SOLUTIONS TO QUALIFYING EXAM PROBLEMS IN ANALYSIS

AS GIVEN BY THE RICE UNIVERSITY MATHEMATICS DEPARTMENT

Contents

Preface

The following pages contain solutions to problems from qualifying exams in Analysis given at Rice University dating back to 1999. The solutions are due to the combined efforts of Amanda Knecht, Eric Samansky, Steve Wallace, and Charlie Bingham with great aid from Professors Gao, Hardt, and Jones. These solutions have been compiled, typeset, and are currently being maintained by Charlie Bingham (bingham@math.rice.edu).

These solutions were last updated on March 1, 2004. A good number of mistakes were remedied. Most were only minor, but a few solutions were complete nonsense. In addition January 2004 exam solutions were added. Corrections are still welcome.

Onward to 1999, Year of the Rabbit...

JAN 1999

Problem 1: Evaluate

$$\int_{|z|=1} (e^{2\pi z} + 1)^{-2} dz$$

where the integral is taken in the counterclockwise direction.

Solution 1: The zeros of $e^{2\pi z} + 1$ are $\frac{i}{2}$ and $-\frac{i}{2}$. The possible poles of the integrand function are thus $\frac{i}{2}$ and $-\frac{i}{2}$. So let's determine the residues incurred at the two points respectively. Expanding $e^{2\pi z}$ in its power series about $\frac{i}{2}$ we obtain:

$$(e^{2\pi z}+1) = [-1-2\pi(z-\frac{i}{2})+\frac{-4\pi^2}{2}(z-\frac{i}{2})^2+...+1] = (z-\frac{i}{2})(-2\pi-2\pi^2(z-\frac{i}{2})+...)$$

$$\Rightarrow (e^{2\pi z}+1)^{-2} = (z-\frac{i}{2})^{-2}(-2\pi - 2\pi^2(z-\frac{i}{2})+...)^{-2} = (z-\frac{i}{2})^{-2}P(z)$$

Note: here we may observe that $\frac{i}{2}$ is a pole of order 2 since $P(\frac{i}{2}) = \frac{1}{4\pi^2}$. Now recall the binomial formula:

$$(1+w)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} w^k$$

So setting $w = (\pi(z-\frac{i}{2})+...)$ we have the following:

$$P(z) = (-2\pi)^{-2}(1+\pi(z-\frac{i}{2})+...)^{-2} = (-2\pi)^{-2}(1+\pi(z-\frac{i}{2})+\cdots)^{-2} =$$

$$= \frac{1}{4\pi^2}(1-2\pi(z-\frac{i}{2}) + \text{higher order terms})$$

Recall from above that $(e^{2\pi z}+1)^{-2} = (z-\frac{i}{2})^{-2}P(z)$ Hence the residue at $\frac{i}{2}$ is $(\frac{1}{4\pi^2})(-2\pi) = -\frac{1}{2\pi}$. Since all of the above is unaltered for the calculation of the residue at $z = \frac{-i}{2}$, we may conclude:

$$\int_{|z|=1} (e^{2\pi z}+1)^{-2} dz = 2\pi i \sum (\text{residues of } (e^{2\pi z}+1)^{-2} \text{ with } |z|<1) =$$

$$= 2\pi i(-\frac{1}{2\pi} - \frac{1}{2\pi}) = -2i$$

□

(Another) Solution 1:

Problem 2: Give an example of a subset of $\mathbb{R}$ having uncountably many connected components. Can such a subset be open? Closed?

Solution 2: An example of such a subset is the Ternary Cantor Set (the Middle Thirds Cantor Set). This set is the compliment of a union of open intervals, hence it is closed. So we need only address the second question: Can such a subset be open? An open subset of $\mathbb{R}$ must contain an open interval (a, b). By density of $\mathbb{Q}$ in $\mathbb{R}$, each open interval (a, b) must contain a rational number. Since $\mathbb{Q}$ is countable, we can have at most a countable number of open connected components. Hence a subset with uncountable connected components can't be open. $\square$

Problem 3: Let f be a meromorphic function on $\mathbb{C}$ which is analytic in a neighborhood of 0. Suppose its Taylor Series at 0 is:

$$\sum_{k=0}^{\infty} a_k z^k$$

With $a_k \geq 0$. Let $r = \min\{|z_0| : f \text{ has a pole at } z_0\} < \infty$. Prove f has a pole at $z = r$ where $r \in \mathbb{R}$.

Solution 3: By elementary arguments and the hypothesis $a_k \geq 0$ we obtain the following:

$$(*) \quad \sum_{k=0}^{\infty} a_k z^k \leq \left| \sum_{k=0}^{\infty} a_k z^k \right| \leq \sum_{k=0}^{\infty} | a_k z^k | = \sum_{k=0}^{\infty} a_k | z^k |$$

We know we have a minimum modulus pole on $|z| = r$, say z_0. Since power series representations diverge at poles, utilizing $(*)$ we have

$$\infty \;=\; \sum_{k=0}^{\infty} a_k z_0^k \;\leq\; \sum_{k=0}^{\infty} a_k |\, z_0^k| \;=\; \sum_{k=0}^{\infty} a_k r^k$$

This implies the power series diverges at $z = r$ as well. Therefore $z = r$ is a pole. $\square$

Problem 4: Let $f(x)$ be $\mathbb{R}$-valued, defined $\forall x \geq 1$, satisfying $f(1) = 1$ and:

$$f'(x) = 1/(x^2 + f(x)^2)$$

Prove $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f(x) \leq 1 + \pi/4$.

Solution 4: By the Fundamental Theorem of Calculus, we have:

$$f(x) = f(1) + \int_1^x f'(t)dt = 1 + \int_1^x f'(t)dt$$

Now, since $f'(t) > 0$ $\forall t$, the function $\int_1^x f'(t)dt$ is increasing in x and hence has a limit $\forall x$. So taking a limit on both sides yields:

$$\lim_{x\to\infty} f(x) = 1 + \lim_{x\to\infty} \int_1^x 1/(t^2 + f(t)^2)dt$$

Furthermore, $f'(t) \geq 0$ $\forall t$ implies:

$$\int_1^x 1/(t^2 + f(t)^2)dt \leq \int_1^x 1/(t^2 + 1)dt$$

$$\Rightarrow \lim_{x\to\infty} f(x) \leq 1 + \lim_{x\to\infty} \int_1^x 1/(t^2 + 1)dt = 1 + (\arctan(\infty) - \arctan(1)) = 1 + \pi/4$$

$\square$

Problem 5: Suppose $f : [0, 1] \to \mathbb{R}$ is continuous.

a. Prove that $\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0$

b. Prove that $\lim_{n\to\infty} n \int_0^1 x^n f(x)dx = f(1)$

Solution 5: (a) Note that f is continuous on $[0,1]$, and x^n is continuous on $[0,1]\forall n$. Hence their product is continuous on $[0,1]$, a compact subset of $\mathbb{R}$. Therefore $x^n f$ is integrable on $[0,1]$. Furthermore, since $0 \leq x^n \leq 1$ on $[0,1]$, $x^n f$ is dominated by f. Thus by Lebesgue's Dominated Convergence Theorem, we have:

$$\lim_{n\to\infty}\int_0^1 x^n f(x)dx = \int_0^1 \lim_{n\to\infty} x^n f(x)dx = \int_0^1 0dx = 0$$

(b) Let $u = x^n$. Then $x = \sqrt[n]{u}$ and $du = nx^{n-1}$. Substitution into the integral yields (bringing the constant n inside):

$$\int_0^1 (\sqrt[n]{u})f(\sqrt[n]{u})du$$

Note that the domain of integration is not altered by the change of variables. Since composition of continuous functions preserves continuity, $f(\sqrt[n]{u})$ is continuous. So by the same argument as in (a), we can interchange the limit and the integral s.t.

$$\lim_{n\to\infty}\int_0^1 (\sqrt[n]{u})f(\sqrt[n]{u})du = \int_0^1 \lim_{n\to\infty}(\sqrt[n]{u})f(\sqrt[n]{u})du = \int_0^1 f(1)du = [f(1)u]_0^1 = f(1) \quad \square$$

Problem 6: Let α be a complex number and ϵ a positive number. Prove that the function $f(z) = \sin z + \frac{1}{z-\alpha}$ has infinitely many zeros in the strip $|Imz| < \epsilon$.

Solution 6: This problem stinks of smelly cheese. So our first instinct should be to apply Rouche's Theorem. Since we know where the zeros of $\sin x$ lie, we should apply Rouche to $f(z) = \sin z + \frac{1}{z-\alpha}$ and $g(z) = \sin z$ with appropriate contour.

Let $Z = \{z_i\}$ be the set of zeros of $f(z)$. Let $\{x_i\}$ be the subset of Z on the the $\mathbb{R}$ interval $[0, 2\pi]$. Since f is non-constant, there are only finitely many, say N zeros on the interval. So let $U = \{U_i\}_{i=1}^N$ be an open cover of the interval $[0, 2\pi]$ such that each U_i does not contain any other zero (on or off the $\mathbb{R}$ line). Furthermore, there exists a contour γ_0 contained in U such that γ_0 is the oriented boundary of a rectangle with vertices $\delta i, 2\pi + \delta i, -\delta i, 2\pi - \delta i$ shifted slightly to the right or left to avoid zeros on the vertical edges of both f and g. We can also take $\delta < \epsilon$.

γ_0 is a compact subset of $\mathbb{C}$ hence it achieves its infimum. By construction, f is non-zero on γ_0. So let $m = \min_{\gamma_0} f(z)$. Now recall that $\sin z$ has period 2π. Therefore we may shift γ_0 sufficiently far to the right to some contour γ_1 (or left for that matter) so that we have

$$\left| \frac{1}{z - \alpha} \right| \leq m$$

on γ_1. We can do this since α is fixed and m is unchanged by 2π shifting. We may now apply Rouche's Theorem to obtain that f and g have the same number of roots inside γ_1. Note that there is at least one zero of $\sin z$ inside γ_1 since $\sin x$ has period 2π. We may obtain the result be repeating this last step on $\{\gamma_k\}$ the set of $2k\pi$ shifts of γ_1. $\square$

Aug 1999

Problem 1: Prove that, for any family $\mathcal{F}$ of upper semi-continuous functions on $\mathbb{R}$, the function

$$g(x) = \inf\{f(x)|f \in \mathcal{F}\}$$

is upper semi-continuous.

Solution 1: f is upper semicontinuous if and only if for any $t > f(x)$ there exists a neighborhood of x such that $t > f(x)$ in that neighborhood. Let $t > g(x)$. Then $t > f(x)$ for at least one function $f \in \mathcal{F}$. But then that function has a neighborhood as mentioned above. There we have $t > f(y) \geq g(y)$. Hence g is upper semicontinuous by the above equivalence. □

Problem 2: Suppose that f is a holomorphic function on $\{z : |z| < 3R\}$, $f(0) = 0$, $M_R = sup_{|z|\leq R}|f(z)|$, and $N_R = \sup_{|z|\leq \mathbb{R}} |f'(z)|$.
a. Estimate M_R (from above) in terms of N_R.
b. Estimate N_R (from above) in terms of M_{2R}.

Solution 2: (a) Take any contour γ along a radii of the circle of radius R. Let $z_0 \in |z| = R$. Then:

$$f(z_0) = f(0) + \int_\gamma f'(z)dz \quad \Rightarrow \quad f(z_0) \leq 0 + RN_R$$

Since this holds for arbitrary z_0 on $|z| = R$, we have $M_R \leq RN_R$.

(b) Since f is holomorphic on $|z| < 3R$, Cauchy's Integral Formula gives us:

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z_0}d\zeta$$

where γ is the boundary of $|z| \leq 2R$ positively oriented, and z_0 on $|z| = R$. So restricting our attention to $|z| = 1$ we have $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$. Differentiating on both sides we have:

$$f'(z) = \frac{d}{dz}\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{d}{dz}\frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{-f(\zeta)}{(\zeta - z)^2}d\zeta$$

Note that the second equality above is justified by the fact that the integral is on a compact subset of the domain of analyticity of an analytic function. Now taking absolute values on both sides of the equality, we obtain:

$$|f'(z)| = \left|\frac{1}{2\pi i}\int_\gamma \frac{-f(\zeta)}{(\zeta - z)^2}d\zeta\right| \leq \frac{1}{2\pi}\int_\gamma \left|\frac{-f(\zeta)}{(\zeta - z)^2}d\zeta\right| = \frac{1}{2\pi}\int_\gamma \frac{|f(\zeta)|}{|(\zeta - z)|^2}|d\zeta| \leq$$
$$\leq \frac{2\pi 2R M_{2R}}{2\pi(2R - R)^2} = \frac{2M_{2R}}{R}$$

Thus we have $N_R \leq 2M_{2R}/R$. ☐

Problem 3: Suppose that $f(x)$ is defined on $[-1, 1]$, and that $f''(x)$ is continuous. Show that the series

$$\sum_{n=1}^{\infty}(n(f(\frac{1}{n}) - f(-\frac{1}{n})) - 2f'(0))$$

converges.

Solution 3: Utilizing the Taylor series expansion about 0, we have:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(t)}{6}x^3$$
$$f(-x) = f(0) - f'(0)x + \frac{f''(0)}{2}x^2 - \frac{f'''(s)}{6}x^3$$

for some $s, t \in [-1, 1]$. Thus we have the following:

$$f(x) - f(-x) = 2f'(0)x + \frac{x^3}{6}[f'''(s) + f'''(t)]$$

Now let $x = 1/n$ to obtain:

$$f(1/n) - f(-1/n) = \frac{2f'(0)}{n} + \frac{1}{6n^3}[f'''(s) + f'''(t)]$$

$$\Rightarrow A_n := [n(f(1/n)) - f(-1/n)) - 2f'(0)] = \frac{1}{6n^2}[f'''(s) + f'''(t)]$$

Since $f'''(x)$ is continuous on $[-1,1]$, $f'''(x)$ is bounded by some M so $[f'''(s) + f'''(t)] \leq 2M$. Thus $A_n \leq \frac{M}{3n^2} \Rightarrow \sum_1^\infty A_n$ converges. □

Problem 4: Let f be an analytic function such that $f(z) = 1+z+z^2+...$ for $|z| < 1$. Define a sequence of real numbers $a_0, a_1, a_2, ...$ by

$$f(z) = \sum_{n=0}^{\infty} a_n(z+2)^n$$

What is the radius of convergence of this new series $\sum_{n=0}^{\infty} a_n(z+2)^n$?

