Math 451-551 HW #3: Due in Gradescope on Friday 1/30/26 NAME:

Please start each question on a separate page from the previous question.

1. Lee 3-4 [Second Edition]
   Show that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

2. Lee Exercise 3.7, Proposition 3.6 [Second Edition] Let M and N be smooth manifolds and let $p \in M$. Show that

   (a) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$,

   (b) if $F : M \to N$ is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

3. Show that if M and N are smooth manifolds and if $p \in M$ and $q \in N$, then there is a canonical isomorphism

   $$T_{(p,q)}(M \times N) = T_pM \oplus T_qN.$$

   It is painful to describe this isomorphism in full detail with respect to derivations or linear combinations of partial derivatives with respect to coordinate charts. (Maybe think about what is happening if you are so inclined.)

   Set up to show there is a canonical isomorphism:
   Let $(x, y) \in M \times N$. Consider the canonical inclusion maps $i^M : x \mapsto (x, q)$ and $i^N : y \mapsto (p, y)$ as well as the projection maps $\pi^M : (x, y) \mapsto x$ and $\pi^N : (x, y) \mapsto y$.

   Define $\Phi : T_pM \oplus T_qN \to T_{(p,q)}(M \times N)$ by

   $$\Phi(v, w) = di^M_p(v) + di^N_p(w)$$

   and define $\Psi : T_{(p,q)}(M \times N) \to T_pM \oplus T_qN$ by

   $$\Psi(v) = (d\pi^M_{(p,q)}(v), d\pi^M_{(p,q)}(w)).$$

   Deduce that $\Psi \circ \Phi$ is the identity on $T_pM \oplus T_qN$. (This can be done by showing and then using the fact that $d\pi^N \circ di^M = 0$ and $d\pi^M \circ di^N = 0$.). Now use linear algebra facts to conclude that Ψ and Φ are isomorphisms.

4. Please spend some time reading Chapters 3-4 of Lee. Please read through the computations in coordinates, especially change of coordinates and watch the module in canvas if you want to see an example worked out by Dr. Nick Castro when he covered for me last year when I had surgery. If you don't feel confident about your memories of the Inverse and Implicit function theorems in Euclidean space, please review Theorem C.34 and C.40. Write down something interesting you learned or an example you liked.

* Which problems provided a worthwhile learning experience? How many hours did you spend on it?