Floer theories and Reeb dynamics

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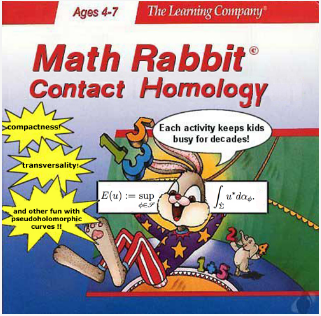


IAS Member's Colloquium, February 2023

Classical mechanics

The **phase space** $\mathbb{R}^{2n}$ of a system consists of the position and momentum of a particle.

Euler-Lagrange: The equations of motion minimize action
$\rightsquigarrow$ n second order differential equations.

Hamilton-Jacobi: The n Euler-Lagrange equations
$\rightsquigarrow$ $2n$ first order equations.

Motion is governed by conservation of **energy**, a Hamiltonian H.

- Flow lines of $X_H = -J_0 \nabla H$ are solutions.
- Phase space is (secretly) a symplectic manifold.
- Regular energy level surfaces give rise to **contact manifolds**.
- Flow lines of the **Reeb vector field** are solutions.

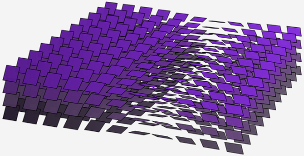
Contact geometry shows up in...

Geodesic flow, optics, thermodynamics, surface dynamics, three body problems ...

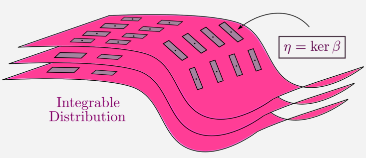
Contact structures

Definition

A **contact structure** is a maximally nonintegrable hyperplane field.



$\xi = \ker(dz - ydx)$



The kernel of a 1-form λ on Y^{2n+1} is a contact structure whenever

- $\lambda \wedge (d\lambda)^n$ is a volume form $\Leftrightarrow$ $d\lambda|_\xi$ is nondegenerate.

Darboux's Theorem

Let λ be a contact form on Y^{2n+1} and $p \in Y$. Then there are coordinates on $U_p \subset Y$ such that $\lambda|_{U_p} = dz - \sum_{i=1}^n y_i dx_i$.

Locally all contact structures look the same!

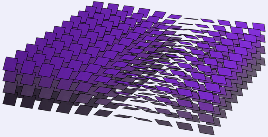
$\rightsquigarrow$ no local invariants like curvature.

Reeb vector fields

Definition

The **Reeb vector field** R on (Y, λ) is uniquely determined by

- $\lambda(R) = 1$
- $d\lambda(R, \cdot) = 0$



$\lambda = dz - ydx, \quad R = \frac{\partial}{\partial z}$

The **Reeb flow** $\varphi_t : Y \to Y$ is defined by $\frac{d}{dt}\varphi_t(x) = R(\varphi_t(x))$.

The Reeb flow preserves the contact form and contact structure.

A closed **Reeb orbit** (modulo reparametrization) satisfies

$$\gamma : \mathbb{R}/T\mathbb{Z} \to Y, \quad \dot{\gamma}(t) = R(\gamma(t)), \tag{1}$$

and is **embedded** whenever (1) is injective.

Reeb orbits on a contact 3-manifold

Given an embedded **Reeb orbit** $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$,
the linearized flow along γ defines a symplectic linear map

$$d\varphi_t : (\xi|_{\gamma(0)}, d\lambda) \to (\xi|_{\gamma(t)}, d\lambda)$$

$d\varphi_T$ is called the **linearized return map**.

If 1 is not an eigenvalue of $d\varphi_T$ then γ is **nondegenerate**. λ is **nondegenerate** if all Reeb orbits associated to λ are nondegenerate.

For $\dim Y = 3$, nondegenerate orbits are either **elliptic** or **hyperbolic** according to whether $d\varphi_T$ has eigenvalues on S^1 or real eigenvalues.