Solution 4: $1 + z + z^2 + ... = 1/(1-z)$ $\forall z = \frac{1}{n}$ as $n \to \infty$. So by uniqueness of power series representations of analytic functions, $f(z) = 1/(1-z)$ on $\mathbb{D}$. In fact, $f(z) = 1/(1-z)$ on the latter's domain of analyticity, which is $\mathbb{C}\backslash\{1\}$. Hence the radius of convergence for the power series representation of $f(z)$ about -2 is $d(-2,1) = 3$. □

Problem 5: Suppose that $f_1, f_2, ...$ are nonnegative continuous functions on $[0,1]$ with $\int_0^1 f_n(x)dx \leq M$.

a. Show that there exists a point $a \in [0,1]$ with $f_1(a) \leq 2M$ and $f_2(a) \leq 2M$.

b. Does there exist a better estimate? That is, a number $N < M$ so that

$$\inf_{0 \leq a \leq 1} \max\{f_1(a), f_2(a)\} \leq 2N$$

for all such f_1, f_2. If so, find the smallest such N. If not, give a counterexample.

c. Show that there always exists an $a \in [0,1]$ so that $f_n(a) \le M$ for *infinitely many n*.

Solution 5

(a) Suppose not. Then $f_1(a) + f_2(a) > 2M$ for all a. Hence we have:

$$\int_0^1 [f_1(x) + f_2(x)]dx > 2M$$

$\Rightarrow$ either $\int_0^1 f_1 > M$ or $\int_0^1 f_2 > M$, a contradiction to our hypothesis.

(b) Draw a picture, or: Fix $N < M$, say $M - N = \epsilon$. So $2M - 2N = 2\epsilon$. Now let:

$$f_{1,\,\epsilon} = \begin{cases} (2M-\epsilon) & \text{if} x \in [0, \frac{1}{2}] \\ \frac{-(2M-\epsilon)x}{\delta} + \frac{(2M-\epsilon)(\frac{1}{2}+\delta)}{\delta} & \text{for } x \in [\frac{1}{2}, \frac{1}{2}+\delta] \\ 0 & \text{otherwise} \end{cases} .$$

$$f_{2,\,\epsilon} = \begin{cases} \frac{(2M-\epsilon)x}{\delta} - \frac{(2M-\epsilon)(\frac{1}{2}-\delta)}{\delta} & \text{for } x \in [\frac{1}{2}-\delta, \frac{1}{2}] \\ (2M-\epsilon) & \text{for} x \in [\frac{1}{2}, 1] \\ 0 & \text{otherwise} \end{cases} .$$

Note that $f_{1,\,\epsilon}$ and $f_{2,\,\epsilon}$ are are reflections about $x = 1/2$ of each other. From this observation we note: $\int_0^1 f_{1,\,\epsilon}(x)dx = \int_0^1 f_{2,\,\epsilon}(x)dx$. Since the area under these curves is the area of a rectangle and a right triangle with base; height $\frac{1}{2}$; $(2M-\epsilon)$ and δ; $2M-\epsilon$ respectively, we have the following:

$$\int_0^1 f_{1,\,\epsilon}(x)dx = \int_0^1 f_{2,\,\epsilon}(x)dx = \frac{1}{2}(2M-\epsilon) + \frac{\delta}{2}(2M-\epsilon) = \frac{1}{2}(2M-\epsilon)(1+\delta)$$

We want the above functional restricted to our two functions to be bounded above by M. i.e. we want $\frac{1}{2}(2M-\epsilon)(1+\delta) \le M$. So we simplify the inequality:

$$\frac{1}{2}(2M-\epsilon)(1+\delta) = \frac{1}{2}(2M + 2M\delta - \epsilon - \epsilon\delta) \le M$$

$$\Rightarrow\ 2M\delta - \epsilon - \epsilon\delta \le 0\ \Rightarrow\ \delta(2M-\epsilon) - \epsilon \le 0$$

Then let $\delta = \epsilon/(2M-\epsilon)$. We thus have equality above which is sufficient to satisfy the hypothesis of concern. So substituting the δ value computed above into our definition of $f_{1,\,\epsilon}$ and $f_{2,\,\epsilon}$ yields a pair of counterexample functions for any $N < M$.

(c) The statement is false. Consider the following counterexample:

$$\text{For } n \geq 2 \text{ define } f_n(x) = \begin{cases} M(1-nx) & \text{if } x \in [0,1/n] \\ M(-1+\frac{n}{2}x) & \text{if } x \in [1/n,2/n] \\ M & \text{if } x \in [2/n,1] \end{cases} .$$

$$\text{Now let } g_n(x) = \frac{f_n(x)}{1-\frac{1}{n}}$$

Note that $\forall n \int_0^1 g_n(x)dx = M$ and for each $a \in [0,1]$, $g_n(a) \leq M$ for only finitely many n. So g_n serves as a counterexample to the statement in (c). $\square$

Problem 6: Prove that there is no one-to-one conformal map of the punctured disc $G = \{z \in \mathbb{C} : 0 < |z| < 1\}$ onto the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

Solution 6: Suppose there exists such a map, $g : \mathbb{D}\backslash\{0\} \to A$. Then g has an analytic continuation G on $\mathbb{D}$. $G(0)$ must lie in A (which is the *open* annulus) since $0 \in int\mathbb{D} \Rightarrow G(0) \in int\bar{A}$ by the Open Mapping Theorem. But g onto implies that there exists some $z_0 \in \mathbb{D}\backslash\{0\}$ such that $g(z_0) = G(0)$. Now take open disjoint neighborhoods about z_0 and 0, U_{z_0} and U_0 respectively. Push them forward to the range space. Since $G(0) = g(z_0)$, $G(U_0) \cap g(U_{z_0}) \neq \emptyset$. Therefore, points in our two domain neighborhoods share image points. Since $G|_{\mathbb{D}\backslash\{0\}} = g$, we have a contradiction to confomality (conformal maps are injective). $\square$

Jan 2000 A

Problem 1: Suppose $g(x) = \lim_{n\to\infty} g_n(x)$ for $x \in [0,1]$ where $\{g_n\}$ are positive and continuous on $[0,1]$. Also suppose that:

$$\int_0^1 g_n dx = 1$$

a. Is it always true that $\int_0^1 g(x)dx \leq 1$?
b. Is it always true that $\int_0^1 g(x)dx \geq 1$?

Solution 1: a. Yes. This is a direct application of Fatou's Lemma:

$$\int \liminf_{n\to\infty} g_n(x)dx \leq \liminf_{n\to\infty} \int g_n(x)dx$$

$$\Rightarrow \int_0^1 g(x)dx \leq 1$$

□

b. No. Consider the following family of functions on $[0,1]$:

$$g_n(x) = \begin{cases} n^2x - (\frac{n^2}{2} - n) & \text{if } x \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}] \\ -n^2x + (\frac{n^2}{2} + n) & \text{if } x \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases} .$$

The graphs of these functions look like triangles with height n peaks at $x = \frac{1}{2}$ and with bases on $[\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}]$. The ratio of the base to the height is $2\ \forall n$ but $\lim_{n\to\infty} f_n = \delta_{\frac{1}{2}} = 0$ almost everywhere. Hence $\int_0^1 f(x) = \int_0^1 \delta_{\frac{1}{2}} = 0$. □

Problem 2: Let f be $\mathbb{C}$-valued in $D(0,1)$ such that $g = f^2$ and $h = f^3$ are both analytic. Prove f is analytic in $D(0,1)$.

Solution 2: Define $k = h/g = f^3/f^2 = f$. Now we must be careful to show that this function k is in fact analytic. The only possible singularities of a quotient of two analytic functions is at the zeros of the denominator function. So suppose $f^2(z_0) = 0$, then $f^3(z_0) = 0$. Let m be the multiplicity of the zero of f^3 at z_0, and n be the multiplicity of the zero of f^2 at z_0. That is,

$$f^3(z) = \alpha_m(z - z_0)^m + \dots \quad \text{and} \quad f^2(z) = \beta_n(z - z_0)^n + \dots$$

Now $(f^3)^2 = f^6 = (f^2)^3 \quad \Rightarrow \quad 2m = 3n \quad \Rightarrow \quad m > n$. Therefore the limit of $k(z)$ as $z \to z_0$ exists (in fact it is zero). Thus z_0 is a removable singularity of k, and so k is indeed analytic there by the Riemann Removable Singularity Theorem. Since z_0 was arbitrary, we are done. $\square$

Problem 3: Construct an open set $U \subset [0, 1]$ such that U is dense in $[0, 1]$, the Lebesgue measure $\mu(U) < 1$, and that $\mu(U \cap (a, b)) > 0$ for any interval $(a, b) \subset [0, 1]$.

Solution 3: Order $\mathbb{Q} \cap [0, 1]$, say $\{q_1, q_2, ...\}$. Let A_i be the interval $(q_i - \frac{1}{2^{i+2}}$, $q_i + \frac{1}{2^{i+2}})$. Let $B_i = A_i \cap [0, 1]$. Note that B_i is open $\forall i$ and $\mu(B_i) \leq \frac{1}{2^{i+1}}$. So we have: $\sum_1^\infty \mu(\cup B_i) \leq \frac{1}{4} + \frac{1}{8} + ... = \frac{1}{4}/(1 - \frac{1}{2}) = \frac{1}{2}$. Hence $\mu(\cup B_i) \leq \frac{1}{2}$. $\mathbb{Q} \subset \cup B_i$, so for any open interval $(a, b) \subset [0, 1]$, $\exists q \in \mathbb{Q}$ such that $q \in (a, b)$. And by construction, we have: $\mu((a, b) \cap (\cup B_i)) > 0$. $\square$

Problem 4: Let $f, g_1, g_2, ...$ be entire functions. Assume that the kth derivatives at 0 satisfy:

a. $|g_n^{(k)}| \leq |f^{(k)}|$ for all n and k;

b. $\lim_{n\to\infty} g_n^{(k)}(0)$ exists for all k.

Prove that the sequence $\{g_n\}$ converges uniformly on compact sets and thats its limit is an entire function.

Solution 4: The $\{g_n\}$ are entire, so they have power series representations at 0:

$$\begin{aligned} g_1 &= \alpha_{10} + \alpha_{11}z + \alpha_{12}z^2 + \dots \\ g_2 &= \alpha_{20} + \alpha_{21}z + \alpha_{22}z^2 + \dots \\ &\vdots \qquad\quad \vdots \qquad\qquad \vdots \\ g_n &= \alpha_{n0} + \alpha_{n1}z + \alpha_{n2}z^2 + \dots \\ &\vdots \qquad\quad \vdots \qquad\qquad \vdots \end{aligned}$$

Also let

$$f(z) = \beta_0 + \beta_1 z + \beta_2 z^2 + \dots$$

be the power series representation of f at 0. Observe that $g_n^{(k)}(0) = (k!)(\alpha_{nk})$ and $f^{(k)}(0) = (k!)(\beta_k)$. By hypothesis (ii), $g_n^{(k)}(0)$ converge for each k. Therefore $\lim_{n\to\infty} \alpha_{nk}$ exists for each k since the sequence is merely a constant multiple $(k!)$ times the given sequence. Let $\alpha_i = \lim_{n\to\infty} \alpha_{ni}$. From our observation above we have $\alpha_i \leq \beta_i$. Now we claim that $g_n(z)$ converges uniformly on compacta to $g(z)$ where $g(z) = \sum_{i=0}^{\infty} \alpha_i z^i$. That is, given $\epsilon > 0$ and $|\,z| \leq R$, $\exists N$ such that $n \geq N \Rightarrow |\,g(z) - g_n(z)| < \epsilon$ for all z such that $|\,z| \leq R$. More precisely, restricting to our compact domain we desire the following:

$$n \geq N \quad \Rightarrow \quad \left| \sum_{\imath=0}^{\infty} \alpha_i z^i - \sum_{\imath=0}^{\infty} \alpha_{in} z^i \right| < \epsilon$$

And since

$$\left| \sum_{\imath=0}^{\infty} \alpha_i z^i - \sum_{\imath=0}^{\infty} \alpha_{in} z^i \right| = \left| \sum_{\imath=0}^{\infty} (\alpha_i - \alpha_{in}) z^i \right| \leq \sum_{\imath=0}^{\infty} |\,(\alpha_i - \alpha_{in}) z^i|$$

it is sufficient to bound the following by ϵ for $n \geq N$

$$\sum_{\imath=0}^{\infty} |\,\alpha_i - \alpha_{in}|\,|\,z|^i \qquad (*)$$

So let $\epsilon > 0$ and $R \in \mathbb{R}$ be given. Let $|\,z| \leq R$. Now, the power series expansions of analytic functions converge absolutely in the interior of the function's domain of analyticity. Since f is entire, on $|z| \leq R$ we have $\exists K \in \mathbb{N}$ such that the second inequality holds below.

$$\sum_{i=K}^{\infty} |\beta_i z^i| \leq \sum_{i=K}^{\infty} |\beta_i| R^i < \frac{\epsilon}{4}$$

Substituting the estimate $|\alpha_i - \alpha_{in}| \leq 2|\beta_i|$ into $(*)$ we obtain

$$\sum_{i=K}^{\infty} |\alpha_i z^i - \alpha_{ni} z^i| \leq 2 \sum_{i=K}^{\infty} |\beta_i z^i| < \frac{\epsilon}{2}$$

We now have an $\epsilon/2$-bound for the infinite tail of the sum which is independent of n and z, so all we need do is bound the first K terms by $\epsilon/2$ independent of z. Since the α_{ni} converge to the α_i,

$$\exists N_i \text{ such that } n \geq N_i \Rightarrow |\alpha_i - \alpha_{ni}| < \frac{\epsilon}{2KR^i}$$

Put $N = \max_{i=0}^{K-1}\{N_i\}$. Then $n \geq N$ ensures:

$$\sum_{i=0}^{K-1} |\alpha_i - \alpha_{ni}| |z|^i < \sum_{i=0}^{K-1} \frac{\epsilon}{2KR^i} R^i = \sum_{i=0}^{K-1} \frac{\epsilon}{2K} = \frac{\epsilon}{2}$$

Combining our two bounds, we may indeed conclude that proved that the limit function g is entire as well. □

Problem 5: Let f be a continuous, strictly increasing function from $[0, \infty)$ onto $[0, \infty)$ and let $g = f^{-1}$ be the inverse of f. Prove that:

$$\int_0^a f(x)dx + \int_0^b g(x)dx \geq ab$$

for all positive numbers a and b.

