Math 222: Zero Content and Theory, optional problems

1. (Folland 4.1) Prove that if f is integrable on $[a, b]$ then $|f|$ is integrable on $[a, b]$, and

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx.$$

Suggestion: You can assume f is bounded on $[a, b]$. To prove that $|f|$ is integrable, show that

$$S_P|f| - s_P|f| \leq S_P f - s_P f.$$

For the inequality $|\int f| \leq \int |f|$, observe that $\pm f \leq |f|$ and use Theorem 4.9(c), namely that if f, g are integrable on $[a, b]$ with $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f \leq \int_a^b g$. You'll want to use the following lemma:

Lem. 4.5: If f is a bounded function on $[a, b]$, the following conditions are equivalent:

(a) f is integrable on $[a, b]$.

(b) For every $\epsilon > 0$ there is a partition P of $[a, b]$ such that $S_P f - s_P f < \epsilon$.

2. (Folland 4.1) Let $f(x) = 1$ if x is rational, $f(x) = 0$ if x is irrational. Show that f is not Riemann integrable on any interval.

Fun Fact: This characteristic/indicator function of the rationals is called the Dirichlet function, $f(x) = \chi_{\mathbb{Q}}(x)$. In Math 425 after defining the Lebesgue integral, you'll see $\int_{\mathbb{R}} \chi_{\mathbb{Q}}(x) = 0$ because it is zero except on the set of rational numbers, which are countable, and hence comprise a set of zero Lebesgue measure in $\mathbb{R}$.

Def: Recall that a set $Z \subset \mathbb{R}^n$ is said to have **zero content** if for every $\varepsilon > 0$ there is $\{R_1, ..., R_M\}$ such that $Z \subset \cup_1^M R_m$ and $\sum_{i=1}^M \mathrm{vol}(R_m) \leq \varepsilon$.

3. (Folland 4.1) Let $\{x_k\}$ be a convergent sequence in $\mathbb{R}$. Show that the set $\{x_1, x_2, ...\}$ has zero content in $\mathbb{R}$.

4. (Angenent) Let $f : [a, b] \to \mathbb{R}$ be a nondecreasing function (e.g. if $x \leq x'$ then $f(x) \leq f(x')$). Show that the graph of f, i.e. the set

$$G = \{(x, f(x)) \mid a \leq x \leq b\}$$

has zero content in $\mathbb{R}^2$. Suggestion: For any (large) integer N, consider the rectangles

$$R_j = [x_{j-1}, x_j] \times [f(x_{j-1}), f(x_j)], \quad 1 \leq j \leq N,$$

where $x_j = a + \frac{j}{N}(b - a)$. There is a nice formula for the total area of the rectangles $R_1, ..., R_N$.

5. (Folland 4.2) Let $f : [a, b] \to \mathbb{R}$ be an integrable function.

(a) Show that the graph of f in $\mathbb{R}^2$ has zero content.

Suggestion: Given a partition P of $[a, b]$, interpret $s_P f - S_P f$ as a sum of areas of rectangles that cover the graph of f.

(b) Suppose $f \geq 0$, and let $S = \{(x, y) : x \in [a, b],\ 0 \leq y \leq f(x)\}$. Show that S is measurable, e.g. S is bounded and and its boundary ∂S has zero content in $\mathbb{R}^2$, and that its area (which we more precisely defined below) equals $\int_a^b f(x)dx$.

Def: We can precisely define **area** as follows:

$$\text{area}(S) = \iint_S 1\, dA := \iint_R \chi_S dA,$$

where R is any rectangle containing S. This means, that given any bounded set $S \subset \mathbb{R}^2$, to compute $\iint_S 1\ dA$, we enclose S in a large rectangle R and consider a partition P of R, which produces a grid of small rectangles that cover S. The lower sum for this partition is simply the sums of the areas of the small rectangles that are contained in S, whereas the upper sum is the sum of the areas of the small rectangles that intersect S.

Taking the suprememum of the lower sums and the infimum of the upper sums yields quantities which we define to be the **inner area** and **outer area** of S:

$$\underline{A}(S) = \underline{\int}_R (\chi_S) := \sup_P s_P(\chi_S), \quad \overline{A}(S) = \overline{\int}_R (\chi_S) := \inf_P S_P(\chi_S).$$

When $\underline{A}(S) = \overline{A}(S)$, that is when the characteristic function χ_S is integrable, this common value of the inner and outer areas is defined to be the **area** of S.

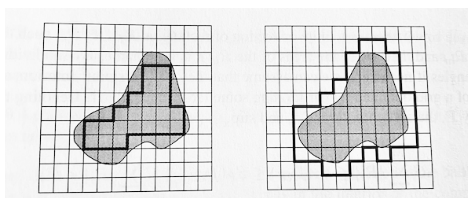

Figure 1: Approximations to the inner and outer areas of a region

Remarks: From the next set of exercises, we can conclude that the inner and outer areas of a bounded set S coincide precisely when the outer area of the boundary ∂S is zero.

But this merely means the condition that the boundary ∂S should have zero content in $\mathbb{R}^2$. In short, the inner and outer area of S coincide precisely when S is measurable. (A set $S \subset \mathbb{R}^2$ is said to be **measurable** whenever S is bounded and ∂S has zero content in $\mathbb{R}^2$.).

This is the explanation for the name **measurable**: The measurable sets in $\mathbb{R}^2$ are those with a well-defined area. One analogously defines the higher dimensional area, otherwise known as volume, for measurable sets in any dimension $\mathbb{R}^n$.

6. (Folland 4.2, Jones 10.A) Let S be a bounded set in $\mathbb{R}^2$. Show that S and its interior, S^{int} have the same inner area.
Suggestion: For any rectangle contained in S, there are slightly smaller rectangles contained in S^{int}.

7. (Folland 4.2, Jones 10.A) Let S be a bounded set in $\mathbb{R}^2$. Show that S and its closure $\overline{S}$ have the same outer area.
Suggestion: For any rectangle that intersects $\overline{S}$, there are slightly larger rectangles that intersect S.

8. (Folland 4.2, Jones 10.A) Let S be a bounded set in $\mathbb{R}^2$. Show that the inner area of S plus the outer area of ∂S equals the outer area of S.
Suggestion: Use the preceding two exercises.