Later, we consider an almost complex structure J on $T(\mathbb{R} \times Y)$:

- J is $\mathbb{R}$-invariant
- $J\xi = \xi$, rotates ξ positively with respect to $d\lambda$
- $J(\partial_s) = R$, where s denotes the $\mathbb{R}$ coordinate

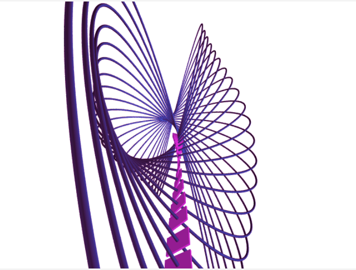
Reeb orbits on S^3

$$S^3 := \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \lambda = \frac{i}{2}(u d\bar{u} - \bar{u} du + v d\bar{v} - \bar{v} dv).$$

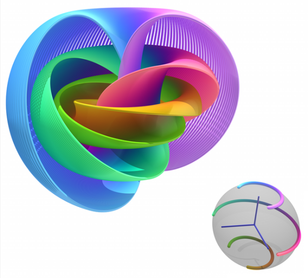
The orbits of the Reeb vector field form the Hopf fibration!

$$R = iu\frac{\partial}{\partial u} - i\bar{u}\frac{\partial}{\partial \bar{u}} + iv\frac{\partial}{\partial v} - i\bar{v}\frac{\partial}{\partial \bar{v}} = (iu, iv).$$

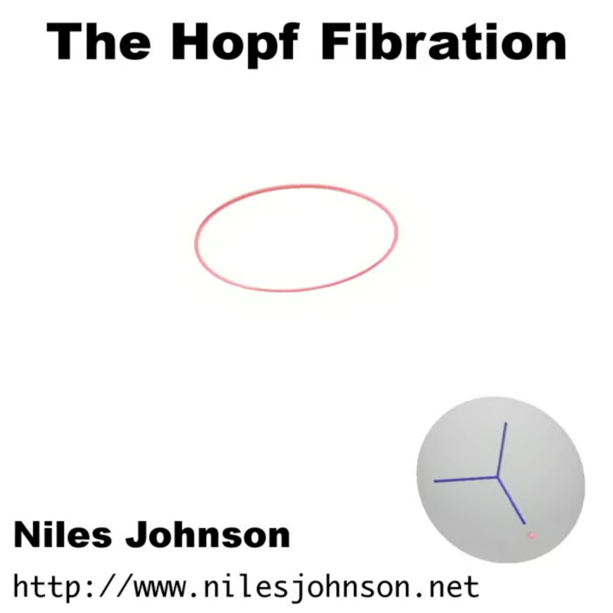
The flow is $\varphi_t(u,v) = (e^{it}u, e^{it}v)$.



Patrick Massot



Niles Johnson, $S^3/S^1 = S^2$


The Hopf Fibration
Niles Johnson
http://www.nilesjohnson.net

Existence of periodic orbits

The Weinstein Conjecture (1978)

Let Y be a closed oriented odd-dimensional manifold with a contact form λ. Then the associated Reeb vector field R_λ has a closed orbit.

- Weinstein (convex hypersurfaces)
- Rabinowitz (star shaped hypersurfaces)
- Star shaped is secretly contact!
- Viterbo, Hofer, Floer, Zehnder ('80's fun)
- Hofer (S^3)
- Taubes (dimension 3)

Tools $>$ 1985: **Floer Theories** and **Gromov's** pseudoholomorphic curves.

A new era of contact geometry

Helmut Hofer:

> *Ja ok, so why did I come into symplectic and contact geometry? So it turned out I had the flu and the only thing to read was Rabinowitz's paper where he proves the existence of periodic orbits on star-shaped energy surfaces. It turned out to contain a fundamental new idea, to study a different action functional for loops in the phase space rather than for Lagrangians in the configuration space. Which actually if we look back, led to the variational approach in symplectic and contact topology, which is reincarnated in infinite dimensions in Floer theory and has appeared in every other subsequent approach.*
>
> *...Ja, the flu turned out to be really good.*

Morse theory

Let $f \in C^\infty(M;\mathbb{R})$ be nondegenerate and g be a "reasonable" metric.
$\rightsquigarrow (f,g)$ is **Morse-Smale.**

