Algebraic torsion of linear boundaries of concave plumbings

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IAS summer collaborators 2024

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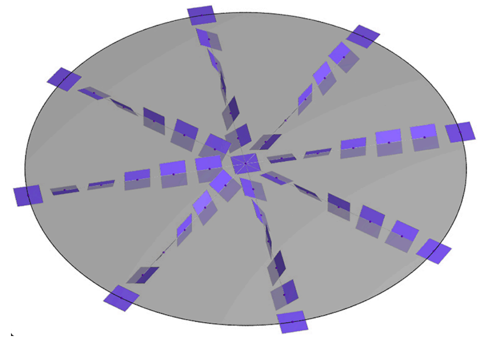
1. Motivation



Eventually find examples of (Y^3, ξ) contact mflds which are tight but not fillable

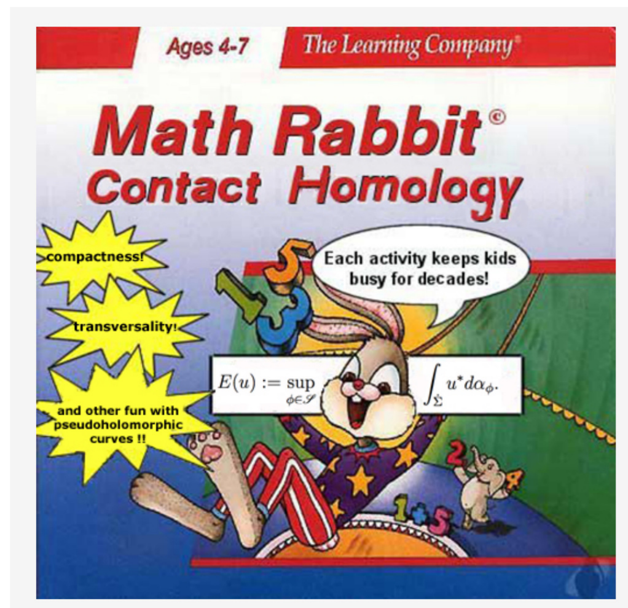
Develop methods for embedded contact homology to study topologically interesting mflds

Understand algebraic torsion

2011 Latschev + Wendl (SFT)
Appendix by Hutchings (ECH)

2023 Kutluhan Matić van Horn-Morris Wand (HF)

vs Moreno-Zhou (SFT)



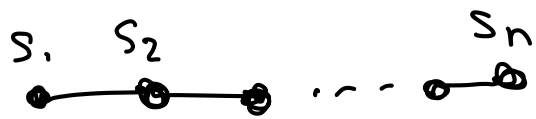
We consider the concave boundary of linear plumbings of disk bundles

- finite connected graph (usually a tree)
- vertices (g_i, s_i) w/ valency d_i, signed edges

genus of base of disk bundle

self intersection / euler #

- swap fibers of one bundle w/ disks in base



always assume or. preserving

give $D_i \times D^2$ to $D_{i+1} \times D^2$

$$Q_\Gamma = \begin{pmatrix} s_1 & 1 & 0 & 0 \\ 1 & s_2 & 1 & \\ 0 & 1 & \ddots & 1 \\ 0 & & 1 & s_n \end{pmatrix}$$ plumbing graph

- symplectic str

↳ on each disk bundle set up zero section to be symp. surface then disk bundle is a std small symp nhood

If $\int_{D_1} \omega_1 = \int_{D_2} \omega_2 = \int_{D^2 \text{ fibers}} \omega$ then plumbing gluing is a symplecto

⇝ have vector of areas of symp areas of each zero section $(a_1, \dots, a_n)$

$\partial(W,\omega) = (Y,\lambda)$, $\quad \omega|_{\text{nbhd } \partial} = d\tilde{\lambda}$, $\tilde{\lambda}|_Y = \lambda$ contact form

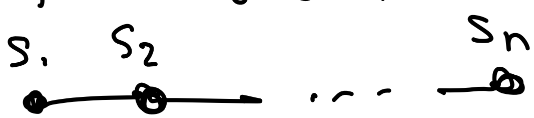
(Y,λ) is convex if $\lambda \wedge d\lambda$ or. agrees w/ or of $Y = \partial W$ else concave

⇝ equivalent to existence of Liouville VF V s.t.
$i_V \omega = \lambda$ near bdry and $V \pitchfork \partial W$

when V is outward pointing  ⇝ convex

inward pointing  concave

plumbing graph



$$Q_\Gamma = \begin{pmatrix} s_1 & 1 & 0 & & 0 \\ 1 & s_2 & 1 & & \\ 0 & 1 & \ddots & & 1 \\ \vdots & & & \ddots & \\ 0 & & & 1 & s_n \end{pmatrix}$$

Thm Gay-Stipsicz Q_Γ neg def ⇒ convex bdry

Li-Mak: concave str if $\exists\, z \in \mathbb{R}^n_{>0}$ s.t.
via G-S $\quad \boxed{-Q_\Gamma z = (a_1, \dots, a_n)}$ symp areas of base spheres

