Complex Analysis Problems

(adapted by D. Gokhman from UC Berkeley Mathematics Ph.D. preliminary examinations 1977–1988)

1. [77/Winter/I.3]

   (i) Let $P_{n-1}(z) = \dfrac{z^n - 1}{z - 1}$. Find $P_{n-1}(1)$.

   (ii) Let $Q_k, k = 1, ...n$ be the vertices of a regular polygon inscribed in the unit circle. Let d_k be the distance between Q_k and Q_1. Show that $\prod_{k=2}^{n} d_k = n$.

2. [77/Winter/I.4] [82/3/2/A.1] Prove the Fundamental Theorem of Algebra: Every nonconstant polynomial with complex coefficients has a complex root. (Hint: use one of the following (a) Liouville's theorem, (b) the Maximum Modulus Principle, (c) Rouché's theorem.)

3. [77/Winter/I.10] Consider the domain $\Omega = \{z\colon\ |z| > 1\}$. Find a complex differentiable 1-form ω on Ω that is closed, i.e. $d\omega = 0$, but that is not exact, i.e. there is no complex differentiable function $F(z)$ on Ω such that $dF = \omega$. Conclude that Ω is not simply connected. What is $\pi_1(\Omega)$?

4. [77/Winter/II.5] Let $f(z)$ be a meromorphic function. A complex number w is called a period of f if $f(w + z) = f(z)$ for all z.

   (i) Show that if w_1 and w_2 are periods, so are $n_1w_1 + n_2w_1$ for all integers n_1, n_2.

   (ii) Show that there are at most finitely many periods of f in any bounded region of the complex plane.

5. Evaluate the following improper integrals (justify your method):

   (i) [77/Winter/II.6] $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 2x}{x^4 + 1}\, dx$

   (ii) [78/2/27/I.4] $\displaystyle\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 dx$

   (iii) [79/2/26/I.7] $\displaystyle\int_{0}^{\infty} \frac{x^2 + 1}{x^4 + 1}\, dx$

   (iv) [79/9/13/II.6] $\displaystyle\int_{0}^{\infty} \frac{x^2}{1 + x^{10}}\, dx$ (Answer: $\dfrac{\pi}{10\sin(3\pi/10)}$)

   (v) [80/9/15/A.5] $\displaystyle\int_{0}^{\infty} \frac{x^{m-1}}{1 + x^n}\, dx$, where n, m are integers such that $0 < m < n$.

(vi) [81/2/26/A.7] $\int_{-\infty}^{\infty} \frac{x \sin x}{(1+x^2)^2}\, dx$

(vii) [86/9/A–B.9] $\int_{0}^{\infty} \frac{\log x}{(x^2+1)(x^2+4)}\, dx$

6. [78/2/27/I.3] Let $f(z)$ be a nonconstant complex differentiable function. Prove that $f(\mathbf{C})$ is dense in $\mathbf{C}$. (It is not sufficient to just quote Picard's theorem.)

7. [78/2/28/II.1] Show that there is a complex differentiable function defined on the set $\Omega = \{z \in \mathbf{C}\colon\ |z| > 4\}$ whose derivative is

$$\frac{z}{(z-1)(z-2)(z-3)}.$$

Is there a complex differentiable function on Ω whose derivative is

$$\frac{z^2}{(z-1)(z-2)(z-3)}?$$

8. [78/2/28/II.2] Prove that the uniform limit of a sequence of complex analytic functions is complex analytic. Is the analogous theorem true for real analytic functions?

9. Let r be a constant such that $r^2 \neq 1$. Evaluate

(i) [78/9/14/I.4]

$$\int_0^{2\pi} \frac{d\theta}{1 - 2r\cos\theta + r^2}.$$

(ii) [85/9/19–20/12]

$$\int_0^{2\pi} e^{e^{\mathrm{i}\theta}}\, d\theta.$$

(iii) [87/1/A–B.20]

$$\int_0^{\pi} \frac{\cos 4\theta}{1+\cos^2\theta}\, d\theta.$$

(iv) [87/9/3–4.9]

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta.$$

10. [78/9/14/I.5] [79/6/6/I.6] Let $f(z)$ be a complex polynomial $f(z) = a_0 + a_1 z + \ldots a_n z^n$. Prove that

$$\frac{1}{2\pi\mathbf{i}} \int_{|z|=r} z^{n-1} \left|f(z)\right|^2\, dz = a_0 \overline{a_n} r^{2n}$$

11. [78/9/15/II.1] [79/2/26/I.1] Suppose $f(z)$ is complex differentiable and bounded on the punctured unit disc $\{z\colon 0 < |z| < 1\}$. Prove that f is differentiable on the unit disc $\{z\colon |z| < 1\}$.

