Symplectic Embeddings on the Virtual BeECH

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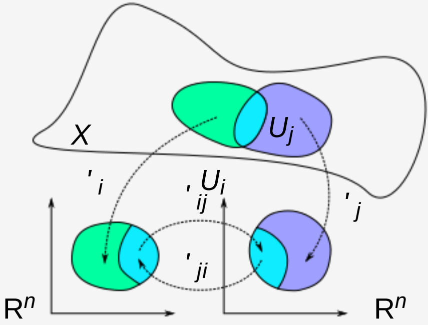
Manifolds

Prof. Jo

Definition

A **smooth n-manifold** X is a topological space that looks locally like $\mathbb{R}^n$ and admits a global differentiable structure.



A **smooth atlas** on X has

Charts (U_i, φ_i) for which the U_i cover X.

The $\varphi_i : U_i \to \mathbb{R}^n$ are diffeomorphisms onto an open subset of $\mathbb{R}^n$.

The **transition maps** are given by

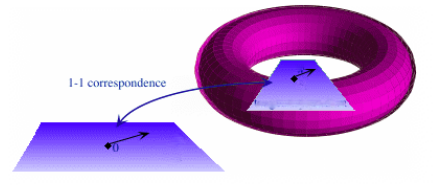
$$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_i \cap U_j)} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j).$$

We need 4 dimensions in order to be embedded!

Tangent spaces and Differential forms Prof. Jo

Definition

The *tangent space* of X^n, denoted T_pX, is a vector space “at” a point p of the manifold diffeomorphic to R^n.



Definition

A *1-form* is a linear function: $T_pX \ \ ! \ \ \mathrm{R}$.

Directional derivatives $D_v f(p) = r f(p) \ \ \frac{v}{jvj}$ and fds from ${}^{\mathrm{H}}_{C} fds$.

Differential forms are a coordinate independent approach to calculus. Great for defining integrals over curves, surfaces, and manifolds!

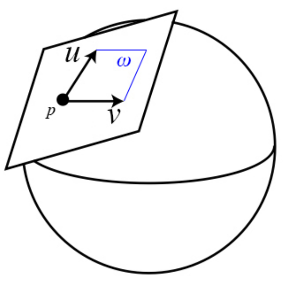
Linear approximation of an owl

What is a symplectic manifold?

Leo

Definition

A 2-form $!$ on a manifold X is a smooth choice of anti-symmetric bilinear functions $!_p : T_pX \quad T_pX \; ! \; \text{R}$ for each $p \; 2 \; X$.



At the infinitesimal level, $!$ measures oriented area spanned by vectors u and v at a point p.

Definition

A symplectic manifold is a pair $(X^{2n}; !)$ such that $!$ is a smooth 2-form satisfying

Closedness: $d! = 0$

Nondegeneracy: $!^n$ is nonvanishing, i.e. a volume form.

Examples

$dx \; \hat{} \; dy$ on R^2

$\text{P}^{\,n}_{i=1} dx_i \; \hat{} \; dy_i$ on R^{2n}

Examples of symplectic manifolds

Leo

Given $\mathbb{C}^n$ with $\omega_0 = \frac{\sqrt{-1}}{2}\sum_{j=1}^n dz_j \wedge d\bar{z}_j$,

consider the symplectic manifolds with boundary

Ball: $B^{2n}(r) := \{z \in \mathbb{C}^n \mid \pi|z_1^2| + \dots + \pi|z_n^2| \leq r\}$

Cylinder: $Z^{2n}(R) := B^2(R) \times \mathbb{C}^{n-1}$

Ellipsoid: $E(a,b) := \left\{z \in \mathbb{C}^2 \,\middle|\, \frac{\pi|z_1^2|}{a} + \frac{\pi|z_2^2|}{b} \leq 1\right\}$

Polydisc: $P(a,b) := \{z \in \mathbb{C}^2 \mid \pi|z_1^2| \leq a,\ \pi|z_2^2| \leq b\}$

Symplectomorphisms and Symplectic Embeddings Leo

A **diffeomorphism** is a smooth bijective map with smooth inverse.

