algebras

Weak Hopf algebras

Generalization of Hopf algebras.

Definition

A *weak Hopf algebra* over $\Bbbk$ is a quintuple $(H, m, u, \Delta_{\mathsf{wk}}, \varepsilon_{\mathsf{wk}}, S)$ such that

1. (H, m, u) is a $\Bbbk$-algebra,
2. $(H,\ \Delta_{\mathsf{wk}},\ \varepsilon_{\mathsf{wk}})$ is a $\Bbbk$-coalgebra,
3. $\Delta_{\mathsf{wk}}(ab) = \Delta_{\mathsf{wk}}(a)\,\Delta_{\mathsf{wk}}(b)$ for all $a, b \in H$,
4. $\varepsilon_{\mathsf{wk}}(abc) = \varepsilon_{\mathsf{wk}}(ab_1)\,\varepsilon_{\mathsf{wk}}(b_2c) = \varepsilon_{\mathsf{wk}}(ab_2)\,\varepsilon_{\mathsf{wk}}(b_1c)$ for all $a, b, c \in H$,
5. $\Delta^2_{\mathsf{wk}}(1_H) = (\Delta_{\mathsf{wk}}(1_H) \otimes 1_H)(1_H \otimes \Delta_{\mathsf{wk}}(1_H)) = (1_H \otimes \Delta_{\mathsf{wk}}(1_H))(\Delta_{\mathsf{wk}}(1_H) \otimes 1_H)$.
6. $S : H \to H$ is a $\Bbbk$-linear map satisfying

$$S(h_1)h_2 = \varepsilon_s(h), \qquad h_1S(h_2) = \varepsilon_t(h), \qquad S(h_1)h_2S(h_3) = S(h).$$

Weak Hopf algebras

Example: Groupoid algebras $\Bbbk\mathcal{G}$ is weak Hopf with

$$\Delta_{\mathsf{wk}}(g) = g \otimes g, \quad \varepsilon_{\mathsf{wk}}(g) = 1, \quad S(g) = g^{-1}$$

Definition

An element Λ in H is called a *left integral* if $h\Lambda = \varepsilon_t(h)\Lambda$ for all $h \in H$, where $\varepsilon_t(x)_{=}\varepsilon_{\mathsf{wk}}(1_1 x)$. An integral Λ is called *non-degenerate* if the linear map

$$\Psi_\Lambda : H^* \to H,\ \phi \mapsto \phi(\Lambda_1)\Lambda_2$$

is a bijection.

Böhm et al. 1999 proved the following for a weak Hopf algebra H:

1. H is self-injective.
2. H is Frobenius if and only if H has a non-degenerate left integral λ.

Results

Theorem (Hernandez-Walton-Y.)

Let H be a weak Hopf algebra. Then the following statements hold.

1. *H is non-counital Frobenius with comultiplication Δ.*
2. *Δ is counital if and only if H is Frobenius (e.g., precisely when H has a non-degenerate integral).*

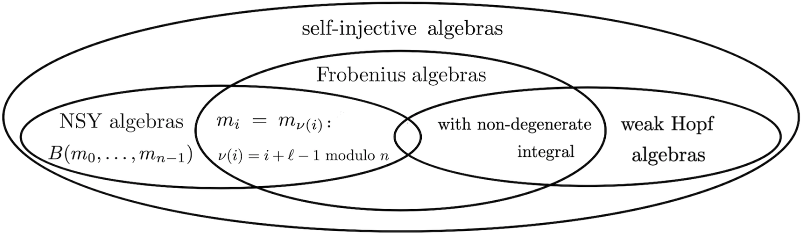
- Every weak Hopf algebras admits a non-zero left integral Λ.
- Comultiplication: $\Delta(h) := \Lambda_1 \otimes S(\Lambda_2)h$
- When Λ is non-degenerate, $\exists\ \lambda \in H^*$ such that $\lambda(\Lambda_1)\Lambda_2 = 1_H$. The, the counit is given by λ.

Example: Groupoid algebra $\Bbbk\mathcal{G}$.

- $\Lambda = \sum_{g \in \mathcal{G}} g$ is a non-degenerate left integral, and

- $\Delta(g) = \sum_{g \in \mathcal{G}} g \otimes (g^{-1} \cdot h),\ \lambda(h) = 1$ if $h = e_i,\ 0$ else.

Conclusion



- Are all quasi-Frobenius algebras non-counital Frobenius?
- What are the physical/topological uses of non-counital Frobenius algebras, akin to the use of Frobenius algebras and $2D$ TQFTs?

Thank you!

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