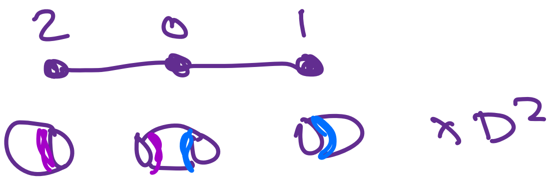
linear plumbings are lens spaces $(L(p,q), \xi_{can})$

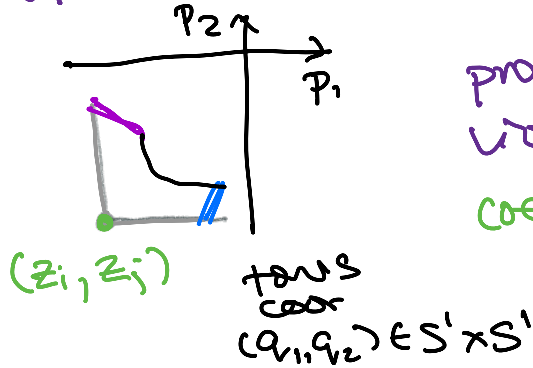
Lisca classified fillings by considering embeddings into closed symp mfld

⇝ nonembeddability (McDuff-Gromov) obstructs realizability of singular curves in algebraic geometry (via blow ups) (degree/genus constraints not enough)

Honda / Giroux also classified fillability of lens spaces

There are explicit local toric models!





product symp str $\omega_\Sigma \times r\,dr \wedge d\theta$

Liouville VF $V_\Sigma + \left(\frac{1}{2} r + \frac{z_v}{r}\right)\partial_r$

coeff ensure Liouville VF lines up

all $z_i < 0 \Rightarrow$ concave

Lemma There is an explicit global toric str for linear plumbings

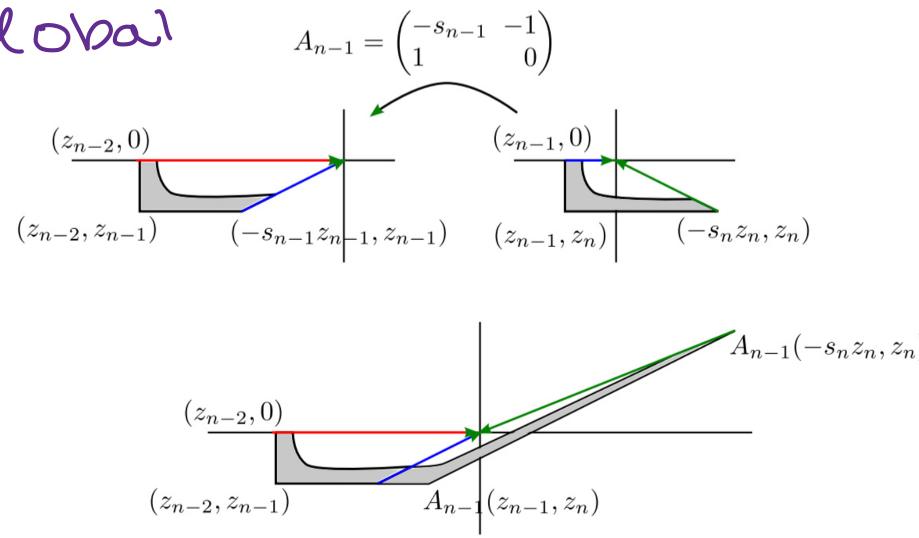


FIGURE 9. Gluing the L-shape $(0, s_n)$ to the L-shape $(0, s_{n-1})$

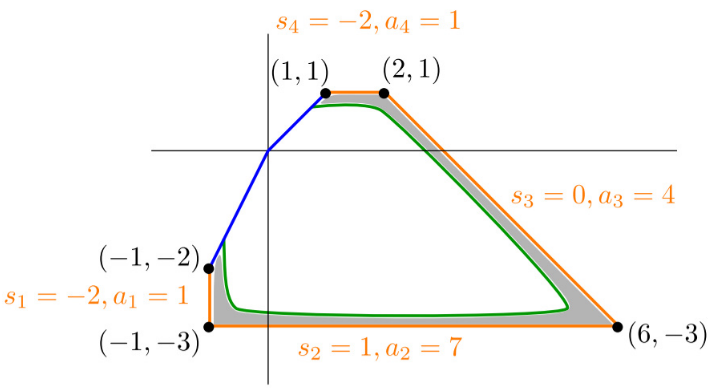


FIGURE 10. The global toric moment map image of the plumbing $(-2, 1, 0, -2)$ with symplectic areas $(1, 7, 4, 1)$.