12. [78/9/15/II.3] Let ω be a complex differentiable 1-form on $\mathbf{C}$. Prove that if ω is closed, i.e. $d\omega = 0$, then ω is exact, i.e. there exists a function $F(z)$ such that $dF = \omega$.

13. Give examples of conformal maps as follows:

    (i) [78/9/15/II.4] from $\{z\colon |z| < 1\}$ onto $\{z\colon \operatorname{Re} z < 0\}$.
    (ii) [78/9/15/II.4] from $\{z\colon |z| < 1\}$ onto itself, with $f(0) = 0$, $f(1/2) = \mathbf{i}/2$.
    (iii) [78/9/15/II.4] from $\{z\colon z \neq 0, 0 < \arg(z) < 3\pi/2\}$ onto $\{z\colon z \neq 0, 0 < \arg(z) < \pi/2\}$.
    (iv) [80/5/28/I.7] from $\{z\colon |z| < 1, \operatorname{Re} z > 0\}$ onto $\{z\colon |z| < 1\}$.

14. [78/9/15/II.5] Suppose $h(z)$ is entire, $h(0) = 3 + 4\mathbf{i}$, and $|h(z)| \leq 5$ if $|z| < 1$. What is $h'(0)$?

15. [79/2/26/I.6] Suppose λ is a real number, $\lambda > 1$. Prove that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution with positive real part.

16. [79/2/27/II.6] For which $z \in \mathbf{C}$ does $\displaystyle\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n^2}{z^n} \right)$ converge?

17. [79/2/27/II.7] [83/2/10–11/15] Let P, Q be complex polynomials such that $\deg Q \geq \deg P + 2$. Prove that there exists $r > 0$ such that if γ is a closed curve outside $\{z\colon |z| = r\}$, then
$$\int_\gamma \frac{P(z)}{Q(z)}\, dz = 0.$$

18. [79/6/6/I.7] Find an open set in $\mathbf{C}$ where the formula
$$f(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$
can be used to define an analytic function.

19. [79/6/7/II.5] Show that
$$\sum_{n=0}^{\infty} \frac{z}{(1+z^2)^n}$$
converges for all complex numbers z exterior to the lemniscate $|1 + z^2| = 1$.

20. [79/6/7/II.6] Let $g_n(z), n = 1, 2...$ be a sequence of entire functions having only real zeros. Suppose $g_n(z) \to g(z)$ as $n \to \infty$ uniformly on compact subsets of $\mathbf{C}$. Prove that $g(z)$ has only real zeros.

21. (i) [79/9/12/I.1] Let $f(z) = z^{100} + 8z^{10} - 3z^3 + z^2 + z + 1$. How many zeros (counting multiplicities) does f have in the closed unit disc $\{z\colon |z| \le 1\}$?

(ii) [80/5/29/II.8] How many zeros has the complex polynomial $3z^9 + 8z^6 + z^5 + 2z^3 + 1$ in the annulus $\{z\colon 1 < |z| < 2\}$?

(iii) [85/9/19–20/5] How many zeros has the complex polynomial $z^4 + 3z^2 + z + 1$ in the right half plane?

(iv) [81/9/16/A.8] How many zeros does the function $f(z) = 3z^{100} - e^z$ have inside the unit circle (counting multiplicities)? Are the zeros distinct? (give a reason)

22. (i) [79/9/12/I.2] Let $g(z)$ be analytic in the right half-plane $\{z\colon \operatorname{Re} z > 0\}$ such that $|g(z)| < 1$. If $g(1) = 0$ how large can $|g(2)|$ be?

(ii) [85/9/19–20/13] Suppose $f(z)$ is analytic on the right half plane $H = \{z\colon \operatorname{Re} z > 0\}$ and $|f(z)| \le 1$ for $z \in H$. Suppose that $f(1) = 0$. What is the largest possible value of $|f'(1)|$? Prove your answer.

23. [79/9/13/II.5] [81/5/28/B.17] Suppose $f(z), g(z)$ are entire and $|f(z)| \le |g(z)|$ for all z. Prove that there is a constant c such that $f(z) = cg(z)$.

