\[ \begin{aligned} x' &= 2x - 2x^2 - xy \\ y' &= 2y - xy - 2y^2 \end{aligned} \]

- **x-nullclines:** \( 2x - 2x^2 - xy = 0 \); \( x(2 - 2x - y) = 0 \).
  We get two lines: \( x = 0 \) and \( 2x + y - 2 = 0 \).

- **y-nullclines:** \( 2y - xy - 2y^2 = 0 \); \( y(2 - x - 2y) = 0 \).
  We get two lines: \( y = 0 \) and \( x + 2y - 2 = 0 \).

**Equilibrium points:** \( (0,0) \), \( (0,1) \), \( (1,0) \), \( (\frac{2}{3}, \frac{2}{3}) \).

**Jacobian** is:
\[
J(x, y) = \begin{bmatrix} 2 - 4x - y & -x \\ -y & 2 - x - 4y \end{bmatrix}
\]

For \( (0,0) \):
\[
J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}
\]
Eigenvalue \( \lambda = 2 \) (repeated root), 
\( (0,0) \) is a degenerate nodal source for the linearized system, so its type for the non-linear system cannot be determined. (pplane shows it to be a nodal source)

For \( (0,1) \):
\[
J(0,1) = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}
\]
Eigenvalues -2, 1; saddle point

For \( (1,0) \):
\[
J(1,0) = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}
\]
Eigenvalues -2, 1; saddle point

For \( (\frac{2}{3}, \frac{2}{3}) \):
\[
J\left(\frac{2}{3}, \frac{2}{3}\right) = \begin{bmatrix} -\frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{4}{3} \end{bmatrix}
\]
Positive eigenvalues \( \Rightarrow \) nodal sink
\[ x' = (2 - 2x - y)x \]
\[ y' = (2 - x - 2y)y \]

\((2/3, 2/3)\) is a nodal sink for the nonlinear system as determined from the linearization method.

\[ u' = Au + Bv \]
\[ v' = Cu + Dv \]

\[ A = -1.3333 \quad B = -0.66667 \]
\[ C = -0.66667 \quad D = -1.3333 \]
\[ x' = 1.2x - xy \]
\[ y' = -0.5y + xy \]

Equilibrium points: \((0,0); (0.5,1.2)\)

\[ J(x, y) = \begin{bmatrix}
1.2 - y & -x \\
-0.5 + x & y
\end{bmatrix} \]

For \((0,0)\):
\[ J(0,0) = \begin{bmatrix}
1.2 & 0 \\
0 & -0.5
\end{bmatrix} \quad \lambda_1 = 1.2, \quad \lambda_2 = -0.5 \Rightarrow \text{saddle point} \]

For \((0.5,1.2)\):
\[ J(0.5,1.2) = \begin{bmatrix}
0 & -0.5 \\
1.2 & 0
\end{bmatrix} \quad \lambda = \pm i \sqrt{0.6} \]

The linearization gives us a center, hence the type of \((0.5,1.2)\) cannot be determined.
\( x' = 1.2x - xy \)
\( y' = -0.5y + xy \)

\((0.5, 1.2)\) seems to be a center for the nonlinear system, same type as for linearized system.
\[ x' = (x + 9y)(1-y) \]
\[ y' = -x - 5y \]

Equilibrium points: 
\[ (x + 9y)(1-y) = 0 \]
\[ -x - 5y = 0 \]
\[ \Rightarrow (0,0) \text{ and } (-5,1) \]

\[ J(x, y) = \begin{bmatrix} 1-y & 9-18y-x \\ -1 & -5 \end{bmatrix} \]

For \((0,0)\): 
\[ J(0,0) = \begin{bmatrix} 1 & 9 \\ -1 & -5 \end{bmatrix} \]
Eigenvalue \(\lambda = -2\) (repeated)

The linearized system gives us a degenerate nodal sink, hence the type of \((0,0)\) cannot be determined.
However, because \(\lambda = -2 < 0\), the equilibrium point \((0,0)\) is asymptotically stable.

For \((-5,1)\): 
\[ J(-5,1) = \begin{bmatrix} 0 & -4 \\ -1 & -5 \end{bmatrix} \]
Saddle point, unstable

\[ z' = y - z \]
\[ y' = x - z \]
\[ z' = x^2 + y^2 - 2z \]

Equilibrium point \((0,0,0)\)

\[ J(x, y, z) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 2x & 2y & -2 \end{bmatrix} \]
\[ J(0,0,0) = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -2 \end{bmatrix} \]

Characteristic polynomial \(p(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 0 & 0 & -2-\lambda \end{bmatrix} \]
\[ = - (\lambda+1)(\lambda+2)(\lambda-1) \]

We have a positive eigenvalue, hence \((0,0,0)\) is unstable.
21. a) \( x_1' = r_1 x_1 \) where \( r_1 = -a_1 + b_1 x_2 \)

\((-a_1 \text{ because } x_1 \text{ will die out in the absence of } x_2\)  
\((b_1 x_2 \text{ because } x_2 \text{ is a prey for } x_1)\)

\(x_2' = r_2 x_2\)  
\(r_2 = -a_2 - b_2 x_1 + c_2 x_3\)

\((-a_2 \text{ because } x_2 \text{ dies out in the absence of } x_2\)  
\((-b_2 x_1 \text{ because } x_1 \text{ preys on } x_2\)  
\(+c_2 x_3 \text{ because } x_2 \text{ preys on } x_3\)

\(x_3' = r_3 x_3\)  
\(r_3 = a_3 - b_3 x_2 - c_3 x_3\)

