Math 451-551 HW 1, due in Gradescope on Friday 1/16/26 NAME:

1. Consider spherical coordinates on $\mathbb{R}^3$ (not including the line $x = y = 0$) ρ, ϕ, θ defined in terms of the Euclidean coordinates x, y, z by

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

   (a) Express $\partial/\partial\rho$, $\partial/\partial\phi$, and $\partial/\partial\theta$ as linear combinations of $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$. (The coefficients in these linear combinations will be functions on $\mathbb{R}^3 \setminus \{x = y = 0\}$.)

   (b) Express $d\rho$, $d\phi$, and $d\theta$ as linear combinations of dx, dy, and dz.

2. Lee 1-8 [Second Edition]
By identifying $\mathbb{R}^2$ with $\mathbb{C}$, we can think of the unit circle S^1 as a subset of the complex plane. An angle function on a subset $U \subset S^1$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\theta(z)} = z$ for all $z \in U$. Show that there exists an angle function on an open subset $U \subset S^1$ if and only if $U \neq S^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for S^1 with its standard smooth structure.

3. Please spend some time reading Chapters 1-2 of Lee. Write down something interesting you learned or an example you liked. Remember that you'll want the angle function in a future homework set to show that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$. In Chapter 1, it's not absolutely necessary to read about manifolds with boundary. I'll revisit these notions when we get to integration on manifolds. In Chapter 2, you can omit as much of the reading about partitions of unity as you would like. We won't use partitions of unity until we talk about integration and Riemannian metrics (at least 1 month away). In Chapter 2, the book alludes to the difficulties in distinguishing between smooth structures up to diffeomorphism. This is a subtle point I hope to explain more about as the semester progresses.

   Please see the next page for hints to the above homework problems

* Which problems provided a worthwhile learning experience? How many hours did you spend on it?

Problem 1: There are two slick chain rule tricks to make HW 1 #1 cute instead of a grotesque brute force computation

To compute the partials, use the chain rule:

Suppose $f(x, y, z)$ is an arbitrary differentiable function, where x, y, z are differentiable functions depending on ρ, ϕ, θ

Then:

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial \rho}$$

Thus

$$\frac{\partial}{\partial \rho} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \rho}$$

Since we are just multiplying terms we can change the order to get:

$$\frac{\partial}{\partial \rho} = \frac{\partial x}{\partial \rho}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \rho}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \rho}\frac{\partial}{\partial z}$$

This should be more reasonable to compute and you don't have to think too deeply about trig identities to get some reasonable expressions.

For the second part you use the interpretation of the chain rule for differentials from multivariable calculus:

$$\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} \text{row of derivatives of } x \text{ wrt each of } \rho, \phi, \theta \\ \text{row of derivatives of } y \text{ wrt each of } \rho, \phi, \theta \\ \text{row of derivatives of } z \text{ wrt each of } \rho, \phi, \theta \end{pmatrix} \begin{pmatrix} d\rho \\ d\phi \\ d\theta \end{pmatrix}$$

Use what you did in the first part to fill in the derivatives in the 3x3 matrix in terms of x, y, z. Invert this matrix to solve for $d\rho, d\phi, d\theta$ (this should be straightforward if you did the first part correctly)

Problem 2:

$\Rightarrow$ Suppose $U = S^1$ and use topology facts to conclude that $\theta(U)$ is actually a closed interval and deduce that S^1 is homeomorphic to a closed interval. Derive a contradiction. (Maybe consider what happens if you remove a point.)

$\Leftarrow$ A milder coordinate computation than the first problem.

Problem 3: I moved the original problem 3 about

$$TS^1$$

to Homework 2. In its place is a more straightforward exercise from Chapter 2 of Lee that should fall out of the definitions of smooth maps.