24. [80/5/28/I.6] Evaluate the integral of $\sqrt{z^2 - 1}\, dz$ around the circle $\{z\colon |z| = 2\}$, where the branch of the square root function is chosen so that $\sqrt{2^2 - 1} > 0$.

25. [80/5/29/II.9] Let $f(z)$ be a meromorphic function on $\mathbf{C}$. Suppose f is analytic at 0 and the coefficients of its power series expansion at 0 are real and nonnegative. In addition, suppose f has a pole of order $r > 0$ and no pole of f has order $< r$. Prove that f has a pole at $z = r$.

26. [80/9/15/A.3] Do there exist functions $f(z), g(z)$ that are analytic at $z = 0$ and satisfy

(a) $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^2}, \qquad n = 1, 2...$

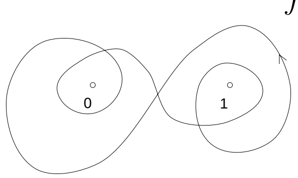
(b) $g\left(\frac{1}{n}\right) = g\left(-\frac{1}{n}\right) = \frac{1}{n^3}, \qquad n = 1, 2...?$

27. [80/9/16/B.11] Let $f(z)$ be an analytic function defined for $|z| \le 1$ and let $u(x, y) = \operatorname{Re} f(z)$, $z = x + \mathbf{i}y$. Prove that the integral of the 1-form $\frac{\partial u}{\partial y}\, dx - \frac{\partial u}{\partial x}\, dy$ around the unit circle is zero.

28. [80/9/16/B.18] Suppose that $f(z)$ is analytic inside and on the unit circle $|z| = 1$ and satisfies $|f(z)| < 1$ for $|z| = 1$. Show that the equation $f(z) = z^3$ has exactly three solutions (counting multiplicities) inside the unit circle.

29. [81/2/26/A.6] Suppose the complex polynomial $\sum_{k=0}^{n} a_k z^k$ has n distinct roots $r_1, \ldots r_n \in \mathbf{C}$. Prove that if $|b_k - a_k|$ is sufficiently small, then $\sum_{k=0}^{n} b_k z^k$ has n roots which are smooth functions of $b_0, \ldots b_n$.

30. [81/2/27/B.11] Evaluate $\int_\gamma \frac{e^z - 1}{z^2(z-1)}\, dz$, where γ is the closed curve shown below



31. (i) [81/5/27/A.7] Evaluate $\int_\gamma \frac{dz}{\sin \frac{1}{z}}$, where γ is the circle $|z| = 1/5$, positively oriented.

    (ii) [86/2/20–21/20] Evaluate $\int_\gamma \left(e^{2\pi z} + 1\right)^{-2} dz$, where γ is the circle $|z| = 1$, positively oriented.

    (iii) [86/9/A–B.18] Evaluate $\int_\gamma \frac{z^{11}}{12z^{12} - 4z^9 + 2z^6 - 4z^3 + 1}\, dz$, where γ is the circle $|z| = 1$, positively oriented.

32. [81/5/28/B.14] Suppose $f : [0,1] \to \mathbf{C}$ is continuously differentiable and as $t \to 0^+$ we have $f(t) \to 0$ and $f'(t) \to C \neq 0$. Show that $g(t) = |f(t)|$ is continuously differentiable for sufficiently small $t > 0$ and $g'(t)$ has a limit at 0^+. Evaluate this limit.

33. [81/5/28/B.19] Suppose n is a natural number and α, β nonzero reals. Show that the number of roots of $z^{2n} + \alpha z^{2n-1} + \beta^2 = 0$ that have positive real part is

    (a) n, if n is even,

    (b) $n - 1$, if n is odd.

34. [81/9/17/B.12] Let a, b be real constants and let

$$u(x,y) = \frac{a^2 + b^2 + x^2 - y^2}{(a-x)^2 + (b-y)^2}.$$

Show that u is harmonic and find an entire function $f(z)$ whose real part is u.

35. [81/9/17/B.15] Let $g(z)$ be a holomorphic map of the unit disc $D = \{z\colon\ |z| < 1\}$ into itself that is not the indentity map $f(z) = z$. Show that g can have at most one fixed point.

36. [82/3/3/B.11] Decide, without too much computation, whether $(\tan z)^{-2} - z^{-2}$ has a finite limit as $z \to 0$. If the limit exists, compute it.