Solution 5: Since f is monotone increasing, f^{-1} is monotone increasing. We have one of the following $f(a) = b, f(a) > b$ or $f(a) < b$. However, since $(f^{-1})^{-1} = f$ we may assume without loss of generality that $f(a) \leq b$. Thus we have

$$\int_0^a f(x)dx + \int_0^b g(x)dx = \left(\int_0^a f(x)dx + \int_0^{f(a)} g(x)dx\right) + \int_{f(a)}^b g(x)dx =$$

$$= \left(\int_0^a f(x)dx + \int_0^{f(a)} g(x)dx\right) + \left(\int_{f(a)}^b a\,dx + \int_{f(a)}^b (g(x)-a)dx\right) =$$

$$= \left(\int_0^a f(x)dx + \int_0^{f(a)} g(x)dx + \int_{f(a)}^b a\,dx\right) + \int_{f(a)}^b (g(x)-a)dx =$$

$$= (ab) + \int_{f(a)}^b (g(x)-a)dx \geq ab$$

□

Problem 6: Suppose that $f(z)$ is analytic and satisfies $f(\frac{1}{z}) = f(z)$ for all $z \in \mathbb{C}\backslash\{0\}$.
a. Write down the general *Laurent Expansion* for f.
b. Show that the coefficients of this expansion are all real if this f has real values on the unit circle $|z| = 1$

Solution 6:
(a) The condition $f(z) = f(\frac{1}{z})$ forces the i^{th} and $-i^{th}$ coefficients to be equal:

$$f(z) = \alpha_0 + \sum_1^\infty \alpha_i[(z-z_0)^i + (z-z_0)^{-i}]$$

(b) For all k we have the following identity:

$$(*) \quad \alpha_k = \frac{1}{2\pi i}\int_{|\zeta|=1} \frac{f(\zeta)}{\zeta^{k+1}}\, d\zeta$$

Since $f(z) \in \mathbb{R}$ on $|z| = 1$, trivially $\overline{f(z)} = f(z)$ there. Also on $|z| = 1$ we have

$$\frac{1}{z} = \frac{1}{z}\cdot\frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{\|z\|^2} = \overline{z}$$

Using this fact we obtain the following:

$$d\overline{z} = \frac{-1}{z^2} dz$$

Now conjugating both sides of $(*)$ and incorporating the above facts yields:

$$\overline{\alpha}_k = \overline{\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta^{k+1}} \, d\zeta} = -\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\overline{f(\zeta)}}{\overline{\zeta^{k+1}}} \, d\overline{\zeta} = -\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta^{-k-1}} \cdot \frac{-1}{\zeta^2} d\zeta =$$

$$= \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta^{-k+1}} d\zeta = \alpha_{-k} = \alpha_k$$

From the above string of equalities, we obtain $\overline{\alpha}_k = \alpha_k$. Hence $\alpha_k \in \mathbb{R}$ for all $k \in \mathbb{Z}$. □

Jan 2000 B

Problem 1: Does there exist a function $f(z)$ that is holomorphic near the origin and that satisfies:

$$f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^3} \text{ , } n = 1, 2, 3, ...?$$

Why or why not?

Solution 1: No. By hypothesis, $f(z)$ agrees with z^3 on the set $\{\frac{1}{n}\}_1^\infty$ which has 0 as a limit point. By uniqueness of power series expansions, $f(z) = z^3$ near zero. But $\frac{1}{n^3} \neq \frac{-1}{n^3}$. □

Problem 2: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with $\int_{-\infty}^{\infty} |f(x)| dx < \infty$. Show that there is a sequence $x_n \in \mathbb{R}$ such that $x_n \to \infty$, $x_n f(x_n) \to 0$ and $x_n f(-x_n) \to 0$ as $n \to \infty$.

Solution 2: Assume to the contrary that there does not exist such a sequence $\{x_n\}$. Then either $\liminf_{x\to\infty} x \,|\, f(x)| > 0$ or $\liminf_{x\to\infty} x \,|\, f(-x)| > 0$. So either $\exists M_+ \in \mathbb{N}$

or $\exists M_- \in \mathbb{N}$ such that

$$\begin{aligned} x \geq M_+ \quad &\Rightarrow \quad x\,|\,f(x)| \;\geq\; k > 0 \\ &\Rightarrow \quad |\,f(x)| \;\geq\; k/x \end{aligned}$$

or

$$\begin{aligned} x \geq M_- \quad &\Rightarrow \quad x\,|\,f(-x)| \;\geq\; c > 0 \\ &\Rightarrow \quad |\,f(-x)| \;\geq\; c/x \end{aligned}$$

But then we would have

$$\int_{M_+}^{\infty} |\,f(x)|\,dx \;\geq\; \int_{M_+}^{\infty} \frac{k}{x} dx = \infty$$

or

$$\int_{M_-}^{\infty} |\,f(-x)|\,dx \;\geq\; \int_{M_-}^{\infty} \frac{c}{x} dx = \infty$$

contradicting integrability. □

Problem 3: Suppose that $f(z) = a_0 + a_1 z + a_2 z^2 + ...$

$$\text{and} \quad g(z) = b_{-2}z^{-2} + b_{-1}z^{-1} + b_0 + b_1 z + b_2 z^2 + ...$$

where the two series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ converge for $|z| < 2$. Find $\int_\Gamma f(z)g(z)dz$ in terms of the a_n, b_n where Γ is the positively-oriented unit circle.

Solution 3:

$$f(z)g(z) = a_0 b_{-2} z^{-2} + a_0 b_{-1} z^{-1} + a_1 b_{-2} z^{-1} + ...$$

Hence by the Residue Formula:

$$\int_\Gamma f(z)g(z)dz = 2\pi i(a_0 b_{-1} + a_1 b_{-2})$$

□

Problem 4: Suppose f_n is a sequence of nonnegative measurable functions on $\mathbb{R}$, and $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n^2(x)dx = 0$.

a. Show that the Lebesgue measure $\lambda\{x : f_n(x) > \delta\} \to 0$ as $n \to \infty$ for every $\delta > 0$.
b. Show that $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)g(x)dx = 0$ for every $g \in L^2(\mathbb{R})$.
c. Find a sequence g_n of nonnegative measurable functions on $\mathbb{R}$ such that $\lim_{n\to\infty} \lambda\{x : g_n(x) > \delta\} = 0$ for every $\delta > 0$ AND $\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)g(x)dx = 0$ for every $g \in L^2(\mathbb{R})$ BUT $\int_{-\infty}^{\infty} g_n^2(x)dx = 1$.

Solution 4:

(a) Recall the Chebyshev Inequality for L^2:

$$\lambda\{x : f_n(x) > \delta\} \;\leq\; \frac{\|f_n\|_2^2}{\delta^2}$$

Fix $\delta > 0$ then the RHS $\to 0$ as $n \to \infty$ since $\|f_n\|_2^2 \to 0$ by hypothesis.

(b) Via a basic inequality and Hölder's Inequality we obtain:

$$\int_{-\infty}^{\infty} f_n(x)g(x)dx \leq \|f_n g\|_1 \leq \|f_n\|_2 \|g\|_2$$

Taking a limit as $n \to \infty$ yields:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)g(x)dx \leq \lim_{n\to\infty} \|f_n\|_2 \|g\|_2 = 0$$

The last equality follows from the fact that $\|g\|_2 < \infty$ and the hypothesis on f_n.

(c) Define $g_n : \mathbb{R} \to \mathbb{R}$ by

$$g_n(x) = \chi_{[n,2n]} \cdot \frac{1}{\sqrt{n}} \qquad \text{for } n = 1, 2, 3, \ldots$$

Note that

$$\int_{-\infty}^{\infty} g_n^2(x)dx = \frac{1}{n} \cdot n = 1$$

Also note that $g_n(x)g(x)$ is dominated by the restriction of g to a compact domain $[n, 2n]$. Since $L^2 \subset L^1$ on compact subsets of $\mathbb{R}$, we obtain the following via Lebesgue's Dominated Convergence Theorem (since $g_n(x)g(x)$ is dominated by $.g(x)$)

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g_n(x)g(x)dx = \int_{-\infty}^{\infty} \lim_{n\to\infty} g_n(x)g(x)dx = \int_{-\infty}^{\infty} 0 \cdot g(x)dx = 0 \text{ a.e.}$$

□

Problem 5:

a. Prove that if f is a holomorphic map from the unit disk $\mathbb{D} = \{z : |z| < 1\}$ to itself with $f(0) = 0$, then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

b. Which of these f admit a point $a \neq 0$ in $\mathbb{D}$ with $|f(a)| = |a|$?

c. Let h be a holomorphic map of the unit disk $\mathbb{D}$ into itself which is not the identity map of $\mathbb{D}$. Show that h can have at most one fixed point.

Solution 5: Note that parts (a)-(c) are known as the Schwarz Lemma.

(a) Define $g : \mathbb{D} \to \mathbb{D}$ by

$$g(z) = \begin{cases} f(z)/z & x \neq 0 \\ f'(0) & x = 0 \end{cases} .$$

Note that this function is analytic in $0 < |\,z| < 1$ Observe that $\lim_{z\to 0} g(z) = f'(0) = g(0)$. Since f is analytic at 0, $f'(0)$ is analytic at 0. Hence our $g(z)$ is analytic in $\mathbb{D}$. By the maximum principle for analytic functions

$$|\,g(z)| \leq \frac{|\,f(z)|}{|\,z|} \leq \frac{1}{|\,z|}$$

So $|\,g(z)|$ is bounded by $\frac{1}{r}$ on every closed disk of radius r. Letting $r \to 1$ and applying the Maximum Principle for analytic functions we obtain the inequality $|\,g(z)| \leq 1$ on D. Therefore $|\,f(z)| \leq |\,z|$ there.

(b) Suppose there exists such a point. Then $g(a) = 1$. The Maximum Principle then implies that $|\,g(z)|$ is constant in $\mathbb{D}$. Hence $f(z)$ must be a rotation about the origin. That is, $f(z) = kz$ with $|\,k| = 1$.

(c) Suppose $h : \mathbb{D} \to \mathbb{D}$ is not the identity, and suppose $h(a) = a$ and $h(b) = b$, $a, b \in \mathbb{D}$. Let

$$g(z) = \frac{a - z}{1 - \bar{a}_0 z}$$

Note that $g(a) = 0$ and $g(0) = a$. Furthermore, this mapping is an analytic automorphism of $\mathbb{D}$ such that $g^{-1} = g$ since $|a| < 1$. Now consider the composition $f = g \circ h \circ g : \mathbb{D} \to \mathbb{D}$. After observing that $f(0) = 0$, we apply the Schwarz Lemma to obtain $|f(z)| \leq |z|$. Furthermore $f = g \circ h \circ g \Rightarrow g \circ h = f \circ g$. Therefore

$$f\big(g(b)\big) = g\big(h(b)\big) \quad \Rightarrow \quad f\left(\frac{a-b}{1-\bar{a}b}\right) = \left(\frac{a-b}{1-\bar{a}b}\right)$$

Therefore f satisfies the Schwarz Lemma and has a fixed point other than 0 inside the disk. Hence $f(z) = z$. But this in turn implies that $h = g \circ id \circ g = g \circ g = id$, a contradiction. □

Problem 6: Suppose $h(x) = \lim_{n\to\infty} h_n(x)$ where $h_1 \geq h_2 \geq h_3 \geq \ldots$ is a decreasing sequence of nonnegative continuous functions on $\mathbb{R}$.

a. Give a specific example of such h and h_n where h is *not* continuous at 0.

b. Prove that, in general, such an h is always upper semi-continuous; that is,

$$h(a) \geq \limsup_{x\to a} h(x)$$

for all $a \in \mathbb{R}$.

Solution 6:

(a) Consider the following sequence of functions:

$$h_n(x) = \begin{cases} e^{nx} & x \leq 0 \\ e^{-nx} & x > 0 \end{cases} .$$

Note that $\lim_{n\to\infty} h_n(x) = 0$ for $x \neq 0$ and $\lim_{n\to\infty} h_n(x) = 1$ for $x = 0$.

(b) Lemma: Let $\mathcal{F}$ be any family of upper semicontinuous $\mathbb{R} \to \mathbb{R}$ functions. Then $g(x) = \inf\{f(x) : f \in \mathcal{F}\}$ is upper semicontinuous.

Proof of Lemma: f is upper semicontinuous if and only if for any $t > f(x)$ there exists a neighborhood of x such that $t > f(x)$ in that neighborhood. Let $t > g(x)$.

Then $t > f(x)$ for at least one function in $f \in \mathcal{F}$. But then that function has a neighborhood as mentioned above. There we have $t > f(y) \geq g(y)$. Hence g is upper semicontinuous by the above equivalence.

Now note that $h_n(x)$ is a monotone decreasing sequence of bounded continuous functions. Hence $\inf_{n\to\infty} h_n(x) = \lim_{n\to\infty} h_n(x) = h(x)$. From our Lemma we have that h is upper semicontinuous. □

Aug 2000

Problem 1: Let $f \geq 0$ be a real valued function defined and integrable over a measurable set $E \subset \mathbb{R}$. Prove that given $\epsilon > 0$, there is a $\delta > 0$, such that

$$\left| \int_A f(x)dx \right| < \epsilon$$

for every measurable subset $A \subset E$ with $\lambda(A) < \delta$.