$CM_* = \mathbb{Z}\langle \mathrm{Crit}(f)\rangle$.
$* = \#\{\text{negative eigenvalues } \mathrm{Hess}(f)\}$
$\partial^{\mathrm{Morse}}$ counts $u \in \mathcal{M}_1(x,y)/\mathbb{R}$, flow lines of $-\nabla f$ between critical points

Theorem (Floer '80s, with technical conditions)

Floer $HF_*(M,\omega,H,J) \cong$ *Morse* $H_*(M,(H,\omega(\cdot,J\cdot))) \cong$ *Sing* $H_*(M;\mathbb{Q})$

The Arnold Conjecture (Floer '80s...)

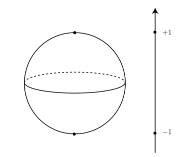
Let (M^{2n},ω) be compact symplectic and $H_t = H_{t+1} : M \to \mathbb{R}$ be a smooth time dependent nondegenerate 1-periodic Hamiltonian. Then

$$\#\{\text{1-periodic orbits of } X_{H_t}\} \geq \sum_{i=0}^{2n} \dim H_i(M;\mathbb{Q})$$

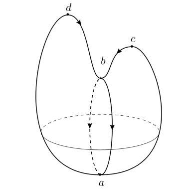
Analytic Necessities:

Transversality (for implicit function theorem $\Rightarrow \mathcal{M}_k(x,y)$ is a manifold)
Compactness (so ∂ is well defined, $\partial^2 = 0$, and invariance holds)

Recollections of spheres



$$C_*(S^2,(f,g)) = \begin{cases} \mathbb{Z}_2 & * = 0,\ 2 \\ 0 & \text{else} \end{cases} \qquad \partial \equiv 0$$



$$C_*(S^2,(f,g)) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & * = 2 \\ \mathbb{Z}_2 & * = 1 \\ \mathbb{Z}_2 & * = 0 \end{cases} \qquad \begin{array}{l} \partial c = \partial d = b \\ \partial b = 2a = 0 \end{array}$$

Theorem (Reeb '46)

If there exists a Morse function on a compact connected M with only two critical points then M is homeomorphic to a sphere.

Theorem (Hutchings-Taubes 2008)

A closed contact 3-manifold admits ≥ 2 embedded Reeb orbits and if there are exactly two then Y is diffeomorphic to S^3 or a lens space.

Embedded contact homology (ECH)

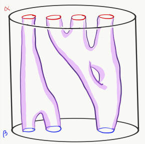
ECH is a gauge theory for (Y^3, λ) and $\Gamma \in H_1(Y;\mathbb{Z})$ due to Hutchings.

$ECC_*(Y, \lambda, \Gamma, J)$ is a $\mathbb{Z}_2$ vector space generated by **Reeb currents** $\alpha = \{(\alpha_i, m_i)\}$:

- α_i is an embedded Reeb orbit, $m_i \in \mathbb{Z}_{>0}$,
- if α_i is hyperbolic, $m_i = 1$,
- $\sum_i m_i[\alpha_i] = \Gamma$.

$*$ is a relative $\mathbb{Z}_d$-grading, d is divisibility of $c_1(\xi) + 2PD(\Gamma)$ in $H^2(Y;\mathbb{Z})$.
$*$ is given by the **ECH index**, a topological index defined via c_1, CZ, and relative self-intersection pairing, wrt $Z \in H_2(Y, \alpha, \beta)$.

$\langle \partial^{ECH} \alpha, \beta \rangle$ counts **currents**, realized by unions of **holomorphic curves**



For generic J,
ECH index one yields
somewhere injective.

-Hutchings' 02 Haiku

Dee squared is zero;
obstruction bundle gluing
is complicated.

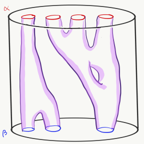
Hutchings-Taubes' 07 & 09 Haiku

Invariance of ECH

$ECC_*(Y, \lambda, \Gamma, J)$ is generated by **Reeb currents** $\alpha = \{(\alpha_i, m_i)\}$ over $\mathbb{Z}_2$

Grading is given by the **ECH index**, a topological index defined via c_1, CZ, and relative self-intersection pairing, wrt $Z \in H_2(Y, \alpha, \beta)$.