37. [82/3/3/B.18] For $\operatorname{Re} z \geq 0$ define
$$F(z) = \int_0^\infty \frac{e^{-zt}}{1+t^4}\, dt.$$
Show that $F(z)$ is
   (a) continuous for $\operatorname{Re} z \geq 0$,
   (b) analytic for $\operatorname{Re} z > 0$.

38. [82/6/2/A.5] Suppose $a_0 \geq a_1 \geq ...a_n > 0$. Prove that the equation $a_0 + a_1 z + ...a_n z^n = 0$ has no roots in the unit disc $|z| < 1$.

39. [82/6/2/A.9] Determine the complex numbers z for which the power series
$$\sum_{n=1}^{\infty} \frac{z^n}{n^{\log n}}$$
and its term-by-term derivatives of all orders converge absolutely.

40. [82/6/2/A.10] For complex numbers $a_1, ...a_t$ prove:
$$\limsup_{n\to\infty} \left| \sum_{k=1}^{t} a_k^n \right|^{\frac{1}{n}} = \max_k |a_k|$$

41. [82/6/3/B.11] Find real differentiable functions $s(y), t(y)$ such that the complex function $f(x + \mathbf{i}y) = e^x\left(s(y) + \mathbf{i}t(y)\right)$ is analytic and $s(0) = 1, t(0) = 0$.

42. [82/6/3/B.15] Suppose $f(z)$ is analytic on the unit disc $D = \{z\colon\ |z| < 1\}$. Prove that there is a sequence z_n in D such that $|z_n| \to 1$ and the sequence $f(z_n)$ is bounded.

43. [82/9/30/I.3] Consider the Laurent expansion $\cot(\pi z) = \sum_{n=-\infty}^{\infty} a_n z^n$ valid on the annulus $1 < |z| < 2$. Compute a_n for $n < 0$. (Recall that $\cot(\pi z)$ has simple poles at integers z with residues $1/\pi$ and no other singularities.)

44. [82/9/30/I.10] Let a and b be complex numbers with nonpositive real parts. Prove the inequality $\left|e^a - e^b\right| \leq |a - b|$.

45. [82/10/1/II.9] Let a and b be nonzero complex numbers. Let $f(z) = az + bz^{-1}$. Determine the image under f of the unit circle $\{z\colon\ |z| = 1\}$.

46. [83/2/10–11/9] Determine all complex analytic functions $f(z)$ defined on the unit disc D that satisfy $f''(1/n) + f(1/n) = 0$ for $n = 2, 3, 4...$ Justify your answer.

47. [83/6/2–3/2] Let $f(z)$ be an entire function such that $\left(1 + |z|^k\right)^{-1} f^{(m)}(z)$ is bounded for some k and m. Prove that $f^{(n)}(z)$ is identically zero for sufficiently large n. How large must n be in terms of k and m?

48. [83/6/2–3/7] Suppose $a > 0$ is a constant. Compute $\int_0^\infty \frac{\log x}{x^2 + a^2}\, dx$. (It *might* be helpful to split the interval of integration at a.)

49. [83/6/2–3/9] Suppose Ω is a bounded domain in $\mathbf{C}$ with boundary consisting of a smooth Jordan curve γ, $f(z)$ is holomorphic in a neighborhood of the closure of Ω and $f(z) \neq 0$ for $z \in \gamma$. Let $z_1, ...z_k$ be the zeros of $f(z)$ in Ω and let $n_1, ...n_k$ be the orders (multiplicities) of these zeros.

    (a) Use Cauchy's integral formula to show that

    $$\frac{1}{2\pi \mathbf{i}} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_{j=1}^{k} n_j.$$

    (b) Suppose that f has only one zero z_1 in Ω with multiplicity $n_1 = 1$. Find a boundary integral involving f whose value is z_1.

50. [83/6/2–3/15] Suppose f is analytic on and inside the unit circle $\gamma = \{z\colon\ |z| = 1\}$. Let ℓ be the length of the image of γ under f. Show that $\ell \geq 2\pi\, |f'(0)|$.

51. [83/6/2–3/16] Suppose Ω is an open subset of $\mathbf{C}$ and $f : \Omega \to \mathbf{C}$ is a smooth (real differentiable) map. Suppose that f preserves orientation and maps any pair of orthogonal curves to a pair of orthogonal curves. Show that f is holomorphic.

52. [83/6/2–3/19] Compute the area of the image of the unit disc $\{z\colon\ |z| < 1\}$ under the map $f(z) = z + z^2/2$.