Definition

Two symplectic manifolds (X, ω) and (X', ω') are **symplectomorphic** if there exists a diffeomorphism $f : (X, \omega) \to (X', \omega')$ s.t. $f^*\omega' = \omega$.

Here $f^*\omega'(\cdot, \cdot) = \omega'(df \cdot, df \cdot)$

Definition

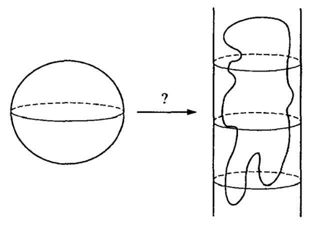
A manifold (X, ω) is said to **symplectically embed** into (X', ω') if there exists an injective smooth map $f : X \overset{s}{\hookrightarrow} X'$ s.t. f is a symplectomorphism onto its image.

Symplectomorphisms are volume-preserving. Are all volume preserving maps are symplectomorphisms?

Rigidity in Symplectic Geometry

Leo

Are symplectic embeddings restricted by more than volume?



Theorem (Gromov Nonsqueezing, 1985)

$B^{2n}(r)$ symplectically embeds into $Z^{2n}(R) = B^2(R) \quad \mathrm{R}^{2n \quad 2}$ if and only if $r \quad R$

Symplectic Capacities

Definition

The **Gromov width** of (X, ω) of dimension $2n$ is the supremum over real numbers r such that $B^{2n}(r)$ embeds symplectically into X.

Symplectic capacity $\rightsquigarrow$ obstructions of symplectic embeddings:
If $c(X, \omega_1) > c(X', \omega')$ then $\nexists\ (X, \omega) \overset{s}{\hookrightarrow} (X', \omega')$.

Definition

A **symplectic capacity**, $c : \begin{Bmatrix} \text{symplectic} \\ \text{manifolds} \end{Bmatrix} \to \mathbb{R}_{\geq 0}$, satisfies:

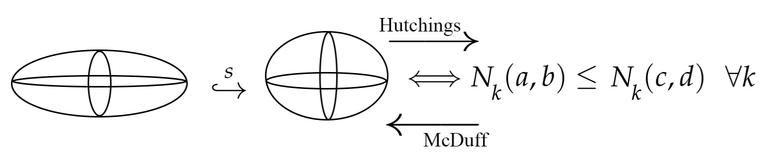
- Monotonicity: If $(X, \omega) \overset{s}{\hookrightarrow} (X', \omega')$ then $c(X) \leq c(X')$.
- Conformality/Scaling: for $a \in \mathbb{R} \setminus 0$, $c(X, a\omega) = |a| c(X, \omega)$
- Weak Normalization: $0 < c(B^{2n}(1)) \leq c(Z^{2n}(1)) < \infty$

Flexibility of Symplectic Embeddings

Leo

$$E(a;b) := \quad \frac{jz_1^2j}{a} + \frac{jz_2^2j}{b} \quad 1$$

Theorem (McDuff, 2011)



$N_k(a;b)$ is k^{th} smallest entry in $(am+bn)_{m;n2Z_{\ 0}}$ with repetitions.

$$N(1;4) \quad = \quad 0\ 1\ 2\ 3\ 4\ 4\ 5\ 5\ 5\ 5$$

$$N(2;2) \quad = \quad 0\ 2\ 2\ 4\ 4\ 4\ 6\ 6\ 6\ 6$$

Thus $E(1;4) \,,\overset{s}{!}\; E(2;2) = B(2)$!

b

$ax + by = L$

a

$N_k(a,b)$ are the **ECH capacities** of $E(a,b)$.

$c_k(E(a,b))$ is the smallest L such that $k+1$ lattice points are contained in the region of $\mathbb{R}^2$ bounded by $ax + by = L$ and the x- and y-axes.

