Bonus Math 222 in class problems   Week: April 12, 2021   Name:

Volumes of n-balls in $\mathbb{R}^n$

1. Folland & Jones 10:29
   Use "double polar coordinates" in $\mathbb{R}^4$:

$$\begin{aligned} x_1 &= r\cos\theta, \\ x_2 &= r\sin\theta, \\ x_3 &= s\cos\varphi, \\ x_4 &= s\sin\varphi \end{aligned}$$

   to compute the 4-dimensional volume of the ball $x_1^2 + x_2^2 + x_3^2 + x_4^2 \leq R^2$. Answer: $\frac{\pi^2}{2}R^4$

2. (Challenging) Jones 10:29
   Using "$n \times$ polar coordinates" in $\mathbb{R}^{2n}$, prove that the $2n$-dim volume of the ball $x_1^2 + ... + x_{2n}^2 \leq 1$ is

$$\mathrm{vol}_{2n}(B(0,1)) := \frac{\pi^n}{n!}.$$

   (Extra Spicy) Modifying this choice of coordinates appropriately for odd dimensions (one possibility is coming up with the analogue of higher dimensional spherical coordinates):

$$\mathrm{vol}_{2n+1}(B(0,1)) := \frac{2(n!)(4\pi)^n}{(2n+1)!}.$$

   Suggestions: Using $\varphi_1, \varphi_2, ..., \varphi_{2n-1} \in [0,\pi]$, and $\varphi_{2n} \in [0, 2\pi]$, set $x_1 = \rho\cos(\varphi_1)$ and for $i \in [1, n]$,

$$\begin{aligned} x_{2i} &= \rho\sin(\varphi_1)\cdots\sin(\varphi_{2i-1})\cos(\varphi_{2i}) \\ x_{2i+1} &= \rho\sin(\varphi_1)\cdots\sin(\varphi_{2i-1})\sin(\varphi_{2i}) \end{aligned}$$

   You then should get

$$\mathrm{Jac}_{2n+1} = \rho\sin(\varphi_1)\cdots\sin(\varphi_{2n-1})\mathrm{Jac}_{2n}$$

   You can then use this to prove by induction that closed form expression for the volume element in $(2n+1)$-spherical coordinates is:

$$d^nV = \left|\det\frac{\partial(x_i)}{\partial(\rho,\varphi_j)}\right| = \rho^{2n}\sin^{2n-1}(\varphi_1)\sin^{2n-2}(\varphi_2)\cdots\sin(\varphi_{2n-1})d\rho\; d\varphi_1\cdots d\varphi_n$$

   Sanity check: When $n = 1$, how do these hyperspherical coordinates compare with our 3-D spherical coordinates? If you get stuck, see also §1-3 of `https://en.wikipedia.org/wiki/N-sphere`.
   The other means of attack is to prove the recursive formula using some tricks (as in Jones §10):

$$\mathrm{vol}_n(B(0,1)) := \frac{2\pi}{n}\mathrm{vol}_{n-2}(B(0,1)).$$

Surface Area of $(n-1)$-spheres in $\mathbb{R}^n$:

3. Find a parametrization for S^3 in $\mathbb{R}^4$. (Use double polar coordinates, then do another polar coordinate transformation to deal with the (r, s) variables $r^2 + s^2 = 1$. Then compute the Gram matrix) Show that the 3-D volume of S^3 is $2\pi^2$ (and if our 3-sphere has radius R then it has 3-D volume $2\pi^2R^3$).

4. Find a parametrization for S^4 in $\mathbb{R}^5$. (Use the extra spicy odd dimensional spherical coordinates and set $\rho = 1$. You'll need to compute the Gram matrix to find the 4-D volume element - this won't be the Jacobian computed in #2). Show that the 4-D volume of S^4 is $\frac{8}{3}\pi^2$ (and if our 4-sphere has radius R then it has 4-D volume $\frac{8}{3}\pi^2R^4$.