53. [85/9/19–20/2] Prove that for every $\lambda > 1$, the equation $ze^{\lambda - z} = 1$ has exactly one root in the disk $\{|z| < 1\}$ and that this root is real.

54. [86/2/20–21/2] Let γ be a simple closed contour enclosing the points $0, 1, ...k$ in the complex plane. Evaluate the integrals

    $$I_k = \int_\gamma \frac{dz}{z(z-1)...(z-k)}, \qquad k = 0, 1, ...$$

$$J_k = \int_\gamma \frac{(z-1)...(z-k)}{z}\,dz, \qquad k = 1, 2, ...$$

55. [86/2/20–21/11] Define $f_n\colon \mathbf{R} \to \mathbf{C}$ by $f_n(t) = \pi^{-1/2}(t-\mathbf{i})^n/(t+\mathbf{i})^{n+1}$. Show that these functions are orthonormal.

56. [86/2/20–21/18] Let $f, g_1, g_2, ...$ be entire functions such that

    (i) $\left|g_n^{(k)}(0)\right| \le \left|f^{(k)}(0)\right|$ for all n and k;

    (ii) $\lim_{n\to\infty} g_n^{(k)}(0)$ exists for all k.

    Prove that the sequence g_n converges uniformly on compact subsets and that its limit is an entire function.

57. [86/9/2.L]et the points a, b, c lie on the unit circle and suppose $a + b + c = 0$. Prove that these points are the vertices of an equilateral triangle.

58. [86/9/A–B.4] Show that the polynomial $z^5 - 6z + 3$ has 5 distinct complex roots, of which exactly 3 are real.

59. [87/1/A–B.3] Suppose f is a complex valued function on the open unit disc $D = \{z\colon |z| < 1\}$ such that $g = f^2$ and $h = f^3$ are both analytic on D. Prove that f is analytic on D.

60. [87/1/A–B.6] Suppose f is an analytic function on the open unit disc such that $|f(z)| \le C/(1-|z|)$ for all z in the disc, where C is a positive constant. Prove that $|f'(z)| \le 4C/(1-|z|)^2$.

61. [87/1/A–B.8] Prove that if a nonconstant polynomial $p(z)$ with complex coefficients has all of its roots in the half plane $\operatorname{Re} z > 0$, then all of the roots of its derivative are in the same half plane.

62. [87/1/A–B.15] Prove or disprove: If the function f is entire and maps every unbounded sequence to an unbounded sequence, then f is a polynomial.

63. [87/9/3–4.1] Prove that $(\cos\theta)^p \le \cos(p\theta)$ for $0 \le \theta \le \pi/2$ and $0 < p < 1$.

64. [87/9/3–4.10] If $f(z)$ is analytic on the open unit disc and $|f(z)| \le 1/(1-|z|)$, show that the Maclaurin coefficients a_n of f satisfy

$$|a_n| = \left|\frac{f^{(n)}(0)}{n!}\right| \le (n+1)(1+\frac{1}{n})^n < e(n+1).$$

65. [87/9/3–4.17] Let u be a positive harmonic function on $\mathbf{R}^2$. Show that u is constant.

66. [87/Spring/A–B.4] True or false: a function $f(z)$ is analytic on $|z-a|<r$ and continuous on $|z-a|\le r$ extends, for some $\delta>0$, to a function analytic on $|z-a|<r+\delta$? Give a proof or a counterexample.

67. [87/Spring/A–B.10] Let Ω be an open connected subset of the complex plane. Let n be a positive integer. Suppose $f(z)$ is analytic on Ω, f is not identically zero, and there is a function $g(z)$ analytic in Ω such that $g(z)^n = f(z)$.

   (a) Prove that there exist exactly n such functions $g=\sqrt[n]{f}$.

   (b) Give an example of a continuous real valued function on $[0,1]$ that has more than two continuous square roots on $[0,1]$.

68. [87/Spring/A–B.14] Suppose f is analytic in the open unit disc and real valued on the radii $[0,1)$ and $[0,e^{\pi\mathbf{i}\sqrt{2}})$. Prove that f is constant.

69. [87/Spring/A–B.17] Suppose f is an analytic function that maps the open unit disc D into itself and vanishes at the origin.

   (a) Prove that $|f(z)+f(-z)|\le 2\,|z|^2$ in D.

   (b) Prove that the inequality in (a) is strict, except at the origin, unless f has the form $f(z)=\lambda z^2$, where λ is a constant with $|\lambda|=1$.