Math 222 in class problems Week: March 8, 2021 Name:

1. Folland, Jones 10.F (on HW 7)
   Calculate $\iint_S (x+y)^4(x-y)^{-5}dA$ where S is the square $-1 \le x+y \le 1$, $1 \le x-y \le 3$. Answer: $\frac{4}{81}$

2. Warm-Up
   Prove that the area of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \le 1$ is πab.

3. Stewart (on HW 7)
   Evaluate the integral by making an appropriate change of variables:
$$\iint_E \sin(9x^2+4y^2)dA,$$
   where E is the region in the first quadrant bounded by the ellipse $9x^2+4y^2 = 1$. Answer : $\frac{\pi}{24}(1-\cos 1)$.

4. Jones 10-41
   UPDATED so that E is actually an ellipsoid: Let E be the ellipsoid
$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + ... + \frac{x_n^2}{a_n^2} \le 1,$$
   where the semi-axes $a_1, ..., a_n$ are any positive integers. Prove that
$$\mathrm{vol}(E) = a_1 \cdots a_n \mathrm{vol}_n(B(0,1))$$
   Sanity check: The unit 2-ball is the unit disk, which has area π. Phew!

5. Folland & Jones 10:29
   Use "double polar coordinates" in $\mathbb{R}^4$:
$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= s\cos\varphi, \\ w &= s\sin\varphi \end{aligned}$$
   to compute the 4-dimensional volume of the ball $x^2+y^2+z^2+w^2 \le R^2$. Answer: $\frac{\pi^2}{2}R^4$

6. (Challenging) Jones 10:29
   Using "$n \times$ polar coordinates" in $\mathbb{R}^{2n}$, prove that the $2n$-dim volume of the ball $x_1^2 + ... + x_{2n}^2 \le 1$ is
$$\mathrm{vol}_{2n}(B(0,1)) := \frac{\pi^n}{n!}.$$
   (Extra Spicy) Modifying this choice of coordinates appropriately for odd dimensions (one possibility is coming up with the analogue of higher dimensional spherical coordinates):
$$\mathrm{vol}_{2n+1}(B(0,1)) := \frac{2(n!)(4\pi)^n}{(2n+1)!}.$$
   Suggestions: Using $\varphi_1, \varphi_2, ..., \varphi_{2n-1} \in [0,\pi]$, and $\varphi_{2n} \in [0, 2\pi]$, set $x_1 = \rho\cos(\varphi_1)$ and for $i \in [1,n]$,
$$\begin{aligned} x_{2i} &= \rho\sin(\varphi_1)\cdots\sin(\varphi_{2i-1})\cos(\varphi_{2i}) \\ x_{2i+1} &= \rho\sin(\varphi_1)\cdots\sin(\varphi_{2i-1})\sin(\varphi_{2i}) \end{aligned}$$

You then should get

$$\mathrm{Jac}_{2n+1} = \rho \sin(\varphi_1) \cdots \sin(\varphi_{2n-1}) \mathrm{Jac}_{2n}$$

You can then use this to prove by induction that closed form expression for the volume element in $(2n+1)$-spherical coordinates is:

$$d^n V = \left| \det \frac{\partial(x_i)}{\partial(\rho, \varphi_j)} \right| = \rho^{2n} \sin^{2n-1}(\varphi_1) \sin^{2n-2}(\varphi_2) \cdots \sin(\varphi_{2n-1}) d\rho \; d\varphi_1 \cdots d\varphi_n$$

Sanity check: When $n = 1$, how do these hyperspherical coordinates compare with our 3-D spherical coordinates? If you get stuck, see also §1-3 of https://en.wikipedia.org/wiki/N-sphere.
The other means of attack is to prove the recursive formula using some tricks:

$$\mathrm{vol}_n(B(0,1)) := \frac{2\pi}{n} \mathrm{vol}_{n-2}(B(0,1)).$$

(Frank does this in the book.)