Solution 1: Fix ϵ. Since $f \in L^1(E)$, there exists a simple function s such that $0 \leq s \leq f$ and:

$$\int_E s(t)dt > \int_E f(t)dt \; - \; \epsilon/2$$

Now s assumes a finite number of values, so let $M = \max\{s(x) : \, x \in E\}$. Then

$$\int_A f(t)dt = \int_A s(t)dt + \int_A (f-s)(t)dt$$

$$\Rightarrow \int_A f(t)dt \; < \; M[\lambda(A)] + \epsilon/2$$

So let $\delta = \epsilon/2M$, and we have our result. □

Problem 2: For each of the following conditions, decide if there is a non-constant, holomorphic function defined on the whole complex plane with the given property.

a.

$$real f(z) > 0 \qquad \forall z$$

b.

$$|f(z)| < \frac{1+|z|}{log(1+|z|)} \qquad \forall z$$

c. A function f which has 0 and ∞ as its only asymptotic values. ($\alpha \in \mathbb{C}$ is an asymptotic value of f if there is an unbound path $\gamma \subset \mathbb{C}$ s.t. $\lim_{z\to\infty,\, z\in\gamma} f(z) = \alpha$)

Solution 2:

(a) Since $f(z)$ is entire, $e^{f(z)}$ is entire. Now $\left|e^{f(z)}\right| = e^{ref(z)} > 1$ by hypothesis. Since this function does not have a zero, $1/e^{f(z)}$ is entire as well. But $\left|1/e^{f(z)}\right| < 1$ thus Liouville's Theorem implies $1/e^{f(z)}$ is constant which in turn implies that $f(z)$ is constant. So there does not exist such a function. □

(b) Consider the following:

$$|f(z)| < \frac{1+|z|}{\log(1+|z|)} \ \forall z \ \Rightarrow \ |f(z)| < \frac{1}{\log(1+|z|)} + \frac{1}{\log(1+|z|)}|z| \ \forall z$$

$$\Rightarrow |f(z)| < 1+|z| \ \forall z \text{ such that } |z| \geq (e-1).$$

So by the Extended Liouville Theorem, $f(z)$ is a polynomial of degree at most 1. But we can do better. I.e. our first argument gave up a lot of ground on estimating the growth of $|f(z)|$. Suppose $f(z) = \alpha z + \beta$, for some α, $\beta \in \mathbb{C}$. $|\alpha z + \beta| \leq |\alpha||z| + |\beta|$. If we consider z with large enough modulus, then eventually $1/(\log(1+|z|)$ will be smaller than both $|\alpha|$ and $|\beta|$. Hence for some M,

$$|f(z)| > M \ \Rightarrow \ |f(z)| > \frac{1+|z|}{\log(1+|z|)}$$

Therefore, if an entire function satisfies such an inequality, it must be constant. □

(c) Claim: e^z satisfies the criterion. First, if we take the paths from the origin to infinity along the positive x-axis and negative x-axis we obtain the asymptotic values of ∞ and 0 respectively. Now, suppose $\alpha \in \mathbb{C}$ is some other asymptotic value. Then $|\,\alpha| = k$, $k \in (0, \infty)$. This implies that the asymptotic path must tend towards $x = e^k$ and hence the path approaches ∞ in the imaginary direction. But if so $|e^z|$ argument never approaches a single value. Roughly speaking, along this path e^z asymptotically cycles the circle of modulus e^k. Therefore α is not really an asymptotic value. (This argument could be made more precise by a delta-epsilon argument) □

Problem 3: Let $g(z)$ be analytic in the right half-plane $\{z : real(z) > 0\}$, with $|\,g(z)| < 1$ for all such z. If $g(1) = 0$, how large can $|g(2)|$ be?

Solution 3: There exists an analytic isomorphism of the right half-plane with the open unit disk, $h : \text{RHP} \to \mathbb{D}$ given by $h(z) = -(\frac{1-z}{1+z})$ with inverse $h^{-1}(z) = (\frac{1+z}{1-z})$. So consider $g(h^{-1}(z)) : \mathbb{D} \to \mathbb{D}$. $g(h^{-1}(0)) = g(1) = 0$ so we may apply Schwarz's Lemma to our composition of analytic functions. But first note: $h^{-1} : \frac{1}{3} \mapsto 2$. Hence:

$$|\,g(h^{-1}(z))| \leq |\,z| \;\Rightarrow\; |\,g(2)| \leq \frac{1}{3}$$

□

Problem 4: Let f be a C^2 function on the real line. Assume f is bounded with bounded second derivative. Let

$$A = \sup_{x \in \mathbb{R}} |f(x)| \quad and \quad B = \sup_{x \in \mathbb{R}} |f''(x)|$$

Prove that

$$\sup_{x \in \mathbb{R}} |f'(x)| \leq 2\sqrt{AB}$$

Solution 4: The Mean Value Theorem states that $\forall (a,b) \subset \mathbb{R}$, $\exists c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. A simple estimate provides $f'(c) \leq \frac{2A}{b-a}$. Applying the MVT in an identical manner, $(**)$ picking an $x \in (a,b)$ we also have $f''(y) = \frac{f'(x)-f'(c)}{x-c}$ for some $y \in (x,c)$ (Assume $x < c$ WLOG). Thus we have the following:

$$f''(y) = \frac{f'(x) - f'(c)}{x - c} \;\Rightarrow\; f'(x) = f'(c) + (x-c)f''(y) \;\Rightarrow$$
$$\Rightarrow |\, f'(x)| \leq |\, f'(c)| + |\, x - c|\, |\, f''(y)|$$
$$\Rightarrow |\, f'(x)| \leq \frac{2A}{b-a} + (b-a)B$$

Now at $(**)$ if we select $x = \frac{b-a}{2}$, we obtain a slightly better estimate:

$$\Rightarrow |\, f'(x)| \leq \frac{2A}{b-a} + \frac{b-a}{2}B$$

Setting $l = (b-a)$ and minimizing the RHS of the inequality, we may obtain a best estimate. I.e. let $F(l) = \frac{2A}{l} + \frac{Bl}{2}$ Then

$$F'(l) = \frac{-2A}{l^2} + \frac{B}{2} = 0 \Rightarrow -\frac{2A}{l^2} = -\frac{B}{2} \Rightarrow l^2 = 4\frac{A}{B} \Rightarrow l = 2\sqrt{\frac{A}{B}}$$

This critical value of $F(l)$ is indeed a global minimum for $\mathbb{R}^+$ for $F''(l) = \frac{4A}{l^3} \geq 0\ \forall l$. Plugging this value of l into our inequality yields:

$$|\, f'(x)| \leq \frac{2A\sqrt{B}}{2\sqrt{A}} + \frac{2B\sqrt{A}}{2\sqrt{B}} = 2\sqrt{AB}$$

□

Problem 5: Let $f(x,y)$ be defined on the unit square

$$S = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$$

and suppose that f has the following properties:

$i.$ For each fixed x, the function $f(x,y)$ is an integrable function of y on the unit interval.

$ii.$ The partial derivative $\frac{\partial f}{\partial x}$ exists at every point $(x,y) \in S$, and is a bounded function on S.

Show that:

a. The partial derivative $\frac{\partial f}{\partial x}$ is a measurable function of y for each x.

b.

$$\frac{d}{dx}\int_0^1 f(x,y)dy \;=\; \int_0^1 \frac{\partial f}{\partial x}(x,y)dy$$

Solution 5:

(a) Consider the following function in the variable y (fix x, h):

$$F_h(y) = \frac{f(x+h,y)+f(x,y)}{h}$$

Since h is a constant, F_h is a measurable function by application of (i). Since a limit of measurable functions is also measurable, we obtain $\frac{\partial f}{\partial x}$ is measurable in y for each x by observing:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0}\frac{f(x+h,y)+f(x,y)}{h} = \lim_{h\to 0} F_h(y)$$

(b) All of the following equalities are elementary or definitional except the second, which is an application of Lebesgue's Dominated Convergence Theorem justified by hypothesis (ii).

$$\int_0^1 \frac{\partial f}{\partial x}(x,y)dy \equiv \int_0^1 \lim_{h\to 0}\frac{(f(x+h,y)+f(x,y))dy}{h} = \lim_{h\to 0}\int_0^1 \frac{(f(x+h,y)+f(x,y))dy}{h} =$$

$$= \lim_{h\to 0}\frac{\int_0^1 (f(x+h,y)+f(x,y))dy}{h} \equiv \frac{d}{dx}\int_0^1 f(x,y)dy$$

□

Problem 6: Suppose that $f(z)$ is a non-constant holomorphic function on a connected open set $U \subset \mathbb{C}$. Suppose that V is an open set such that its closure $\bar{V}$ is a compact subset of U, and suppose that $|f(z)|$ is constant, say k, on the boundary of V. Show that f has at least one zero in V.

Solution 6: By the maximum principle, we have $|\,f(z)| < |\,z'|$, for all $z \in V$, z' on ∂V. So $|\,f(z)| < |\,k|$ on V. Hence f maps V into the open disk of radius k. Now suppose there is no point z_0 of V such that $f : z_0 \mapsto 0$. Then $1/f$ is holomorphic on V as well. By the maximum principle and compactness of domain, it obtains a maximum on ∂V. When $1/f$ is maximized, f is minimized, hence f obtains a minimum on ∂V. Therefore, f has constant modulus $V \Rightarrow f$ is constant there $\Rightarrow f$ is constant on U, a contradiction. Whence f indeed has a zero in V. □

Aug 2001

Problem 1: Suppose f is a positive bounded measurable function on $[0, 1]$ and $F(x) = \int_0^x f(t)dt$ for $0 \leq x \leq 1$.
a. Show that F is continuous on $[0, 1]$.
b. Show that F is differentiable at almost every point in $[0, 1]$.

Solution 1:
(a)

$$|\,F(x) - F(y)| = \left| \int_0^x f(t)dt - \int_0^y f(t)dt \right|$$

Let ϵ be given. By hypothesis, $|f| \leq M$. Without loss of generality, let $y \leq x$. Then we have

$$\left| \int_0^x f(t)dt - \int_0^y f(t)dt \right| = \left| \int_0^y f(t)dt - \int_0^y f(t)dt + \int_y^x f(t)dt \right| =$$
$$= \left| \int_y^x f(t)dt \right| \leq M \cdot |\,x - y\,|$$

So let $\delta = \epsilon/2M$. □

(b) This is an application of Lebesgue's Differentiation Theorem. Since f is bounded, $f \in L^1_{loc}(\mathbb{R})$. Then we have $F'(x) = f(x)$ almost everywhere. $\square$

Problem 2: Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function, m and n are integers, and

$$(2 + |z|^m)^{-1}\frac{d^n f}{dz^n}$$

is bounded on $\mathbb{C}$.

a. Prove that f is a polynomial

b. Estimate the degree of f in terms of m and n.

Solution 2:

(a) f is entire so $d^n f/dz^n$ is entire. Furthermore, by the Extended Louiville Theorem,

$$\text{i.e. since } \quad \frac{d^n f}{dz^n} \leq 2M + M|\,z|^m$$

$d^n f/dz^n$ is a polynomial of degree at most m. Integrating backwards, we have f must be a polynomial.

(b) In (a) we integrate backwards n times. Therefore $deg(f) \leq m + n$. $\square$

Problem 3: Suppose f and g are integrable functions on $\mathbb{R}$. Show that the convolution $h(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$ is integrable with

$$\int_{-\infty}^{\infty} h(x)dx \;\; \leq \;\; \int_{-\infty}^{\infty} f(x)dx \int_{-\infty}^{\infty} g(x)dx$$

Solution 3: Consider the absolute value of the given product function. By Fubini's Theorem for non-negative measurable functions, we have:

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |\,f(y)g(x-y)|\,dy = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |\,f(y)g(x-y)|\,dx$$

Let $(|\,f| * |\,g|)(x)$ denote the convolution $\int_{-\infty}^{\infty} |\,f(y)g(x-y)|\,dy$. Now observe

$$\text{LHS} = \int_{-\infty}^{\infty} (|\,f| * |\,g|\,)(x)dx$$

$$\text{RHS} = \int_{-\infty}^{\infty} |\, f(y)|\, dy \cdot \int_{-\infty}^{\infty} |\, g(x-y)|\, dx = \|f\|_1 \, \|g\|_1$$

Since $f, g \in L^1$, $(|\, f| * |\, g|\,)(x)$ exists a.e. and is integrable. Now note the following inequalities

$$f * g \le |\, f * g| = \left| \int_{-\infty}^{\infty} f(y) g(x-y) dy \right| \le \int_{-\infty}^{\infty} |\, f(y) g(x-y) dy| = |\, f| * |\, g|$$

So incorporating all of the above and using the fact that $h \in L^1 \iff |h| \in L^1$, we obtain $\|f * g\|_1 \le \|f\|_1 \, \|g\|_1$ $\square$

Problem 4: Find the following integral:

$$\int_0^{\infty} \frac{dx}{x^4+1}$$

Solution 4: Let $f(z) = \frac{1}{z^4+1}$. Consider the following contour integral

$$\int_{\gamma_R} \frac{1}{z^4+1} dz = \int_{-R}^{R} \frac{1}{x^4+1} dx + \int_{S_R} \frac{1}{z^4+1} dz$$

where γ_R is the contour that runs from $-R$ to R on the x-axis then over the half circle in the upper half-plane of radius R, and S_R is the half-circle component of the contour. By the residue formula, we have

$$(*) \quad \int_{\gamma_R} \frac{1}{z^4+1} dz = 2\pi i \sum (\text{residues of } f(z) \text{ in the UHP})$$

Now since $\left| \frac{1}{z^4+1} \right| \le \frac{1}{|\, z|^4} \le \frac{1}{R^4}$ the following is true:

$$\lim_{R \to \infty} \int_{S_R} \frac{1}{z^4+1} dz = 0$$

Since $(*)$ holds for all R, we obtain via the limit $R \to \infty$ that

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = 2\pi i \sum (\text{residues of } f(z) \text{ in the UHP})$$

So now all we need to do is calculate the residues in the UHP. The resides of f occur at the singularities of f, which are the zeros of $z^4 + 1$. Those zeros are

$e^{\pi i/4}$, $e^{3\pi i/4}$, $e^{5\pi i/4}$, $e^{7\pi i/4}$. Only the first two lie in the UHP. The residues are given by the following function (evaluated at each singularity)

$$\frac{1}{f'(z)} = \frac{1}{4z^3}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}dx \;=\; 2\pi i(\frac{1}{4}e^{-3\pi i/4} + \frac{1}{4}e^{-\pi i/4}) = (\frac{\pi i}{2})(e^{-\pi i/4})(1+e^{-\pi i/2}) =$$

$$= (\frac{\pi i}{2})(\frac{1-i}{\sqrt{2}})(1-i) = \frac{\pi}{\sqrt{2}}$$

□

Problem 5:

a. Show that any sequence f_n of non-negative integrable functions on $[0,1]$ with $\int_0^1 f_n^2 dx \leq \frac{1}{n^3}$ must converge to zero almost everywhere.

b. Is there a sequence g_n of non-negative integrable functions on $[0,1]$ satisfying $\int_0^1 g_n^2 dx \to 0$ which does *not* converge to zero almost everywhere? Explain.