$\langle \partial^{ECH} \alpha, \beta \rangle$ counts **currents**, realized by unions of **holomorphic curves**



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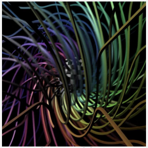


Jason Hise

Theorem (Taubes G&T (2010), no. 5, 2497-3000)

If Y is connected, there is a canonical isomorphism of relatively graded modules

$$ECH_*(Y, \lambda, \Gamma, J) = \widehat{HM}^{-*}(Y, \mathfrak{s}_\xi + \mathrm{PD}(\Gamma))$$

ECH is a topological invariant of Y !

(shift Γ when changing choice of ξ)

Prequantization bundles

Theorem (Boothby-Wang construction '58)

Let (Σ_g, ω) be a Riemann surface such that $\frac{[\omega]}{2\pi}$ admits an integral lift. Let $\mathfrak{p} : Y \to \Sigma_g$ be the principal S^1-bundle with Euler class $e = -\frac{[\omega]}{2\pi}$. Then there is a connection 1-form $-i\lambda$ on Y whose Reeb vector field R is tangent to the S^1-action.

- (Y, λ) is the **prequantization bundle** over (Σ_g, ω).
- The Reeb orbits of R are the S^1-fibers of this bundle.
- $d\lambda = \mathfrak{p}^*\omega$
- $\mathfrak{p}_*\xi = T\Sigma_g$
- The Reeb orbits of R are degenerate.

Use a Morse-Smale $H : \Sigma_g \to \mathbb{R}$, which is C^2 close to 1 to perturb λ. The perturbed Reeb vector field for $\lambda_\varepsilon := (1 + \varepsilon\mathfrak{p}^* H)\lambda$

$$R_\varepsilon = \frac{R}{1 + \varepsilon\mathfrak{p}^* H} + \frac{\varepsilon\tilde{X}_H}{(1 + \varepsilon\mathfrak{p}^* H)^2}$$

Perturbed Reeb dynamics of prequantization bundles

$$R_\varepsilon = \frac{R}{1+\varepsilon\mathfrak{p}^*H} + \frac{\varepsilon\tilde{X}_H}{(1+\varepsilon\mathfrak{p}^*H)^2}$$

Action: $\mathcal{A}(\gamma) := \int_\gamma \lambda_\varepsilon$

- if $\mathcal{A}(\gamma) < L_\varepsilon$ then γ is nondegenerate and a fiber over $p \in \mathrm{Crit}(H)$;
- if $\mathcal{A}(\gamma) > L_\varepsilon$ then γ loops around the tori above the orbits of X_H, or is a larger iterate of a fiber above $p \in \mathrm{Crit}(H)$.

Denote the k-fold cover projecting to $p \in \mathrm{Crit}(H)$ by γ_p^k.

$$CZ_\tau(\gamma_p^k) = RS_\tau(\mathrm{fiber}^k) - \frac{\dim(\Sigma_g)}{2} + \mathrm{ind}_p(H).$$

Using the constant trivialization of $\xi = \mathfrak{p}^*T\Sigma_g$, $RS_\tau(\mathrm{fiber}^k) = 0$.

$$CZ_\tau(\gamma_p^k) = \mathrm{ind}_p(H) - 1.$$

Assume H is perfect. Denote the

- index zero elliptic orbit by e_-,
- index two elliptic orbit by e_+,
- hyperbolic orbits by $h_1, \ldots, h_{2g}$.

ECH from Morse H_*

Theorem (Nelson-Weiler '20, $\mathbb{Z}_2$-grading in Farris '11)

Let $(Y, \xi = \ker\lambda)$ be a prequantization bundle over (Σ_g, ω) of negative Euler class e. Then as $\mathbb{Z}_2$-graded $\mathbb{Z}_2$-modules,

$$\bigoplus_{\Gamma \in H_1(Y;\mathbb{Z})} ECH_*(Y, \xi, \Gamma) \cong \Lambda^* H_*(\Sigma_g; \mathbb{Z}_2).$$

There is an explicit upgrade to a (relatively) $\mathbb{Z}$-graded isomorphism.

