Math 222 HW#10, due Friday 4/30/21 **NAME:**

To earn full credit do #1-5 or #4-7. It should be shorter (and more pleasant) than HW 9.

1. Folland
   Determine whether each of the following vector fields is the curl of a vector field F, and if so, find such an F.
   (a) $G(x, y, z) = \langle x^3 + yz, y - 3x^2y, 4y^2\rangle$
   (b) $G(x, y, z) = \langle xy + z, xz, -(yz + x)\rangle$
   (c) $G(x, y, z) = \langle xe^{-x^2z^2} - 6x, 5y + 2z, z - ze^{-x^2z^2}\rangle$

2. Stewart computation
   Use the Divergence Theorem to calculate the surface integral $\iint_S F \cdot \mathbf{n}\, dS$; that is calculate the flux of F across S for
   $$F(x, y, z) = \langle 3xy^2, xe^z, z^3\rangle$$
   and S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$, & $x = 2$.

3. Folland computation
   Use the Divergence Theorem to calculate the surface integral $\iint_S F \cdot \mathbf{n}\, dS$; that is calculate the flux of F across the ellipsoid S: $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ for
   $$F(x, y, z) = \left\langle \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right\rangle$$

4. Folland
   Let $\mathbf{x} = (x, y, z)$ and $g(\mathbf{x}) = |\mathbf{x}|^{-1} = \frac{1}{\sqrt{x^2+y^2+z^2}}$.
   (a) Compute $\nabla g(\mathbf{x})$ for $\mathbf{x} \neq 0$.
   (b) Show that $\nabla^2 g(\mathbf{x}) = \Delta g = \text{div}(\text{grad } g) = 0$ for $\mathbf{x} \neq 0$.
   Suggestion: To show this you may use the following fun fact (without proof) that for $\mathbf{x} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and a C^2 function $F(\mathbf{x}) = f(r)$ on $(0, \infty)$ with $r = |\mathbf{x}|$ that
   $$\nabla^2 g\mathbf{x} = \frac{\partial^2 F}{\partial x^2} + ... + \frac{\partial^2 F}{\partial x^2} = f''(r) + (n-1)r^{-1}f'(r).$$
   (c) Show by a direct calculation that $\displaystyle\iint_S \nabla g \cdot n\, dS = -4\pi$ if S is any sphere centered at the origin.
   (d) Since $\nabla^2 g = \text{div}(\text{grad } g)$ why do (b) and (c) not contradict the Divergence theorem?

5. Folland (An amusing but rarely useful fact)
   Let R be a regular region in $\mathbb{R}^3$ with piecewise smooth boundary. Show that
   $$\text{Volume}(R) = \frac{1}{3} \iint_{\partial R} F \cdot \mathbf{n}\, dS,$$
   where $F(x, y, z) = \langle x, y, z\rangle$

6. Stewart proof
   Prove the following identity, assuming that R and ∂R satisfy the conditions of the Divergence Theorem, and that the scalar functions and vector fields are C^1:
   $$\iint_{\partial R} (f\nabla g) \cdot \mathbf{n}\, dS = \iiint_R (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV$$

7. Folland

   Prove the following integration by parts formula for triple integrals:

$$\iiint_R f \frac{\partial g}{\partial x} dV = -\iiint_R g \frac{\partial f}{\partial x} dV + \iint_{\partial R} f \ g \ n_x \ dS,$$

   where n_x is the x component of the outward unit normal to ∂R. (Similar formulas hold with x replaced by y and z.)

* Assignment Reflections

  How difficult was this assignment? How many hours did you spend on it? Which problems did you find to provide a worthwhile learning experience? Should I be assigning a similar number of problems, fewer problems, or more problems in the future? Is there a good mix of theory and computations?