Math 451 HW #4: Due in Gradescope on Friday 2/6/25 at 11:59pm NAME:

Please start each question on a separate page from the previous question.

1. Lee 8-10 [Second Edition] (You don't have to read Chapter 8 to do this - it is meant to reassure you in how to compute differentials. Note F_* is the same notation for dF, without a point explicitly specified.
   Let M be the open submanifold of $\mathbb{R}^2$ where both x and y are positive and let $F : M \to M$ be the map
   $$F(x, y) = \left(xy, \frac{y}{x}\right).$$
   Show that F is a diffeomorphism, and compute $F_* X$ and $F_* Y$ where
   $$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \qquad Y = y\frac{\partial}{\partial x}$$
   Note: The definition of the differential/pushforward yields $(F_* Z)_{(s,t)} = dF_{F^{-1}(s,t)} Z_{F^{-1}(s,t)}$.

2. Lee 4-6 [Second Edition]
   Let M be a nonempty smooth compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$. A submersion is a smooth map whose differential is surjective.

3. Lee 5-1 [Second Edition]
   Consider the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^4$ defined by
   $$\Phi(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y).$$
   Show that $(0, 1)$ is a regular value of Φ in the sense of multivariable calculus. *(It turns out that the level set $\Phi^{-1}(0, 1)$ is diffeomorphic to S^2, but this is surprisingly painful to prove, so I won't ask you to do this.)*

4. Lee 5-10 [Second Edition]
   For each $a \in \mathbb{R}$, let M_a be the subset of $\mathbb{R}^2$ defined by
   $$M_a = \{(x, y) \mid y^2 = x(x - 1)(x - a)\}.$$
   For which values of a is M_a an embedded submanifold of $\mathbb{R}^2$? *For which values can M_a be given a topology and smooth structure making it into an immersed submanifold? You don't need to give a rigorous proof for the a values which don't produce embedded submanifolds; this is surprisingly painful.*

* Which problems provided a worthwhile learning experience? How many hours did you spend on it?