Corollary (Nelson-Weiler '20)

For $*$ sufficiently large and $g > 0$, the groups $ECH_*(Y, \xi, \Gamma)$ are isomorphic to $\mathbb{Z}_2^{f(g)}$, where $f(g) = 2^{2g-1}$.

1. Critical points of a perfect H form a basis for $H_*(\Sigma_g; \mathbb{Z}_2)$. Generators of ECC are $e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+}$ where $m_i = 0, 1$.
$\rightsquigarrow$ basis for $\Lambda^* H_*(\Sigma_g; \mathbb{Z}_2)$
2. ∂^{ECH} only counts cylinders corresponding to Morse flows on Σ_g; $\partial^{ECH}(e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+})$ is sum of ways to apply ∂^{Morse} to h_i or e_+.

More details

Theorem (Nelson-Weiler '20)

Let $(Y, \xi = \ker\lambda)$ be a prequantization bundle over (Σ_g, ω) of negative Euler class e. Each $\Gamma \in H_1(Y;\mathbb{Z})$ satisfying $ECH_*(Y,\xi,\Gamma) \neq 0$ corresponds to a number in $\{0, \dots, -e-1\}$,

$$ECH_*(Y,\xi,\Gamma) \cong \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \Lambda^{\Gamma + (-e)d} H_*(\Sigma_g;\mathbb{Z}_2), \qquad d = \frac{M-N}{|e|}$$

$$|\alpha|_* - |\beta|_* = -e(d_\alpha^2 - d_\beta^2) + (\chi(\Sigma_g) + 2\Gamma + e)(d_\alpha - d_\beta) + |\alpha|_\bullet - |\beta|_\bullet$$

$$\begin{aligned} I(\alpha,\beta) = {} & \chi(\Sigma_g)d - d^2 e + 2dN + m_+ - m_- - n_+ + n_- \\ & c_\tau(\alpha,\beta) + Q_\tau(\alpha,\beta) \quad + CZ_\tau^I(\alpha) - CZ_\tau^I(\beta), \; CZ_\tau^I(\gamma) = \textstyle\sum_i \sum_{k=1}^{\ell_i} CZ_\tau(\gamma_i^k) \end{aligned}$$

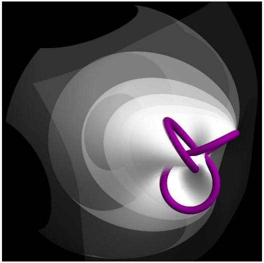
1. There exists $\varepsilon > 0$ so that the generators of $ECC_*^L(Y, \lambda_\varepsilon, J)$ are $e_-^{m_-} h_1^{m_1} \cdots h_{2g}^{m_{2g}} e_+^{m_+}$, e.g. orbits which are fibers over critical points.
2. $\partial^{ECH,L}$ only counts cylinders over Morse flow lines in Σ_g.
3. Finish with a direct limit argument, sending $\varepsilon \to 0$ and $L \to \infty$, by way of the action filtered isomorphism with Seiberg-Witten.

Open book decomposition of (S^3, ξ_{std}) along $T(2,q)$

Definition

An **open book decomposition** of Y^3 is a pair (B, π) where,

- B is an oriented link in Y, aka the **binding**;
- $\pi : Y \setminus B \to S^1$ is a **fibration** of the complement of B such that $\pi^{-1}(\theta) = \mathring{\Sigma}_\theta,\ \partial\Sigma_\theta = B$ for all $\theta \in S^1,\ \Sigma \cong \Sigma_\theta$ is the **page**.
- The **monodromy** ϕ is the self diffeo of the page.