Definition

Given a symplectic 4-manifold (X,ω), its **ECH capacities** are a sequence of real numbers

$$0 = c_0(X,\omega) < c_1(X,\omega) \leq c_2(X,\omega) \leq \ldots \leq \infty$$

such that

$$(X,\omega) \overset{s}{\hookrightarrow} (X',\omega') \Rightarrow c_k(X,\omega) \leq c_k(X',\omega') \, \forall k.$$

Some properties of ECH capacities:

ECH capacities *obstruct* low-dimensional symplectic embeddings:

$$(X;\omega) \overset{s}{\not\hookrightarrow} (X';\omega') \Longleftarrow \exists k\, c_k(X;\omega) > c_k(X';\omega')$$

$c_1(B^4(r)) = r$ and $c_1(Z^4(R)) = R \Longrightarrow$ 4D Gromov nonsqueezing.

$\lim_{k\to\infty} \frac{c_k(X,\omega)^2}{k} = 4\int_X \omega\wedge\omega$, relating ECH capacities to volume.

$c_k(X;\omega)$ measures the ω-area of a surface in X solving a "J-holomorphic curve" PDE with fixed boundary on ∂X; we have

$$(X;\omega) \overset{s}{\hookrightarrow} (X';\omega') \Longrightarrow c_k(X;\omega) \le c_k(X';\omega')$$

because properties of ECH force the surface in X' with fixed boundary to agree in X with the surface for X.

- The same reasoning implies the "Action Inequality" later in the talk.

Symplectic Toric Domains

Dr. Morgar

Symplectic toric domains are defined by a polytope $R^2{}_0$

$$X = f(z_1; z_2) \; 2 \; C^2 \; j \; (\; jz_1j^2; \; jz_2j^2) \; 2 \quad g:$$

$B^4(a) := f \; jz_1j^2 + \; jz_2j^2 \quad ag$

isosceles right triangle with side length a.

$E(a; b) := f \; \frac{jz_1j^2}{a} + \frac{jz_2j^2}{b} \quad 1g$

right triangle with lengths $a; b$.

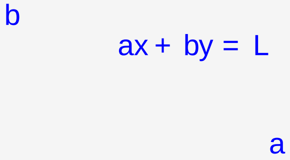
$P(a; b) := f \; jz_1j^2 \quad a; \; jz_2j^2 \quad bg$

rectangle of sides $a; b$.

(a) Ball $B(a)$ (b) Ellipsoid $E(a; b)$ (c) Polydisk $P(a; b)$

The combinatorics of the polytopes tell us about embeddings of their toric domains X !

- The area of equals the volume $R_X\ !_0 \ \hat{}\ !_0$ of X .
- In some cases we can compute $c_k(X\ )$ from the geometry of . When $X\ = E(a;b)$, we have



(d) Ellipsoid $E(a;b)$ (e) $c_k(E(a;b))$ $P(a;b)$

- And we can also compute $c_k(X\ )$ from in more complex ways for more general .
- Soon you'll see even more ways to obstruct embeddings of X using combinatorics of .

Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq A,\ 0 \leq y \leq f(x)\}$, $f \geq 0$ nonincreasing.

Definition

If f is concave, then X_Ω is a convex toric domain. If f is convex, then X_Ω is a concave toric domain.

Theorem (Cristofaro-Gardiner '19, generalizing McDuff '11)

If X_Ω is concave and $X_{\Omega'}$ is convex, then

$$X_\Omega \overset{s}{\hookrightarrow} X_{\Omega'} \iff c_k(X_\Omega) \leq c_k(X_{\Omega'})\ \forall k; \qquad (0.1)$$

However, if X_Ω is convex (e.g. a polydisk), then (0.1) is not an equivalence, only $\Rightarrow$!

So we need other means to obstruct $X_\Omega \overset{s}{\hookrightarrow} X_{\Omega'}$.

Embeddings of toric domains “beyond” ECH capacities Dr. Mor

Definition

If f is concave, then X is a convex toric domain. If f is convex, then X is a concave toric domain.

Polydisks are convex, not concave!

(f) Ball $B(a)$ (g) Ellipsoid $E(a,b)$ (h) Polydisk $P(a,b)$

Our results use the combinatorics of “beyond” the ECH capacities of X to obstruct

$$P(a,1) \xhookrightarrow{s} E(bc,c)$$

based on the relationships between a, b, and c.