Solution 5:

(a) The first equality in the following line is a corollary of Lebesgue's Monotone Convergence Theorem (using non-negativity).

$$\int_0^1 (\sum_0^{\infty} f_n^2(x))dx = \sum_0^{\infty} \int_0^1 f_n^2(x)dx \leq \sum_0^{\infty} \frac{1}{n^3} < \infty$$

$$\int_0^1 (\sum_0^{\infty} f_n^2(x))dx < \infty \;\Rightarrow\; \sum_0^{\infty} f_n^2(x) < \infty \;\; a.e. \;\Rightarrow\; f_n^2(x) \to 0 \;\; a.e.$$

$$\Rightarrow\; f_n(x) \to 0 \;\; a.e.$$

(b) Define $g_n : \mathbb{R} \to [0, \infty)$ by $g_n(x) = \chi_{A_n}$ where $A_1 = [0, 1], A_2 = [0, \frac{1}{2}], A_3 = [\frac{1}{2}, 1], A_4 = [0, \frac{1}{3}], A_5 = [\frac{1}{3}, \frac{2}{3}], A_6 = [\frac{2}{3}, 1], A_7 = [0, \frac{1}{4}], ...$

$$\lim_{n\to\infty} \int_0^1 g_n(x)dx = \lim_{n\to\infty} \lambda(A_n) = 0$$

Yet for each point, no matter how much time passes, there will be infinitely many A_n that cover it. Hence *no* point has a limit. Notice that the integral tends very slowly to 0. This is caused by requiring a covering of $[0, 1]$ for each interval width. □

Problem 6: Suppose $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

a. Is the function $\frac{1}{z}$ the $uniform$ limit of a sequence of polynomials on A. Explain.

b. Is the function $\frac{1}{z-3}$ the $uniform$ limit of a sequence of polynomials on A. Explain.

Solution 6:

(a) No. If polynomials converge uniformly for $|\,z| = R$ then they converge uniformly for all $0 \le r \le R$. But $\frac{1}{z}$ blows up at 0 and hence cannot be such a limit.

(b) Yes. At $z_0 = 0$, $\frac{1}{z-3}$ has a radius of convergence $r = 3$, the distance to its singularity $z = 3$. Since the annulus is inside its radius of convergence, the power series for $\frac{1}{z-3}$ converges absolutely and uniformly on A. Thus let $p_n(z) = \sum_0^n a_n z^n$, the partial sums of the power seires. □

Jan 2002

Problem 1: Let $\Omega = \mathbb{C}\backslash\{\mathbb{R}^-\}^c$. Is there a non-constant bounded holomorphic function on Ω? Justify your answer.

Solution 1: Ω is star-convex (take -1 as the star center) hence simply connected. Thus the Riemann Mapping Theorem applies to Ω. I.e. Ω is analytically isomorphic to the unit disk. Therefore there exists an analytic map $h : \Omega \to \mathbb{D}$. $\square$

Problem 2: A function $f : \mathbb{R} \to \mathbb{R}$ is *convex* if
$f((1-t)a + tb) \leq (1-t)f(a) + tf(b)\ \forall a, b \in \mathbb{R}$ and $0 \leq t \leq 1$.
a. Prove that a convex function is continuous
b. Prove that a twice differentiable function f is convex if $f'' \geq 0$.

Solution 2:
(a) The convexity condition above is equivalent to

$$a < b < c \quad \Rightarrow \quad \frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(b)}{c - b}$$

So let $x \in \mathbb{R}$ be given, and choose $s < l < x < r < t$. Then our equivalent convexity condition implies

$$\frac{f(l) - f(s)}{l - s} \leq \frac{f(x) - f(l)}{x - l} \leq \frac{f(t) - f(x)}{t - x}$$

and

$$\frac{f(x) - f(s)}{x - s} \leq \frac{f(r) - f(x)}{r - x} \leq \frac{f(t) - f(r)}{t - r}$$

Therefore

$$\text{either} \quad |f(x) - f(l)| \leq |x - l| \left| \frac{f(t) - f(x)}{t - x} \right| \quad (*)$$

$$\text{or} \quad |f(x) - f(l)| \leq |x - l| \left| \frac{f(l) - f(s)}{l - s} \right| \quad (**)$$

and

$$\text{either} \quad |f(r) - f(x)| \leq |r - x| \left| \frac{f(t) - f(r)}{t - r} \right| \quad (***)$$

$$\text{or} \quad |f(r) - f(x)| \leq |r - x| \left| \frac{f(x) - f(s)}{x - s} \right| \quad (****)$$

(*) and (****) are independent of l and r respectively thus left and right continuity holds in those cases (let $l \to x$ and $r \to x$). Using the following estimates we similarly have left and right continuity for (**) and (***) respectively

$$|f(l)| \leq \max \left\{ |f(s)| \, , \, \left| f(x) + s \frac{f(t) - f(x)}{t - x} \right| \right\}$$

$$|f(r)| \leq \max \left\{ |f(t)| \, , \, \left| f(x) + t \frac{f(s) - f(x)}{s - x} \right| \right\}$$

(b) DO THIS PART

Problem 3: If f is a nowhere zero, entire holomorphic function, is there necessarily an entire holomorphic function g such that $e^g = f$. Prove your answer.

Solution 3: Define

$$h(z) \equiv \int_{z_0}^{z} \frac{f'(t)}{f(t)} dt$$

Hence $h'(z) = f'(z)/f(z)$. Thus we have

$$\left(\frac{f}{e^h} \right)' = \left(f e^{-h} \right)' = f' e^{-h} + (-h') f e^{-h} = e^{-h} (f' - f') = 0$$

And so f/e^h is constant, which implies $e^h = \alpha f$. So let $g \equiv h + \ln \alpha$. □

Problem 4:

a. Prove that if f_n is a sequence of Lebesgue measurable functions on $[0, 1]$ with

$\lim_{n\to\infty}\int_0^1 f_n^2 dx = 0$, then $\lim_{n\to\infty}\int_0^1 |f_n| dx = 0$.
b. Does it also follow that $\lim_{n\to\infty} f_n(x) = 0$ for a.e. $x \in [0,1]$?
c. Is the implication in (a) still true if $[0,1]$ is replaced by the whole real line $\mathbb{R}$?

Solution 4:
(a) $\lim_{n\to\infty}\int_0^1 f_n^2 dx = 0 \Rightarrow \lim_{n\to\infty}(\int_0^1 f_n^2 dx)^{\frac{1}{2}} = 0$. I.e. $\lim_{n\to\infty}||f||_2 = 0$. Since $||f_n||_1 \leq ||f_n||_2$ on the unit interval, we have $\lim_{n\to\infty}||f||_1 = 0$. $\square$

(b) No. Consider the following family of sets, $\{A_i\}_{i=1}^{\infty}$ where
$A_1 = [0,1]$, $A_2 = [0,\frac{1}{2}]$, $A_3 = [\frac{1}{2},1]$, $A_4 = [0,\frac{1}{3}]$, $A_5 = [\frac{1}{3},\frac{2}{3}]$, $A_6 = [\frac{2}{3},1]$, ...
Let $f_n = \chi_{A_n}$. Then we have $\lim_{n\to\infty}\int_0^1 |f_n(x)| dx = 0$ but $\lim_{n\to\infty} f_n(x)$ does not exist for any $x \in [0,1]$. $\square$

(c) The conclusion does not extend to $\mathbb{R}$. Consider the following counterexample.

$$f_n(x) = \chi_{[0,n]} \cdot \left(\frac{1}{n}\right)$$

Note that

$$\lim_{n\to\infty}\int_0^{\infty} f_n^2(x) dx = \lim_{n\to\infty}\left(\frac{1}{n^2}\right)(n) = \lim_{n\to\infty}\frac{1}{n} = 0$$

while

$$\lim_{n\to\infty}\int_0^{\infty} |\, f_n(x)|\, dx = 1$$

$\square$

Problem 5: Does there exist an analytic function mapping the annulus

$$A = \{z : 1 \leq |z| \leq 4\} \ \textit{onto} \ B = \{z : 1 \leq |z| \leq 2\}$$

and taking $C_1 \to C_1$, $C_4 \to C_2$, where C_r is the circle of radius r? Why or why not?

Solution 5: Define $g(z) = z^2$. Consider $g : B \to A$. $g : C_1 \mapsto C_1$ and $g : C_2 \mapsto C_4$. Then:

$$\left| \frac{(g \circ f)(z)}{z} \right| = 1 = \left| \frac{z}{(g \circ f)(z)} \right|$$

But then the maximum modulus principle implies that $g \circ f = c$, a constant with $|c| = 1$. In turn this implies that $f(z) = c\sqrt{z}$. But $\sqrt{z}$ is not a surjective mapping from A to B for $\sqrt{z} : A \hookrightarrow B \cap \overline{UHP}$. Since $|c| = 1 \Rightarrow c\sqrt{z}$ is some rotation of the half annulus, we cannot possibily find a surjective map from A to B. $\square$

Problem 6: Suppose that $f \in L^1(\mathbb{R})$. Find a necessary and sufficient condition such that

$$\lim_{t \to 0} \frac{||f + tg||_1 - ||f||_1}{t}$$

exists for all $g \in L^1(\mathbb{R})$.

Solution 6: (Missing)

Jan 2003

Problem 1: How many roots of the equation $f(z) = z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane? Hint: Apply the argument principle to a large half disk.

Solution 1: Let γ_R be the contour which travels along the circle of radius R in the right half-plane and then from iR to $-iR$ on the imaginary axis. On the half-circle, i.e. $z = Re^{it}$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$, we have:

$$f(z) = f(Re^{it}) = R^4 e^{4it}(1 + \frac{8}{Re^{it}} + \frac{3}{R^2 e^{2it}} + \frac{8}{R^3 e^{3it}} + \frac{3}{R^4 e^{4it}}) = R^4 e^{4it}(1 + \zeta)$$

where $|\zeta| \leq 22/R < \epsilon$ for any ϵ for R large enough. So the argument of $f(Re^{it})$ is approximately the argument of e^{4it}. As we move along the half-circle, t goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, thus the argument of $f(z)$ changes by approximately 4π along this part of the contour. Now lets consider the argument of $f(z)$ along the imaginary axis section of γ. I.e. $z = iy$ where $y \in [-R, R]$. So $f(iy) = y^4 - 8iy^3 - 3y^2 + 8iy + 3$. Hence:

$$ref(iy) = y^4 - 3y^2 + 3 \quad \text{and} \quad imf(iy) = -8y^3 + 8y$$

So for R large enough, $f(iR)$'s real part (quartic) dominates its imaginary part (cubic). Hence for R large enough the change in argument on the vertical segment is approximately zero. Therefore by the Argument Principle, the number of roots in the RHP is $\frac{1}{2\pi}(4\pi + 0) = 2$. □

Problem 2: Suppose that f and g are entire functions, and $|f(z)| \leq |g(z)|$ for all z with $|z| \geq 1000$. Show that f/g is a rational function.

Solution 2: g is entire but g may have some zeros. But there can only be finitely many. For if there are infinitely many in $\Omega = \{z : |z| \leq 1000\}$, the set of zeros has a limit point there. So then $g = 0$ on Ω. Hence $g = 0$ on $\mathbb{C}$ which implies $f = 0$ on $\mathbb{C}$. So let $\{\alpha\}_1^N$ be the set of roots of g in Ω. Let $q(z) = (z - \alpha_1)(z - \alpha_2)...(z - \alpha_N)$. Then qf/g is entire. Furthermore, by hypothesis $|\, qf/g| \leq |\, q|$ for all z such that $|z| \geq 1000$. The Extended Liouville Theorem implies qf/g is a polynomial, p of degree at most N. Therefore $f/g = p/q$ is a rational function. □

Problem 3: Suppose that f is a nonvanishing holomorphic function on the punctured disk $\Omega \equiv \{z : 0 < |z| < 2\}$, and $\lim_{z \to 0} |f(z)| = +\infty$.

a. Prove that $\lim_{z \to 0} z^N f(z) = \alpha$ for some positive integer N and nonzero complex number α.

b. Find a formula for α in terms of an integral over the unit circle of some expression in terms of f and N.
c. Find a formula for N in terms of an integral over the unit circle of some expression in terms of f and f'.

Solution 3:
(a) Consider $g(z) \equiv 1/f(z)$. Since f is non-vanishing on Ω, $g : \Omega \to \mathbb{C}$ is holomorphic. Furthermore by hypothesis $\lim_{z\to 0} = 0$. Hence 0 is a removable singularity for g. So g may be extended to a holomorphic function $G : \Omega \cup \{0\} \to \mathbb{C}$ given by:

$$G(z) = \begin{cases} g(z) & \text{on } \Omega \\ 0 & \text{at } 0 \end{cases} .$$

Hence for some unique positive n we have that $G(z) = z^N \cdot Q(z)$ where Q is holomorphic on $\Omega \cup \{0\}$ and $Q(0) \neq 0$. $G(z)$ is non-vanishing on Ω implies that $Q(z)$ is non-vanishing on $\Omega \cup \{0\}$. It thus follows that $1/Q(z)$ is a holomorphic function on $\Omega \cup \{0\}$. Therefore on $\Omega \cup \{0\}$

$$1/Q(z) = \sum_{j=0}^{\infty} \beta_j z^j$$

For $0 < |\,z| < 2$

$$\begin{aligned} f(z) &= \frac{1}{G(z)} \\ &= z^{-N} \cdot \frac{1}{Q(z)} \\ &= z^{-N} \cdot \left(\sum_{j=0}^{\infty} \beta_j z^j \right) \\ &= \sum_{j=-N}^{\infty} \beta_{j+N} z^j \end{aligned}$$

By uniqueness of Laurent expansions, this series agrees with f on Ω so it must indeed be it's Laurent expansion. Our desired result follows by observing that

$$\lim_{z\to 0} \big(z^N \cdot f(z) \big) = \lim_{z\to 0} \left(z^N \cdot \sum_{j=-N}^{\infty} \beta_{j+N} z^j \right) = \beta_0$$

Note: This proof was tailored from the development in *Function Theory of One Complex Variable* by Greene & Krantz. Their development of meromorphic functions is excellent.