(Henry Blanchette)

The right handed torus knot is the binding of an open book decomposition of (S^3, ξ_{std})

$$T(2,q) = \left\{(z_1, z_2) \in S^3 \mid z_1^2 + z_2^q = 0\right\},$$

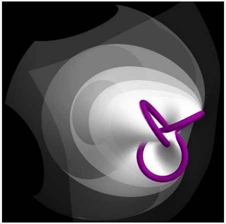
with the Milnor fibration projection map

$$\pi : S^3 \setminus T(2,q) \to S^1, \quad (z_1, z_2) \mapsto \tfrac{z_1^2 + z_2^q}{|z_1^2 + z_2^q|}.$$

The page Σ is a surface of genus $\frac{q-1}{2}$ (q odd).
The monodromy ϕ is $2q$-periodic.

http://people.reed.edu/~ormsbyk/projectproject/posts/milnor-fibrations.html

Open book decomposition of S^3 along $T(2, q)$



- Seifert fiber space $Y(0; (q, (1-q)/2), (2, 1))$, $e_{\mathbb{R}} = -\frac{1}{2q}$
- S^1-orbibundle over $\mathbb{CP}^1_{2,q}$ (*Reeb VF is tangent to fibers*)
- Knot filtered ECH realizes the relationship between action and linking of orbits.
- $\mathcal{F}_B(B^m\alpha) = m \operatorname{rot}(B) + \ell(\alpha, B)$,

Theorem (Nelson-Weiler '23)

Let B_0 be the standard transverse $T(2, q)$ torus knot for q odd in (S^3, ξ_{std}) with $\operatorname{rot}(B_0) = 2q$. *Then*

$$ECH_{2k}^{\mathcal{F}_{B_0} \leq K}(S^3, \xi_{std}, B_0, 2q) = \begin{cases} \mathbb{Z}/2 & K \geq N_k(2, q) = \{0, 2, q, 4, 2+q, ...\}_k \\ 0 & \textit{otherwise} \end{cases}$$

and $ECH_*^{\mathcal{F}_{B_0} \leq K} = 0$ *in all other gradings* $*$.

Corollary

kECH + ECH Weyl Law $\Rightarrow$ quantitative bounds on arbitrary Reeb currents.

Quantitative bounds on arbitrary Reeb currents

Corollary (NW '23)

Suppose that λ is a contact form for (S^3, ξ_{std}) so that $B = T(2, q)$ is an elliptic Reeb orbit with rotation number $2q + \epsilon$. Then for every $\delta > 0$, if $k \gg 0$, there exists a Reeb current α not including B, and $m \in \mathbb{N}_{\geq 0}$, such that

$$\frac{(\mathcal{A}(\alpha) + m\mathcal{A}(B))^2}{2k} \leq \mathrm{vol}(S^3, \lambda) + \delta.$$

$$\ell(\alpha, B) + (2q + \epsilon)m \geq N_k(2 + \epsilon/2, q + \epsilon/q)$$

In previous construction,

$$\mathrm{vol}(S^3, \lambda_0) = \tfrac{1}{2q}.$$

$$\text{ECH Weyl Law} \Rightarrow \lim_{k\to\infty} \frac{(c_k(S^3,\lambda_0))^2}{2k} = \lim_{k\to\infty} \frac{\left(N_k\left(\frac{1}{2}, \frac{1}{q}\right)\right)^2}{2k} = \frac{1}{2q}$$

Work in Progress (NW '23)

For any appropriately constructed open book decomposition (Y_ψ, λ_ψ) along $B = T(2,q)$ with rotation number $2q + \epsilon$ of (S^3, ξ_{std}), there is an associated Reeb orbit γ, which is not the binding, s.t.

$$\frac{\text{action}(\gamma)}{\text{linking of } \gamma \text{ with } T(2,q)} \leq \sqrt{\frac{1}{2q + \epsilon}\text{Vol}(\lambda_\psi)}.$$

The corresponding symplectomorphisms ψ of the page have periodic orbits whose total mean actions are at most $\text{Cal}(\psi) = \text{Vol}(\lambda_\psi)$, *assuming* $\text{Cal}(\psi) \leq \frac{1}{2q + \epsilon}$.

Set up allows for any symplectomorphism ψ of a genus $\frac{q-1}{2}$ surface with boundary on $T(2,q)$ which is freely isotopic to a $2q$-periodic diffeomorphism that is rotation by $\frac{2\pi}{2q + \epsilon}$ near boundary.