Known results for polydisks $\overset{s}{\hookrightarrow}$ ellipsoids

Theorem (Hutchings, 2016)

Let $1 \le a \le 2$ and $b \in \mathbb{Z}_{>0}$. Then $P(a,1) \overset{s}{\hookrightarrow} E(bc,c)$ if and only if $a + b \le bc$.

Figure: $a + b \le bc$ means that the polydisk $P(a,1)$ can be directly included into the ellipsoid $E(bc,c)$.

New results for polydisks $\overset{s}{\hookrightarrow}$ ellipsoids

Yirong

Theorem (Ning-Yang, 2020)

Let d_0 3 be a prime number. Let 1 a $(2d_0$ $1)=d_0$, $c > 0$ and $b = p=2$ for some odd integer p $4d_0 + 1$. Then $P(a;1) \overset{s}{\hookrightarrow} E(bc;c)$ if and only if $a + b$ bc.

Figure: Each dot represents some prime d_0 3 and the shaded regions show the applicability of our theorem. With an increasing restriction on p, the theorem works for more a 1 approaching $a = 2$.

When p is small

For p 13, we can provide sharp obstructions for $1 a < 2$.

Theorem (Ning-Yang, 2020)

Let 1 a $4=3$, $c > 0$ and $b = p=2$ for some odd integer $p > 2$.
Then $P(a;1)!\overset{s}{,}$ $E(bc;c)$ if and only if $a + b$ bc.

Theorem (Ning-Yang, 2020)

Let 1 a $3=2$, $c > 0$ and $b = p=2$ for some odd integer p 7.
Then $P(a;1)!\overset{s}{,}$ $E(bc;c)$ if and only if $a + b$ bc.

Convex generators

Definition

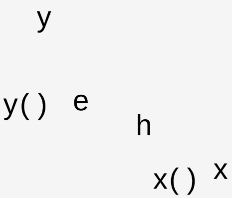
A convex generator is a convex integral path such that:

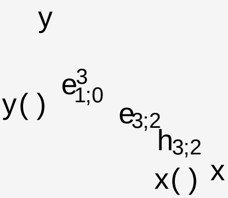
- Each edge of is labeled `e' or `h'.
- Horizontal and vertical edges can only be labeled `e'.

Definition

If be a convex generator, then its ECH index is defined to be

$$I(\ ) = 2(\ L(\ )\ \ 1)\ \ h(\ ) :$$





Action and minimal convex generators

The symplectic action of a convex generator wrt $P(a;1)$ is

$$A_{P(a;1)}(\ ) = \ x(\ ) + \ ay(\ ) :$$

The symplectic action of with respect to $E(bc;c)$ is

$$A_{E(bc;c)}(\ ) = \ r; \text{ where } cx + bcy = r \text{ is tangent to } .$$

A convex generator with $I(\ ) = 2\ k$ is minimal for $E(bc;c)$ if:

All edges of are labeled e`.

uniquely minimizes $A_{E(bc;c)}$ among convex generators with $I = 2k$ and all edges labeled e`.

Key: $e^{d_0}_{p;2}$ is minimal for $E(pc=2;c)$ for any $c > 0$, $d_0 \ 1$.

Remark

If $I(\ ) = 2\ k$ and is minimal for X then $A\ (\ ) = \ c_k(X\ )$.

Necessary conditions for J-curve in cobordism Yirong

Definition (Hutchings, 2016)

Let $\Lambda ; \Lambda^{0}$ be convex generators s.t. all edges of Λ^{0} are labeled 'e'. We write $\Lambda \quad_{P(a;1);E(bc;c)} \Lambda^{0}$ whenever:

1. (Index requirement)

   $I(\Lambda) = I(\Lambda^{0})$;
2. (Action inequality)

   $A_{P(a;1)}(\Lambda) \quad A_{E(bc;c)}(\Lambda^{0})$;
3. (J-holomorphic curve genus inequality)

   $x(\Lambda) + y(\Lambda) \quad h(\Lambda)\text{=}2 \quad x(\Lambda^{0}) + y(\Lambda^{0}) + m(\Lambda^{0}) \quad 1.$

We abbreviate ' ' for ' $_{P(a;1);E(bc;c)}$' between generators when when $a; b$ and c are specified without ambiguity.

