Math 222 HW#8, due Monday 4/12/21 NAME:

For full HW credit, do 1-4 or 3-5.

1. Let S be the annulus $1 \leq x^2 + y^2 \leq 4$. Compute $\displaystyle\int_{\partial S} (xy^2\ dy - x^2y\ dx)$ both directly and by using Green's theorem. (Folland: $15\pi/2$)

2. Use Green's theorem as in Folland 5.2 Example 3 (Prof Jo Slide 27) to calculate the area under one arch of the cycloid described parametrically by $\mathbf{r}(t) = \langle R(t - \sin t), R(1 - \cos t)\rangle$. (Folland: $3\pi R^2$). See also https://en.wikipedia.org/wiki/Cycloid. UPDATED: corrected cycloid!

3. Find the simple closed curve C that maximizes the value of the line integral $\displaystyle\int_C y^3\ dx + (3x - x^3)dy$.

4. Fleshing out Jones 12.D
   Let $S = \{(x, y) \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\}$, where f is a non-negative C^1 function on $[a, b]$. Explain how the formula $A = -\int_{\partial S} ydx$ for the area of S in Folland 5.2 Example 3 (Prof Jo Slide 27) leads to the familiar formula $A = \int_a^b f(x)dx$. (Your argument should be self-contained, e.g. not require the grader to hunt through Jones' book.)

5. This exercise shows how Green's theorem can be used to deduce a special case of the Change of Variables Theorem in $\mathbb{R}^2$. Let U, V be connected open sets in $\mathbb{R}^2$, and let $G : U \to V$ be a one to one transformation of class C^1, whose Jacobian matrix $DG(u)$ is invertible for all $u \in U$. Moreover, let S be a regular region in V with piecewise smooth boundary, let A be its area, and let $T = G^{-1}(S)$.

   (a) The Jacobian determinant $\det DG$ is either everywhere positive or everywhere negative on U. Why?

   (b) Suppose $\det DG(u) > 0$ for all $u \in U$. Write $A = \int_{\partial S} y\ dx$ as in Folland 5.2 Example 3 (Prof Jo Slide 27), make a change of variable to transform this line integral into a line integral over ∂T, and apply Green's theorem to deduce that $A = \iint_T \det DG\ dA$.

   (c) By a similar argument (which you don't need to write up), show that if $\det DG(u) < 0$ for all $u \in U$ then $A = -\iint_T \det DG\ dA = \iint_T |\det DG|\ dA$. Explain however, why the minus sign arises here and not in 5(b).

* Assignment Reflections
How difficult was this assignment? How many hours did you spend on it? Which problems did you find to provide a worthwhile learning experience? Should I be assigning a similar number of problems, fewer problems, or more problems in the future? Is there a good mix of theory and computations?