\(a_3 - c_3 x_3 \text{ is the logistic growth of } x_3\)  
\(-b_3 x_2 \text{ because } x_2 \text{ preys on } x_3\)

b) \( x_1' = (-1+2x_2)x_1\)  
\(x_2' = (-0.5 - 2x_1 + 4x_3)x_2\)  
\(x_3' = (2-x_2 - 2x_3)x_3\)

The positive equilibrium is obtained from:

\[-1+2x_2 = 0\]
\[-0.5 - 2x_1 + 4x_3 = 0\]
\[-2 - x_2 - 2x_3 = 0\]

\[x_1 = \frac{5}{4}, \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{3}{4}\]

\[J(x_1, x_2, x_3) = \begin{bmatrix} -1+2x_2 & 2x_1 & 0 \\ -2x_2 & -\frac{1}{2} - 2x_1 + 4x_2 & 4x_2 \\ 0 & -x_3 & 2 - x_2 - 4x_3 \end{bmatrix} \]

and \(J\left(\frac{5}{4}, \frac{1}{2}, \frac{3}{4}\right) = \begin{bmatrix} -1 & \frac{5}{4} & 0 \\ 0 & -\frac{1}{2} & 2 \\ 0 & -\frac{3}{4} & -\frac{3}{2} \end{bmatrix}\)

The eigenvalues are \(-1.0610, -0.2195 \pm 1.8671i\).  
All have negative real part \(\Rightarrow (\frac{5}{4}, \frac{1}{2}, \frac{3}{4})\) is asymptotically stable.
\[ x' = (2-x-y) x \]
\[ y' = (3-3x-y) y \]

Notice that \( x = 0 \) (y-axis) is an x-nullcline
\[ y = 0 \] (x-axis) is a y-nullcline

The direction vectors on the y-axis are vertical, hence the y-axis is invariant. The direction vectors on the x-axis are horizontal, hence the x-axis is invariant.

Because of this, any solution that starts in one of the four quadrants stays in that quadrant (the uniqueness theorem tells us that solution curves do not intersect).

B.3 10.3. Similar reasoning.

B.9 10.3. x-nullclines: \( x = 0 \), \( 2x-y = 0 \)
\[ y \text{-nullclines: } y = 0 \text{, } 3-3x-y = 0 \]
Equilibrium points: (0,0), (2,0), (0,3), \( \left( \frac{1}{2} , \frac{3}{2} \right) \)

\[
J(x,y) = \begin{bmatrix}
2-2x-y & -x \\
-3y & 3-3x-2y
\end{bmatrix}
\]
Linearization gives us that:

- \((0,0)\) - nodal source
- \((2,0)\) - nodal sink
- \((0,3)\) - nodal sink
- \((\frac{1}{2}, \frac{3}{2})\) - saddle point

On the \(x\)-axis (y-nullcline) the direction vector is \([2-x, 0]\).

- If \(0 < x < 2\) then \((2-x) > 0\) so it points to the right.
- If \(x > 2\) then \((2-x) < 0\) so it points to the left.

On the \(y\)-axis (x-nullcline) the direction vector is \([0, 3-y]\).

- If \(0 < y < 3\) then \((3-y) > 0\) so it points up.
- If \(y > 3\) then \((3-y) < 0\) so it points down.

For the \(x\)-nullcline \(2-x-y = 0\) the direction vector is

\[
[0, (3-3x-y) y] = [0, (3-3x-2x)(2-x)]
= [0, (1-2x)(2-x)]
\]

- If \(0 < x < \frac{1}{2}\) then \((-2x)(2-x) > 0\) so it points up.
- If \(\frac{1}{2} < x < 2\) then \((-2x)(2-x) < 0\) so it points down.

For the \(y\)-nullcline \(3-3x-y = 0\) the direction vector is

\[
[(2-x-y)x, 0] = [(2-x-3+3x)x, 0] =
= [(2x-1)x, 0].
\]

- If \(0 < x < \frac{1}{2}\) then \((2x-1)x < 0\) so it points to the left.
- If \(x > \frac{1}{2}\) then \((2x-1)x > 0\) so it points to the right.

Remark: Alternatively, one can choose particular values on the nullclines to decide the direction.
Notice that regions I and III are invariant. Every solution in region I approaches the nodal sink \((0,3)\). Every solution in region III approaches the nodal sink \((2,0)\). Plane 6 shows that almost all solutions approach either \((0,3)\) or \((2,0)\). The only exception are the stable solutions across corresponding to the saddle point \((\frac{1}{2}, \frac{3}{2})\).

\[ B^{13/9,2} \]

\[ x' = 1 - y \]
\[ y' = y - x^2 \]

- **x-nullcline:** \( y = 1 \)
- **y-nullcline:** \( y = x^2 \)

**Equilibrium points:** \((1,1), (-1,1)\).

**Jacobian:**
\[
J(x,y) = \begin{bmatrix} 0 & -1 \\ -2x & 1 \end{bmatrix}
\]

For \((1,1)\): \(J(1,1) = \begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix} \Rightarrow \text{saddle point}\)

For \((-1,1)\): \(J(-1,1) = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow \text{spiral source}\)

On the x-nullcline \(y = 1\) the direction vector is \([0, 1-x^2]\), hence
- if \(-1 < x < 1\), \(1-x^2 > 0\), it points up
- if \(x < -1\) or \(x > 1\), \(1-x^2 < 0\), it points down

On the y-nullcline \(y = x^2\), the direction vector is \([1-x^2, 0]\)
- if \(-1 < x < 1\), \(1-x^2 > 0\), it points to the right
- if \(x < -1\) or \(x > 1\), \(1-x^2 < 0\), it points to the left
\[ x' = 1 - y \]
\[ y' = y - x^2 \]
\[ x' = (2 - x - y) x \]
\[ y' = (3 - 3x - y) y \]