The Hutchings criterion

Theorem (The Hutchings criterion, 2016)

Let X_Ω and $X_{\Omega'}$ be convex toric domains. Suppose $X_\Omega \overset{s}{\hookrightarrow} X_{\Omega'}$. Let Λ' be a convex generator which is minimal for $X_{\Omega'}$. Then there exists a convex generator Λ with $I(\Lambda) = I(\Lambda')$, a nonnegative integer n, and product decompositions

$$\Lambda = \Lambda_1 \cdots \Lambda_n \text{ and } \Lambda' = \Lambda'_1 \cdots \Lambda'_n,$$

such that

1. $\Lambda_i \leq_{\Omega,\Omega'} \Lambda'_i$ *for each* $i = 1, \ldots, n$.
2. *Given* $i, j \in \{1, \ldots, n\}$, *if* $\Lambda_i \neq \Lambda_j$ *or* $\Lambda'_i \neq \Lambda'_j$, *then* Λ_i *and* Λ_j *have no elliptic orbit in common.*
3. *If* S *is any subset of* $\{1, \ldots, n\}$, *then* $I\left(\prod_{i\in S}\Lambda_i\right) = I\left(\prod_{i\in S}\Lambda'_i\right)$.

In our case, $X_\Omega = P(a,1)$ and $X_{\Omega'} = E(bc, c)$.

Sketch of Proof

Theorem (Ning-Yang, 2020)

Let d_0 3 *be a prime number. Let* 1 a $(2d_0$ $1)=d_0$, $c > 0$ *and* $b = p=2$ *for some odd integer* p $4d_0 + 1$. *Then* $P(a;1) ,\overset{s}{!} E(bc;c)$ *if and only if* $a + b$ bc.

Key: use Hutchings' criterion to show the non-existence of $P(a;1) ,\overset{s}{!} E(bc;c)$ when $a + b > bc$.

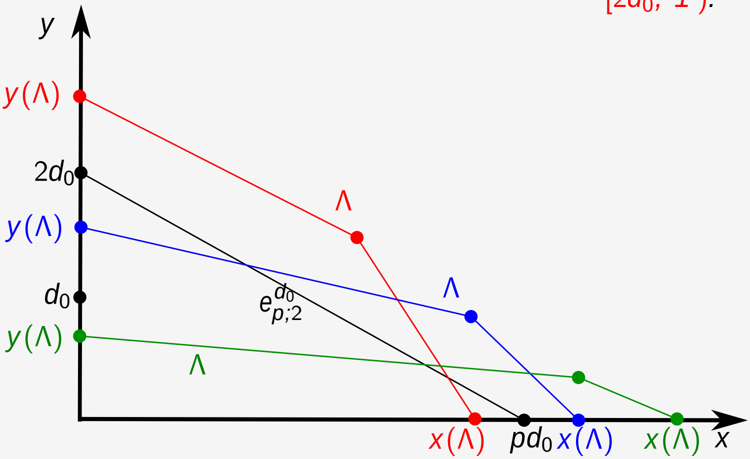
Take $\Lambda^{0} = e^{d_0}_{p;2}$, we need to show the non-existence of Λ such that

- (Trivial factorization) Λ Λ^{0}.
- (Full factorization) $\Lambda = Q_i \Lambda_i$ where Λ_i $e_{p;2}$ for 1 i d_0.
- Λ factors into 2 k d_0 1 factors.

Eliminating the trivial factorization

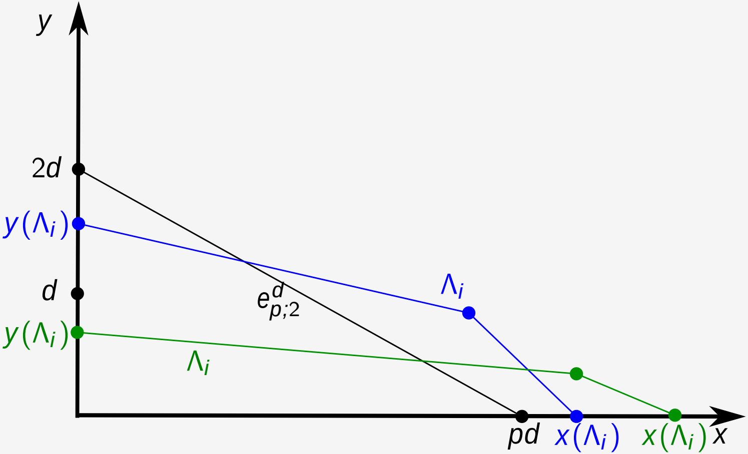
Split $\Lambda \quad \Lambda^0 = e^{d_0}_{p;2}$ into three cases: $y(\Lambda)$ 2