Generalizes Hutchings '16 for disk maps; Weiler '18 for annulus maps.

Related work for 'generic' Hamiltonians in Pirnapasov-Prasad '22.

Study symplectomorphisms $\psi : (\mathring{\Sigma}_{(q-1)/2}, d\eta)$, $\partial\mathring{\Sigma} = T(2,q)$ such that ψ is freely isotopic to the right handed $2q$-periodic Nielsen-Thurston rep of Mod $\left(\mathring{\Sigma}_{(q-1)/2}\right)$ and ψ is rotation by $\frac{2\pi}{2q}$ near the boundary.

The **action function** of ψ with respect to η is the unique $f_{\psi,\eta}$ such that $df = \psi^*\eta - \eta$ and $f|_{\partial\mathring{\Sigma}} = \frac{1}{2q}$. (measure of ψ distortion of curves)

The **Calabi invariant** of ψ is the average of the action function

$$\mathrm{Cal}_\eta(\psi) := \int_{\mathring{\Sigma}} f d\eta$$

(kind of rotation number; can show independent of η)

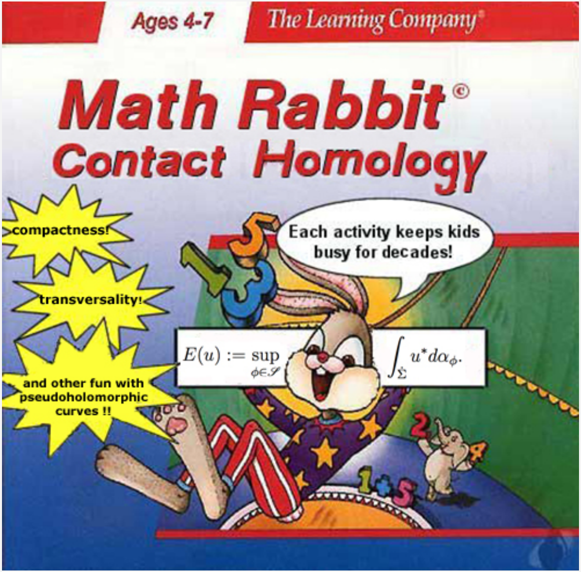
Work in progress to show if $f > 0$ and $\mathrm{Cal}(\psi) < \frac{1}{2q}$, then

$$\inf\left\{ \frac{\mathrm{Action}(\gamma)}{\mathrm{Period}(\gamma)} \,\middle|\, \gamma \text{ is a periodic orbit of } \psi \right\} < \frac{1}{2q}$$

Here a periodic orbit of ψ is a tuple of points $(x_1, ..x_\ell)$ such that $\psi(x_i) = x_{i+1} \mod \ell$. $\mathrm{Action}(\gamma) := \sum_{i=1}^{\ell} f(\gamma_i)$, $\mathrm{Period}(\gamma) = \ell$.

(More interesting: irrational $\mathrm{rot}(B_0) = 2q + \epsilon$)

Thanks!


Ages 4-7
The Learning Company
Math Rabbit
Contact Homology
compactness!
transversality!
and other fun with pseudoholomorphic curves !!
Each activity keeps kids busy for decades!
$E(u) := \sup_{\phi \in \mathscr{S}} \int_{\Sigma} u^* d\alpha_\phi.$

Visualizations

Hopf fibration: https://nilesjohnson.net/hopf.html

Spinors exhibit a sign-reversal that depends on the homotopy class of the continuous rotation of the coordinate system between some initial and final configuration in contrast to vectors and other tensors. https://en.wikipedia.org/wiki/Spinor

In the limit, a piece of solid continuous space can rotate in place like this without tearing or intersecting itself.
(A more extreme example of the **belt trick**.)
https://www.youtube.com/watch?v=LLw3BaliDUQ

Milnor fibrations of torus knots (& **open book decompositions**)

http://people.reed.edu/~ormsbyk/projectproject/posts/milnor-fibrations.html

https://www.unf.edu/~ddreibel/research/milnor/milnor.html

https://sketchesoftopology.wordpress.com/2012/08/24/bowman/