(b) Let γ be the curve that traverses the unit circle once in the positive orientation. Cauchy's Integral Formula states

$$\int_\gamma f(z)dz = 2\pi i \cdot \mathrm{Res}_0(f)$$

Recall that $\mathrm{Res}_0(f)$ is defined as the $i = -1$ coefficient in the Laurent expansion at 0 for f. So to find α in the statement (β_0 in the proof of part (a)), we need only take the following integral

$$\frac{1}{2\pi i}\int_\gamma z^{N-1} f(z)dz$$

(c) The following is a consequence of the Reside Theorem. Assume f is meromorphic in U, with finite number of zeros and poles there. Let $a_1, a_2, \dots, a_r$ be the zeros in U and $b_1, b_2, \dots, b_n$ be the poles in U. Also assume that γ is a closed 0-homologous curve in U, then

$$\int_\gamma \frac{f'}{f} = 2\pi i \left(\sum_{j=1}^{r} m_j \,\mathrm{ord}_{a_j} f - \sum_{k=1}^{n} m_k \,\mathrm{mul}_{b_k} f \right)$$

where m_i is the winding number relative γ at each point a_i (likewise for the poles). In this case, we have that inside $\mathbb{D}(0,2)$ f has no zeros (non-vanishing), has precisely one singularity at 0, and γ winds about 0 once. Hence the formula above becomes:

$$\int_\gamma \frac{f'}{f} = 2\pi i(-\,\mathrm{mul}_0 f)$$

Solving for $\mathrm{mul}_0 f$ we obtain

$$N = \mathrm{mul}_0 f = \frac{-1}{2\pi i}\int_\gamma \frac{f'}{f}$$

□

Problem 4: Calculate

$$\lim_{n\to\infty}\int_0^{1/n}\frac{n}{1+n^2x^2+n^6x^8}dx$$

Show all your work and justify all steps.

Solution 4: Let $y = nx$, then $dy = ndx$ so

$$A = \lim_{n\to\infty}\int_0^{1/n}\frac{n}{1+n^2x^2+n^6x^8}dx \;=\; \lim_{n\to\infty}\int_0^{1}\frac{1}{1+y^2+\frac{1}{n^2}y^8}dy$$

$$\frac{1}{1+y^2+\frac{1}{n^2}y^8} \text{ is dominated by } \frac{1}{1+y^2} \in L^1([0,1])$$

So Lebesgue's Dominated Convergence Theorem implies

$$A = \int_0^1 \lim_{n\to\infty}\frac{1}{1+y^2}dy = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

□

Problem 5: Suppose f is a nonnegative measurable function defined on $\mathbb{R}^k$. Let $a(n) = \int_{\mathbb{R}^k} f^n$, for $n = 1, 2, 3, ...$

a. Suppose that the infinite series $\sum a(n)$ converges. Show that $f < 1$ almost everywhere and that $f/(1-f)$ is in L^1.

b. Prove the converse.

Solution 5:

(a) First suppose that $f \geq 1$ on a set $B \subset \mathbb{R}^k$ with $\lambda(B) = \epsilon$. Then

$$\sum_{n=1}^{\infty} a(n) = \sum_{n=1}^{\infty}\int_{\mathbb{R}^k} f^n \geq \sum_{n=1}^{\infty}\int_B f^n \geq \sum_{n=1}^{\infty}\epsilon = \infty$$

Which is a contradiction to the hypothesis. Hence $f < 1$ *a.e.*. Using this result we obtain that

$$\frac{f}{1-f} = \sum_{n=1}^{\infty} f^n$$

Since f is non-negative, as a consequence of Lebesgue's Monotone Convergence Theorem we have

$$\int_{\mathbb{R}^k} \frac{f}{1-f} = \int_{\mathbb{R}^k} \sum_{n=1}^{\infty} f^n = \sum_{n=1}^{\infty} \int_{\mathbb{R}^k} f^n = \sum_{n=1}^{\infty} a(n) < \infty$$

Therefore $f/(1-f) \in L^1$.

(b) The above arguments are reversible, i.e.

$$\infty > \int_{\mathbb{R}^k} \frac{f}{1-f} = \int_{\mathbb{R}^k} \sum_{n=1}^{\infty} f^n = \sum_{n=1}^{\infty} \int_{\mathbb{R}^k} f^n = \sum_{n=1}^{\infty} a(n)$$

□

Problem 6:

a. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function with $f(0) = 0$, $f(-x) = f(x)$, and $|f''(x)| \leq 3$ for all $x \in \mathbb{R}$. Find the best bound M so that $|f(x)| \leq M$ for all $x \in [-2, 2]$.

b. Suppose that f_n is a sequence of functions satisfying all the conditions on f from (a). Prove that f_n contains at least one subsequence that converges uniformly on every bounded subset of $\mathbb{R}$ to a continuous function.

Solution 6:

(a) First note that the hypothesis $f(-x) = f(x)$ and $f(0) = 0$ implies that $f'(0) = 0$. Now by the Fundamental Theorem of Calculus we have

$$f(x) = f(0) + \int_0^x f'(t)dt = \int_0^x f'(t)dt = \int_0^x \left(f'(0) + \int_0^t f''(s)ds\right) dt$$
$$= \int_0^x \left(\int_0^t f''(s)ds\right) dt$$

Taking absolute value on both sides we obtain:

$$|\,f(x)| \leq \int_0^x \left(\int_0^t |\,f''(s)ds|\right) dt = \int_0^x \left(\int_0^t 3ds\right) dt = \int_0^x 3tdt = \frac{3x^2}{2}$$

Hence any function satisfying the hypothesis is absolutely bounded by 6 on $[-2, 2]$. This bound is shown to be a best estimate by the function $3x^2$.

(b) Let $\mathfrak{F}$ be a family of functions that satisfy the hypothesis, and suppose $f \in \mathfrak{F}$. On any interval $[-R, R]$ we have the following

$$|\,f(y) - f(x)| = \left|\int_0^y f'(t)dt - \int_0^x f'(t)dt\right| = \left|\int_x^y f'(t)dt\right| \leq \int_x^y |\,f'(t)|\,dt \leq$$
$$\leq \int_x^y 3Rdt = 3R\,|\,y - x|$$

The above inequlality implies that $\mathfrak{F}$ is equicontinuous. For given an $\epsilon > 0$ we may set $\delta = \epsilon/6R$ to guarantee that $|\,y - x| < \delta \ \Rightarrow \ |\,f(y) - f(x)| < \epsilon$ for all $f \in \mathfrak{F}$. Hence the Arzela-Ascoli Theorem implies that on every interval $[-R, R]$, $\mathfrak{F}$ has a uniformly convergent subsequence (which converges to a continous function). We may find a convergent subsequence on $\mathbb{R}$ via the following diagonal argument. Let F_n^1 be the convergent subsequence on $[-1,1]$, F_n^2 be the convergent subsequence on $[-2,2]$, . . . , F_n^k be the convergent subsequence on $[-k,k]$, . . . Now define a new subsequence F_n by setting $F_i = F_i^i$. From the construction it is clear that F_n converges uniformly on all bounded subsets of $\mathbb{R}$. □

May 2003

Problem 1: Suppose that $x_1 > 0$ and $x_{n+1} = (2+x_n)^{-1}$ for $n = 1, 2, 3, \ldots$. Prove that the sequence x_n converges and find its limit.

Solution 1: First we may find the fixed points for the mapping by setting the following equality

$$x_n = x_{n+1} \Rightarrow x_n = (2+x_n)^{-1} \ \Rightarrow \ x_n^2 + 2x_n - 1 = 0$$

So we obtain that the fixed points for the map are $-1 \pm \sqrt{2}$. Since $x_1 > 0$ and the map preserves positivity, the possible limit point is the fixed point $-1+\sqrt{2}$. We need only prove that the map indeed converges. Let $M(x_n) = (2+x_n)^{-1}$. Now observe:

$$|\,M(x_{n+1}) - M(x_n)| = \left| \frac{1}{2+x_{n+1}} \quad - \quad \frac{1}{2+x_n} \right| =$$

$$= \left| \frac{x_n - x_{n+1}}{4 + 2x_n + 2x_{n+1} + x_n x_{n+1}} \right| \leq \frac{1}{4} |\, x_{n+1} - x_n|$$

The inequality follows from the fact that x_n, x_{n+1} are positive $\forall n$ (since $x_1 > 0$). So now we invoke Banach's Fixed Point Theorem to obtain that iteration of the mapping does converge to our fixed point. □

Problem 2: Evaluate

$$\int_0^\infty \frac{\log x}{x^2+a^2}dx$$

where $a > 0$.

Solution 2: C_r as usual denotes the circle of radius R. Let γ_1 be the contour that traverses C_R from $z = R$ to $z = -R$ in the positive orientation (counter-clockwise). Let γ_2 be the contour that traverses C_δ from $z = -\delta$ to $z = \delta$ in the negative orientation. Suppose at all stages to come that $\delta < \frac{1}{2}$ and $R > 2$. Let γ_3 be the straight-line contour on the x-axis from δ to R, and γ_4 be the straight-line contour on the x-axis from $-R$ to $-\delta$.

Now consider the contour integral

$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_4} + \int_{\gamma_2} + \int_{\gamma_3}$$

By the Residue Theorem, we know that

$$\int_\gamma f(z)dz = 2\pi i \sum (\text{residues of } f \text{ in the } UHP)$$

Now put

$$f(z) = \frac{\log z}{z^2+a^2}$$

First we show that $\int_{\gamma_1} f(z)dz \longrightarrow 0$ as $\mathbb{R} \to \infty$

$$\int_{\gamma_1} f(z)dz = \int_0^\pi \frac{\log Re^{it}}{R^2e^{2it}+a^2} iRe^{it}dt = \int_0^\pi \frac{\log R + it}{Re^{it} + \frac{a^2}{Re^{it}}} idt$$

Note that the RHS tends to 0 as R approaches ∞.

Next we will show that $\int_{\gamma_2} f(z)dz \longrightarrow 0$ as $\delta \to 0$.

$$\int_{\gamma_2} f(z)dz = \int_0^\pi \frac{\log(\delta e^{it})}{\delta^2 e^{2it}+a^2} i\delta e^{it}dt = \int_0^\pi \frac{(\log\delta)\left(\delta e^{it} + \delta e^{it}(it)\right)}{\delta^2e^{2it}+a^2} idt$$

$$= \int_0^\pi \frac{(\delta\log\delta)\left(e^{it} + ite^{it}\right)}{\delta^2e^{2it}+a^2} idt = (\delta\log\delta)\int_0^\pi \frac{\left(e^{it}+ite^{it}\right)}{\delta^2e^{2it}+a^2} idt$$

Therefore $\lim_{\delta\to 0}(\delta \log \delta) = 0 \;\Rightarrow\; \int_{\gamma_2} f(z)dz = 0$.

Since the Residue Theorem result holds for all δ, R, taking the respective limits as we did above, we obtain that

$$\begin{aligned}\int_\gamma f(z)dz &= \lim_{\delta\to 0}\left(\int_{-\infty}^{-\delta} \frac{\log(-x) + i\pi}{x^2+a^2}dx + \int_\delta^\infty \frac{\log(x)}{x^2+a^2}dx\right)\\ &= 2\int_0^\infty \frac{\log(x)}{x^2+a^2}dx + \int_0^\infty \frac{i\pi}{x^2+a^2}dx\end{aligned}$$

Substituting $2\pi i(\mathrm{Res}_f(ai))$ for $\int_\gamma f(z)dz$ we obtain that

$$(*) \qquad \int_0^\infty \frac{\log(x)}{x^2+a^2}dx = \frac{1}{2}\left(2\pi i(\mathrm{Res}_f(ai)) - \int_0^\infty \frac{i\pi}{x^2+a^2}dx\right)$$

We have

$$\int_0^\infty \frac{i\pi}{x^2+a^2}dx = \frac{i\pi}{a^2}\int_0^\infty \frac{1}{\left(\frac{x}{a}\right)^2+1}dx = \frac{i\pi}{a}\left(\arctan\left(x/a\right)\right)\Big|_0^\infty = \frac{i\pi}{a}\frac{\pi}{2} - 0 = \frac{i\pi^2}{a}$$

And

$$\mathrm{Res}_f(ai) = \left(\log(z)\cdot\frac{1}{2z}\right)\Bigg|_{z=ai} = \frac{\log a + i\pi}{2ai}$$

Substituting these two values into $(*)$ we have

$$\int_0^\infty \frac{\log(x)}{x^2+a^2}dx = \frac{1}{2}\left(\frac{\pi\log a + i\pi^2}{a} - \frac{i\pi^2}{a}\right) = \frac{\pi \log a}{2a}$$

□

Problem 3: Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $\int_{-\infty}^{\infty} |f(x)|dx < \infty$.
a. Show that the Lebesgue measure of $\{x : |f(x)| > t\}$ approaches 0 as $t \to \infty$.
b. Show that the additional assumption $\int_{-\infty}^{\infty} |f'(x)|dx < \infty$ implies that $f(x) \to 0$ as $x \to \infty$.

Solution 3:

(a) Recall the Chebyshev Inequality for $L^1(\mathbb{R})$:

$$\lambda\{x : |f(x)| > t\} \leq \frac{\|f\|_1}{t}$$

Taking the limit as $t \to \infty$ we obtain the desired inequality.