$$\begin{cases} [0; d_0); \\ [d_0; 2d_0); \\ [2d_0; 1): \end{cases}$$



Eliminating factorizations with 2 k d_0 1 factors

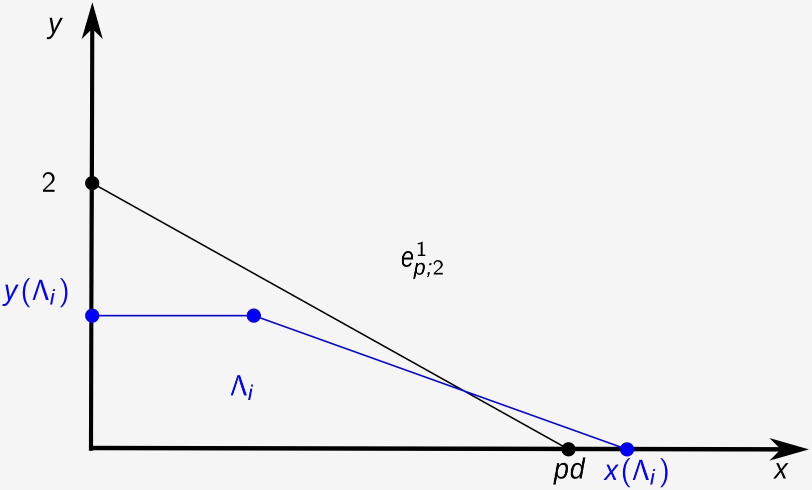
Previous argument can also show that for d d_0 1, Λ_i $e^{d}_{p;2}$ is only possible if $y(\Lambda_i) = d$.



Such Λ_i must contain an $e_{1;0}$ factor! Now use primality of d_0.

Eliminating the full factorization

In this case, we only need to consider $y(\Lambda_i) < 2$. Explicit index computations finishes the proof.



“Optimal” restrictions

Theorem (Ning-Yang)

Let $d_0 \quad 3$ be a prime number. Let $1 \quad a \quad (2d_0 \quad 1){=}d_0$, $c > 0$ and $b = p{=}2$ for some odd integer $p \quad 4d_0 + 1$. Then $P(a; 1) \overset{s}{,!} E(bc; c)$ if and only if $a + b \quad bc$.

The bound $(2d_0 \quad 1){=}d_0 \quad 2$ on a is “optimal” in this sense:

Example

Under the same hypothesis, let $'' > 0$ and take instead

$$a = (2d_0 \quad 1){=}d_0 + '':$$

There always exists a convex generator $\Lambda \quad e^{d_0}_{p;2}$ for $d_0 \quad 2$, when

$$a + b \quad ''{=}2 < bc < a + b;$$

i.e. Hutchings’ criterion cannot provide sharp obstructions.

“Optimal” restrictions

The restriction $p \quad 4d_0 + 1$ is similarly “optimal”:

Example

Under the same hypothesis, consider $a = (2d_0 \quad 1)=d_0$ and take

$$p = 4d_0 \quad 3 \quad 4d_0 + 1:$$

There always exists a convex generator $\Lambda \quad e_{p;2}^{d_0}$ for $d_0 \quad 2$, when

$$a + b \quad \frac{d_0 \quad 1}{2d_0^2} < bc < a + b;$$

again in which case Hutchings’ criterion cannot provide sharp obstructions.

General case when $q > 2$?

When $b = p{=}q$ and $q > 2$, “Beyond ECH” tools are insufficient to disprove the existence of $\Lambda \quad e^{d_0}_{p;q}$ for $a + b > bc$.