(b) The additional hypothesis implies that $f'(x) \to 0$ as $x \to \infty$. Since f is differentiable, the Fundamental Theorem of Calculus gives us: $f(x) = f(x-1) + \int_{x-1}^{x} f'(t)dt$. Thus we obtain:

$$|f(x)| \leq |f(x-1)| + \left| \int_{x-1}^{x} f'(t)dt \right| \leq |f(x-1)| + \int_{x-1}^{x} |\, f'(t)|\, dt$$

Now taking a limit yields the result:

$$\lim_{x\to\infty} |f(x)| \leq \lim_{x\to\infty} (|f(x-1)| + \int_{x-1}^{x} |\, f'(t)|\, dt) = 0 + 0 \cdot 1 = 0$$

Note: the last equality follows from the application of Lebesgue's Dominated Convergence Theorem. We can apply it here because $f'(t)$ is a continuous function integrated over a compact domain. One could alternatively view the limit as $x \to \infty$ on the integral as a limit of a family of continuous functions defined on $[0,1]$. I.e. $g_n : [0,1] \to \mathbb{R}$ given by $g_n(x) = f'(x + (n-1))$. The application of LDCT from this viewpoint is more easily justified. □

Problem 4: Find all entire functions f which satisfy $Re(f(z)) \leq \frac{2}{|z|}$ for $|z| > 1$. Prove your answer.

Solution 4: Since f is entire, e^f is entire. Furthermore, $\left| e^{f(z)} \right| = \left| e^{Ref(z)} \right| \leq e^2$ for $|z| \geq 1$. The maximum modulus principle implies that $e^{f(z)} \leq e^2$ on $D(0,1)$. So e^f is entire *and* bounded on $\mathbb{C}$. By Liouville's Theorem e^f is constant which implies f is constant. □

Problem 5: Suppose that f is a bounded measurable function on $\mathbb{R}$ and $\int_{-\infty}^{\infty} |g(x)|dx < \infty$. Prove that

$$\lim_{t \to 0} \int f(x)[g(x+t) - g(x)]dx = 0$$

Solution 5: Suppose f is bounded on $\mathbb{R}$ by M, we may write:

$$\int f(x)[g(x+t)-g(x)]dx \leq \left| \int f(x)[g(x+t) - g(x)]dx \right| \leq \int |f(x)[g(x+t)-g(x)]|dx \leq$$

$$\leq \int M|[g(x+t) - g(x)]|dx$$

Now since $g \in L^1$ by hypothesis, we have continuity of translation. I.e. if we take the limits on both sides of the string of inequalities above, we obtain:

$$\lim_{t \to 0} \int f(x)[g(x+t) - g(x)]dx \leq \lim_{t \to 0} \int M \,|\, [g(x+t) - g(x)]|\, dx =$$

$$= M \lim_{t \to 0} \int |\, [g(x+t) - g(x)]|\, dx = M \cdot 0 = 0$$

□

Problem 6:

a. State Rouche's Theorem.

b. State Schwarz's Lemma.

c. Suppose f is holomorphic in the unit disk $|z| < 1$ with $|f(z)| \leq 1$ and $f(0) = 0$. Prove that for any integer $n \geq 1$, $f(z) - 2^n z^n$ has precisely n zeros (counting multiplicity) in the disk $|z| < \frac{1}{2}$.

Solution 6:

(a) Rouche's Theorem: Let γ be a closed path homologous to 0 in U and assume that γ has an interior. Let h, g be analytic on U, and $|\, h(z) - g(z)| < |\, h(z)|$ for z on γ.

Then h and g have the same number of zeros in the interior of γ.

(b) Schwarz's Lemma: If $f(z)$ is analytic for $|z| < 1$ and satisfies the conditions $|f(z)| \le 1$, $f(0) = 0$, then $|f(z)| \le |z|$ and $|f'(0)| \le 1$.

(c) Let γ be the boundary of $|z| < \frac{1}{2}$ oriented in the positive direction. Let $h(z) = -2^n z^n$, $g(z) = f(z) - 2^n z^n$. Then $|h(z) - g(z)| = |f(z)|$. Applying the Schwarz Lemma we obtain that $|f(z)| \le \frac{1}{2}$ on $|z| = \frac{1}{2}$. Since $|h(z)| = 2^{n-1} \ge 1$ for $n = 1, 2, 3, \ldots$, Rouche's Theorem implies that $h(z), g(z)$ have the same number of zeros inside γ. Therefore, $f(z) - 2^n z^n$ has precisely n zeros inside $|z| = \frac{1}{2}$. □

Aug 2003

Note: the solutions to this exam's problems were posted by Professor Hardt following the exam. Permission has been given to reproduce his solutions here. Any mistakes below are certainly due to error in reproduction.

Problem 1: Suppose $0 < \alpha < 2$.
a. The *principal value integral* of $\int_0^\infty \frac{x^\alpha}{x - x^3} dx$ is

$$\lim_{\epsilon \to 0} \int_0^{1-\epsilon} \frac{x^\alpha}{x - x^3} dx + \int_{1-\epsilon}^\infty \frac{x^\alpha}{x - x^3} dx$$

Why is this definition necessary?
b. Compute this integral.
c. Show that the answer you obtained in (b) is in agreement with the change of variables $x = \frac{1}{y}$ in the integral.

Solution 1:

(a) Here it is necessary to consider a limit (called an improper integral) because the function $\frac{x^\alpha}{x-x^3}$ is *not* integrable near $x = 1$. It is locally integrable near 0 and ∞. The choice of $1-\epsilon$ and $1+\epsilon$ for approximating integration limits is the simplest (the principal value) as opposed to other choices such as $1-\epsilon$, $1+\epsilon^2$.

(b) Here one can apply the Cauchy integral formula to the principal branch $f(z)$ of $-\frac{z^\alpha}{z(z-1)(z+1)}$ on the domain Ω_ϵ in the upper half plane bounded by the 4 oriented intervals $[-\frac{1}{\epsilon}, -1-\epsilon]$, $[-1+\epsilon, -\epsilon]$, $[-\epsilon, \epsilon]$, $[1+\epsilon, \frac{1}{\epsilon}]$, the upper counterclockwise oriented semicricle $C_{1/\epsilon}$ of radius $\frac{1}{\epsilon}$, and 3 clockwise oreiented upper semicircles D_ϵ^{-1}, D_ϵ^0, D_ϵ^{+1} of radius ϵ centered at $-1, 0, 1$ respectively. Ae $\epsilon \longrightarrow 0$, the integral over the big semicricle $C_{1/\epsilon}$ approaches 0 because there $|f(z)|(\frac{1}{\epsilon}) \leq C \cdot \left(\frac{1}{\epsilon}^{1+\alpha-3}\right) \longrightarrow 0$. The integrals over D_ϵ^{+1} and D_ϵ^{-1} approach $-\pi i$ times the residues of f at $+1$ and -1, which are $-\frac{1}{2}$ and $-\frac{1}{2}e^{\pi i\alpha}$. On the positive X-axis $f(z) = \frac{x^\alpha}{x-x^3}$ and on the negative X-axis $f(z) = \frac{e^{\pi i\alpha}|x|^\alpha}{x-x^3}$. Changing variables $x \mapsto -x$ for the integrals on the negative X-axis now gives

$$0 = \int_{\Omega_\epsilon} f(z)dz = o(\epsilon) + \frac{\pi i}{2}(1+e^{\pi i\alpha}) + (1-e^{\pi i\alpha})(\int_\epsilon^{1-\epsilon} + \int_{1+\epsilon}^{1/\epsilon})\frac{x^\alpha}{x-x^3}dx$$

Letting $\epsilon \longrightarrow 0$ the principal value integral equals

$$\int_0^\infty \frac{x^\alpha}{x-x^3}dx = (\frac{\pi i}{2})\frac{e^{\pi i\alpha}+1}{e^{\pi i\alpha-1}} = \frac{\pi}{2}\cot(\frac{\pi\alpha}{2})$$

□

(c) Consider the following

$$I(\alpha) = \int_0^\infty \frac{x^\alpha}{x-x^3}dx = \int_\infty^0 \frac{y^{-\alpha}}{\frac{1}{y}-\frac{1}{y^3}}\frac{-dy}{y^2} = \int_0^\infty \frac{y^{-\alpha}}{y-\frac{1}{y}}dy =$$

$$= \int_0^\infty \frac{y^{2-\alpha}}{y^3-y}dy = -\int_0^\infty \frac{y^{2-\alpha}}{y-y^3}dy = -I(2-\alpha)$$

and $\cot\frac{\pi(2-\alpha)}{2} = \cot(\pi - \frac{\pi\alpha}{2}) = -\cot(\frac{\pi\alpha}{2})$ agrees. □

Problem 2: Suppose that $f(x, y)$ is continuous on the plane and that there is a finite M so that $|f(x, y) - f(x, z)| \leq M|y - z|$ for all $x, y, z \in \mathbb{R}$.
a. Show that, for any $x \in \mathbb{R}$, the partial derivative $\frac{\partial f}{\partial y}(x, y)$ exists for almost all $y \in \mathbb{R}$.
b. Prove that $\frac{d}{dy} \int_0^1 f(x, y)dx = \int_0^1 \frac{\partial f}{\partial y}(x, y)dx$.
c. Express $\frac{d}{dy} \int_0^{y^2} f(x, y)dx$ in terms of integrals of f and $\frac{\partial f}{\partial y}$.

Solution 2:
(a) For any $x \in \mathbb{R}$, the function $f(x, \cdot)$ is Lipschitz, hence absolutely continuous. So the partial derivative $\frac{\partial f}{\partial y}(x, y)$ exists for almost all $y \in \mathbb{R}$.

(b) For each $y \in \mathbb{R}$, $x \in [0, 1]$, and sequence $\epsilon_i \to 0$ let

$$g_i(x, y) = \frac{f(x, y + \epsilon_i) - f(x)}{\epsilon_i}$$

Then $|\, g_i(x, y)| \leq M$ for all i. So Lebesgue's Dominated Convergence Theorem implies that

$$\lim_{i\to\infty} \epsilon_i^{-1}[\int_0^1 f(x, y + \epsilon_i)dx - \int_0^1 f(x, y)dx] = \lim_{i\to\infty} \int_0^1 g_i(x, y)dx =$$
$$= \int_0^1 \lim_{i\to\infty} g_i(x, y)dx = \int_0^1 \frac{\partial f}{\partial y}(x, y)dx$$

Since the RHS is independent of the sequence $\epsilon_i \to 0$, one finds that the derivative $\frac{d}{dy} \int_0^1 f(x, y)dx$ exists and equals the RHS.

(c) Express $\frac{d}{dy} \int_0^{y^2} f(x, y)dx$ in terms of integrals of f and $\frac{\partial f}{\partial y}$. Letting $F(s, t) = \int_0^s f(x, t)dx$, we see from the Fundamental Theorem and (b) that

$$\frac{\partial F}{\partial s}(s, t) = f(s, t) \text{ and } \frac{\partial F}{\partial t}(s, t) = \int_0^s \frac{\partial f}{\partial y}(x, t)dx$$

So we use the chain rule to compute

$$\frac{d}{dy} \int_0^{y^2} f(x, y)dx = \frac{d}{dy} F(y^2, y) = \frac{\partial F}{\partial s}(y^2, y)\frac{\partial y^2}{\partial y} + \frac{\partial F}{\partial t}(y^2, y)\frac{\partial y}{\partial y}$$
$$= 2yf(y^2, y) + \int_0^{y^2} \frac{\partial f}{\partial y}(x, y)dx \quad \square$$

Problem 3:

a. Show that the direct analog of Rolle's theorem does not apply to holomorphic functions. Do this by exhibiting an entire holomorphic function f such that
$f(0) = f(1)$ and yet $f'(z)$ never takes the value 0.

b. Suppose f is a holomorphic function on the unit disk $\{|z| < 1\}$. Show that f must be constant if $f(a_i) = f(b_i)$ for two sequences a_i, b_i of positive real numbers that satisfy the inequalities:

$$0 < ... < a_{i+1} < b_{i+1} < a_i < b_i < ... < a_1 < b_1 < 1$$

Solution 3:

(a) e^z doesn't vanish and $e^{z+2\pi i} = e^z$. So we can rotate the domain by 90^o and rescale by letting $f(z) = e^{2\pi iz}$. Then $f(0) = 1 = f(1)$ and $f'(z) = 2\pi ie^{2\pi iz} \neq 0$.

(b) Both monotone sequences converge to some some real number c with $0 \leq c < 1$. Writing $f = u + iv$ we find from Rolle's Theorem, points $a_i < c_i < b_i$ so that $\frac{\partial u}{\partial x}(c_i) = 0$. Since $c_i \to c$, we deduce from the real analyticity of $u(\cdot, 0) \equiv 0$ and so u is constant on the X-axis. Similarly v is also constant on the X-axis. But then the holomorphic function f being constant on the X-axis, must itself be constant. $\square$

Problem 4: Suppose $0 < M < \infty$ and, for each positive integer j, $f_j : [0,1] \to [-M, M]$ is a monotone increasing function. Prove that there is a subsequence $f_{j'}$ and a countable subset A of $[0,1]$ so that $f_{j'}(t)$ converges, as $j' \to \infty$, for every $t \in [0,1]\backslash A$.

Solution 4: Suppose $\mathbb{Q} \cap [0,1] = \{a_1, a_2, \dots\}$. A subsequence $f_{\alpha_1(1)}(a_1), f_{\alpha_1(2)}(a_1), \dots$ of the bounded sequence of numbers $f_1(a_1), f_2(a_1), \dots$ converges to a number $f(a_1)$. Inductively, choose a subsequence $f_{\alpha_j(1)}(a_j), f_{\alpha_j(2)}(a_j), \dots$ of the sequence $f_{\alpha_{j-1}(1)}(a_j), f_{\alpha_{j-2}(2)}(a_j), \dots$ convergent to a number $f(a_j)$.
Let $j' = \alpha_j(j)$ and $f(x) = \sup_{a_i<x} f(a_i) = \lim_{\epsilon \to 0} \sup_{x-\epsilon<a_i<x} f(a_i)$. Then f is monotone increasing and the set A of discontinuities of f is at most countable. To see that $\lim_{j\to\infty} f_{j'}(x) = f(x)$ for any $x \in (0,1) \setminus A$, we choose, for $\epsilon > 0$, numbers $a_i < x < a_{\tilde{i}}$ so that $f(a_{\tilde{i}}) - \epsilon < f(x) < f(a_i) + \epsilon$, and then J so that $|f_{j'}(a_i) - f(a_i)| < \epsilon$ and $|f_{j'}(a_{\tilde{i}}) - f(a_{\tilde{i}})| < \epsilon$ for $j \geq J$. For such j it follows that
$f(x) - 2\epsilon < f(a_i) - \epsilon < f_{j'(a_i)} < f_{j'}(x) < f_{j'}(a_{\tilde{i}}) < f(a_{\tilde{i}}) + \epsilon < f(x) + 2\epsilon$.
Thus $|f_{j'}(x) - f(x)| < 2\epsilon$. □

Problem 5:
a. Is there a nonconstant real function h that is continuous on the closed disk $\{z : |z| \leq 1\}$, harmonic on the open disk $\{z : |z| < 1\}$, and vanishes on the upper unit semi-circle?
b. Is there a nonconstant complex function f that is continuous on the closed disk $\{z : |z| \leq 1\}$, holomorphic on the open disk $\{z : |z| < 1\}$, and vanishes on the upper unit semi-circle?

Solution 5:
(a) The Posson Integral Formula shows that, for any continuous function g on the unit circle, one may find a harmonic function on the open ball which is continuous on

the closed ball and has boundary values g. So it suffices to choose any nonconstant g which vanishes on the upper semi-circle.

(b) There is a conformal map from the unit disk to the upper half plane. This takes the upper semi-circle to an interval on the X-axis. Composing with this conformal map thus gives a holomorphic map on the upper half plane which vanishes on this interval. Schwarz reflection about this interval then extends this function to be a holomorphic function whose domain contains the interval and vanishes on the interval. The identity theorem implies that this function, and hence the original function, must vanish identically. $\square$

Problem 6: Assume that $f(x)$ is a Lebesgue measurable function on $\mathbb{R}$. Prove the function defined on $\mathbb{R}^2$ by $F(x,y) = f(x-y)$ is Lebesgure measurable.

Solution 6: We need to show that $F^{-1}\big((a,b)\big)$ is measurable in $\mathbb{R}^2$ for any interval $(a,b) \subset \mathbb{R}$. Note that $F = f \circ P$ where $P(x,y) = x+y$. Also note that $P = p \circ \sqrt{2} \cdot \phi$ where ϕ is a 45^o rotation of the plane and $p(x,y) = x$. So

$$F^{-1}\big((a,b)\big) = (\sqrt{2} \cdot \phi)^{-1}\big(p^{-1}[f^{-1}\big((a,b)\big)]\big)$$

$E = f^{-1}\big((a,b)\big)$ is measurable in $\mathbb{R}$ by the measurability of f,and
$p^{-1}(E) = E \times \mathbb{R}$ is measurable by the definition of Lebesgue measure as a product measure. Moreover, since Lebesgue measurability is preserved under rotation and homothety $F^{-1}\big((a,b)\big) = (\sqrt{2} \cdot \phi)^{-1}(E \times \mathbb{R})$ is measurable. $\square$

Jan 2004

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Problem 1:
(a) Classify all enitre functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\sup_{z \in \mathbb{C}} \frac{|\, f(z)|}{1 + |\, z|^4} < \infty$$

(b) Classify all entire functions $g : \mathbb{C} \to \mathbb{C}$ such that

$$\inf_{z \in \mathbb{C}} \frac{|\, g(z)|}{|\, z|^4} > 0$$

Solution 1: The function $f(\frac{1}{z})$ has an isolated singularity at 0. If this singularity is removable, then f is bounded and so constant by Louiville's Theorem, which is one possibility. If it had a transcendental singularity at 0, then $z^4 f(\frac{1}{z})$ would also have a transcendental singularity at 0 and be unbounded, contradicting the growth assumption on f at ∞. We see that $f(\frac{1}{z})$ must have a pole at 0 so that f is necessarily a polynomial. Also we see that the degree of f is at most 4, and any such polynomial satisfies the hypothesis. Thus $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4$ for some $a_0, \ldots, a_4 \in \mathbb{C}$.

Again $g(\frac{1}{z})$ cannot have a transcendental singularity at 0 because then $z^4 g(\frac{1}{z})$ would be arbitrarily close to zero for some points z near 0. So again g is a polynomial. But now the condition implies that g can vanish only at the origin. So, by the Fundamental Theorem of Algebra, $g(z) = az^m$. The condition $\inf_{z \in \mathbb{C}} |\, a|\, |\, z|^{m-4} > 0$ requires that $m - 4 \geq 0$ (for z near 0) and $m - 4 \leq 0$ (for z near ∞). So $g(z) = az^4$ with

$a \neq 0$. □

Problem 2: Suppose that $f_n : \mathbb{R} \to \mathbb{R}$ is differentiable function for every positive integer n, $M = \sup_{n,x} |\, f_n'(x)| < \infty$ and that $f(x) = \lim_{n\to\infty} f_n(x) \in \mathbb{R}$ exists for all $x \in \mathbb{R}$.

(a) Show that the functions f_n are *uniformly bounded* on each interval $[a, b] \subset \mathbb{R}$.

(b) Is f *continuous* on $\mathbb{R}$? Prove or find a counterexample.

(c) Is f *differentiable* on $\mathbb{R}$? Prove or find a counterexample.

Solution 2:

(a) Since $f(a) = \lim_{n\to\infty} f_n(a)$, $N = \sup_n |\, f_n(a)| < \infty$. Then for any $x \in [a, b]$ the Fundamental Theorem of Calculus gives the uniform bound

$$|\, f_n(x)| \leq |\, f_n(a)| + \left| \int_a^x f_n'(t)dt \right| \leq N + M\,|\,b - a|$$

(b) Yes, as in (a) the Fundamental Theorem of Calculus implies that for $-\infty < x < y < \infty$,

$$|\, f(y) - f(x)| = \lim_{n\to\infty} |\, f_n(y) - f_n(x)| \leq \limsup \int_x^y |\, f_n'(t)|\, dt \leq M(y - x)$$

(c) Not necessarily. One easily obtains an example with $f(x) = |\,x|$ and the graph of $f_n(x)$ being obtained by slightly rounding the graph of $|\,x|$. □

Problem 3: Compute the (improper) integral

$$\int_0^\infty \frac{\sin x}{x(x^2 + 1)} dx$$

Solution 3: This improper integral exists as $\lim_{R\to\infty} I_R$ where

$$I_R = \int_{1/R}^{R} \frac{sinx}{x(x^2+1}dx = \frac{1}{2}\left[\int_{-R}^{-1/R} + \int_{1/R}^{R}\right] \frac{sinx}{x(x^2+1}dx$$

because $\left|\frac{sinx}{x}\right| \leq 1$ and $\frac{1}{x^2+1}$ is integrable on $[0,\infty)$. We want to use the Cauchy Integral Formula, buw we need to choose the $f(z)$ so that the integral on the extra boundary curve will approach 0 as the domain gets larger. [Warning: $|sinz| \leq 1$ is not always true for complex z.] One thing that works is to note that

$$\frac{sinx}{x(x^2+1} = \mathcal{I}m\frac{e^{ix}}{x(x^2+1)}$$

for x real and take

$$f(z) = \frac{e^{iz}}{z(z^2+1)}$$

on the domain Ω_R in the upper half-plane bounded by the 4 curves

$$[-R,-\frac{1}{R}],\ \gamma_R = \{\frac{1}{R}e^{i\theta} : \pi \geq \theta \geq 0\},\ [\frac{1}{R},R],\ \Gamma_R = \{Re^{i\theta} : 0 \leq \theta \leq \pi\}$$

Inside Ω_R, $f(z)$ has a single pole at $z = i$ with residue $\frac{e^{i^2}}{i(i+i)} = -\frac{1}{2e}$. Thus, Cauchy's Residue Formula gives

$$-\frac{\pi}{e} = \mathcal{I}m\left(2\pi i\left(-\frac{1}{2e}\right)\right) = \mathcal{I}m\int_{\partial\Omega_R} f(z)dz \ = \ 2I_R + \mathcal{I}m\int_{\gamma_R} f(z)dz + \mathcal{I}m\int_{\Gamma_R} f(z)dz$$

On Γ_R, $\left|e^{iRe^{i\theta}}\right| = \left|e^{-R\sin\theta}\right| \leq 1$ because $\sin\theta \in [0,1]$. So we see that

$$\left|\int_{\Gamma_R} f(z)dz\right| \leq \frac{1}{R^3}\pi R \ \longrightarrow 0 \text{ as } R \to \infty$$

Finally

$$\int_{\gamma_R} f(z)dz = -\frac{1}{2}\int_{\partial\mathbb{B}_{1/R}} f(z)dz = -\frac{1}{2}(2\pi i)\,\mathrm{Res}_0 f = -\pi i(1)$$

So taking imaginary parts,

$$\lim_{R\to\infty} I_R = \frac{1}{2}\left[-\frac{\pi}{e} + \pi\right] = \frac{\pi}{2}(1 - e^{-1})$$

□

Problem 4:

(a) In the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ how many solutions are there to the equation $z^8 - 5z^3 + z = 2$?

(b) In the radius-2 disk $\{z \in \mathbb{C} : |z| < 2\}$ how many solutions are there to the same equation $z^8 - 5z^3 + z = 2$?

Solution 4:

(a)We apply Rouche's Theorem with $f(z) = z^8 - 5z^3 + z - 2$ and $g(z) = -5z^3$ on the unit disk noting that for $|z| = 1$,

$$|f(z) - g(z)| = \left|z^8 + z - 2\right| \leq |z|^8 + |z| + 2 = 1 + 1 + 2 = 4 < 5(1)^3 = |g(z)|$$

Thus in the unit disk, $f(z)$ has the same number of zeros as $g(z)$ (counting multiplicities), namely 3. So the equation $z^8 - 5z^3 + z = 2$ has 3 solutions in the unit disk.

(b) Here we use the same f but now take $g(z) = z^8$ and note that for $|z| = 2$ one has

$$|f(z) - g(z)| = \left|-5z^3 + z - 2\right| \leq 5(2)^3 + 2 + 2 = 44 < (2)^8 = |g(z)|$$

So the equation $z^8 - 5z^3 + z = 2$ has 8 solutions in the radius-2 disk. □

Problem 5:

(a) Suppose that f is integrable on $[0, 1]$. Show that there exists a sequence of positive numbers $a_n \downarrow 0$ so that $\lim_{n\to\infty} a_n |f(a_n)| = 0$.

(b) Let f_n be a sequence of functions integrable on $[0, 1]$ with $\sup_n \int_0^1 |f_n(x)|\, dx < \infty$. Does there exist a subsequence f_{n_k} of f_n and a sequence of positive numbers $b_k \downarrow 0$ and so that $\lim_{k\to\infty} b_k |f(b_k)| = 0$. If so, prove it. If not, find a counterexample.

Solution 5:

(a) If this were false, then $\epsilon = \liminf_{x\to 0} x |f(x)| > 0$, and there would exist a positive

δ so that $x\,|\,f(x)| \geq \frac{\epsilon}{2}$ whenever $0 < x \leq \delta$. But then

$$\int_0^1 |\,f(x)|\,dx \geq \int_0^\delta |\,f(x)|\,dx \geq \int_0^\delta \frac{\epsilon}{2x}dx = \infty$$

contradicting the integrability of f.

(b) As Professor Jones pointed out, a stronger result is true. One need only assume that each f_n is integrable and one doesn't need to pass to a subsequence f_{n_k} for the conclusion. Here we first choose $\alpha_k \downarrow 0$ so that $\sum_{k=1}^\infty \alpha_k \int_0^1 |\,f_k(x)|\,dx < \infty$, and apply (a) to the integrable function $f(x) = \sum_{k=1}^\infty \alpha_k\,|\,f_k(x)|$ to find points $a_m \downarrow 0$ so that $\lim_{m\to\infty} a_m f(a_m) = 0$. Passing to a subsequence we can make this sequence converge as fast as we want. In particular we can choose inductively $a_{m_k} \downarrow 0$ so that $a_{m_k} f(a_{m_k}) \leq \alpha_k^2$. Letting $b_k = a_{m_k}$, we conclude that

$$b_k f(b_k) \;\leq\; b_k \alpha_k^{-1} f(b_k) \;\leq\; \alpha_k^{-1}\alpha_k^2 = \alpha_k \;\longrightarrow 0 \;\text{ as }\; k \to \infty$$

□

Problem 6: Suppose $1 \leq p \leq \infty$, $f \in L^p([0,1])$, and $h(t)$ is the Lebesgue measure of the set $\{x \in [0,1] : |\,f(x)| > t\}$ for $0 \leq t < \infty$.
(a) Show that $\int_0^\infty h(t)dt < \infty$ if $1 < p \leq \infty$.
(b) Is this still true for $p = 1$? Prove or find a counterexample.

Solution 6: Here this is true for $p = 1$. Since Holder's inequality implies that $L^p([0,1]) \subset L^1([0,1])$, we only need prove the case $p = 1$ and part (a) follows.

For this, we use Fubini's Theorem with the characteristic function of the subgraph

$$A = \{(x,y) : 0 \leq x \leq 1, 0 \leq y < |\,f(x)|\}$$

Let λ denote the 1-dimensional Lebesgue measure. By Fubini's Theorem, A is 2-dimensional Lebesgue measureable function with 2-dimensional measure

$$|A| = \int_0^1 \lambda\{y : (x,y) \in A\}dx = \int_0^1 |f(x)|\,dx < \infty$$

But slicing the other way shows that

$$\int_0^\infty h(y)dy \;=\; \int_0^\infty \lambda\{|f(x)| > y\}dy \;=\; \int_0^\infty \lambda\{x : (x,y) \in A\}dy \;= |A| < \infty$$

One can obtain an alternative proof of (a) but not (b) by using Chebyshev's Inequality to see that

$$h(t) = \lambda\{x \in [0,1] : |f(x)|^p > t^p\} \leq \frac{1}{t^p}\int_0^1 |f(x)|^p\,dx$$

So

$$\int_0^\infty h(t)dt \;\leq\; 1 + \int_1^\infty h(t)dt \;\leq\; 1 + \left(\int_0^1 |f(x)|^p\,dx\right)\int_1^\infty t^{-p}dt \;<\; \infty$$

$\square$