$\rightsquigarrow$ induced contact form has product symmetry

$\rightsquigarrow$ Reeb dynamics are explicit (MB contact form)

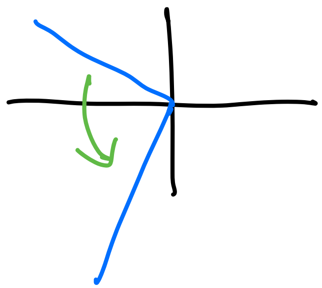
2. Results (Toric methods)

Toric contact mflds were classified by Lerman

$\rightsquigarrow$ must be lens spaces

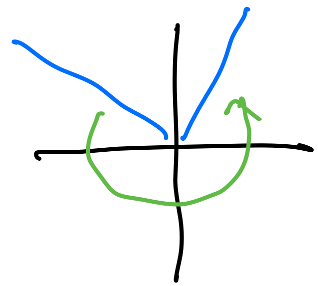
$\rightsquigarrow$ contact mfld up to contactomorphism is determined by 2 rays + angle between them (moment cone)

if $R_1 = (-1,0)$ and $R_2 = (-\ell, -k)$ ptng towards origin then bdry diffeo to $L(k,\ell)$



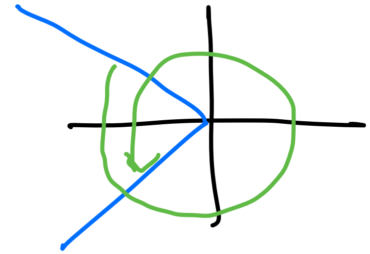
angle $\leq \pi$

tight canonical contact str



angle $> \pi$

OT



adding 2π to angle

$\longleftrightarrow$ 1/2 twist

htpy class of plane field preserved

MNSTRW '24

Theorem 5.4. *The contact structure on the boundary of a linear plumbing $(s_1, \ldots, s_n)$, where $s_i \geq 0$ for at least one $i \in \{1, \ldots, n\}$, is tight if one of the following cases occurs:*

(a) $s_j \leq -2$, for all $j \neq i$,
(b) $s_i = 0$, $s_{i-1} + s_{i+1} \leq -2$ and $s_j \leq -2$, for all $j \neq i, i+1$.

Theorem 5.2. *The contact structure on the boundary of a linear plumbing $(s_1, \ldots, s_n)$, is overtwisted if for some index $i \in \{1, \ldots, n\}$ where $s_i \geq 0$, one of the following cases occurs:*

(a) $s_i s_{i+1} \geq 2$ (or $s_i s_{i+1} \geq 1$ and $n > 2$);
(b) there exists $j \in \{1, \ldots, n\}$ such that $|i - j| > 1$ and $s_j \geq 1$ (or $s_j \geq 0$ and $n > 3$);
(c) $s_i = 0$ and $s_{i-1} + s_{i+1} \geq 1$ (or $s_{i-1} + s_{i+1} \geq 0$ and $n > 3$).

3. Results (ECH methods)

Almost

Theorem 8.1. *Let (Y^3, ξ) be a compact connected contact toric manifold characterized by two real numbers t_1, t_2, which define the corresponding moment cone*

(a) If $t_2 - t_1 < \pi$ then $c(\xi) \neq 0$ and $\mathfrak{at}_{\mathrm{simp}}(Y, \lambda, J) = \infty$;
(b) if $t_2 - t_1 > \pi$ then $c(\xi) = 0$ and $\mathfrak{at}(Y, \lambda, J) = 0$;
(c) if $t_2 - t_1 = \pi$ then $\mathfrak{at}_{\mathrm{simp}}(Y, \lambda, J) > 0$,

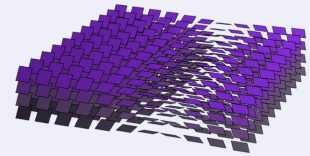
for all ECH data (λ, J).

- ⇝ uses algebraic torsion contact invariant in embedded contact homology
- ⇝ develop a much better understanding of holomorphic curves and planes
- ⇝ rest of talk elucidate ideas and difficulties
- ⇝ It all goes back to Hofer's pseudoholomorphic planes in early 90's for Weinstein Conj.

Reeb orbits on a contact 3-manifold

The **Reeb vector field** R on (Y,λ) is uniquely determined by

- $\lambda(R) = 1$
- $d\lambda(R,\cdot) = 0$



$\lambda = dz - ydx, \quad R = \frac{\partial}{\partial z}$

Given an embedded **Reeb orbit** $\gamma : \mathbb{R}/T\mathbb{Z} \to Y$,
the linearized flow along γ defines a symplectic linear map

$$d\varphi_t : (\xi|_{\gamma(0)}, d\lambda) \to (\xi|_{\gamma(t)}, d\lambda)$$

$d\varphi_T$ is called the **linearized return map**.

If 1 is not an eigenvalue of $d\varphi_T$ then γ is **nondegenerate**. λ is **nondegenerate** if all Reeb orbits associated to λ are nondegenerate.

For $\dim Y = 3$, nondegenerate orbits are either **elliptic** or **hyperbolic** according to whether $d\varphi_T$ has eigenvalues on S^1 or real eigenvalues.

Later, we consider an almost complex structure J on $T(\mathbb{R} \times Y)$:

- J is $\mathbb{R}$-invariant
- $J\xi = \xi$, rotates ξ positively with respect to $d\lambda$
- $J(\partial_s) = R$, where s denotes the $\mathbb{R}$ coordinate

Embedded contact homology (ECH)

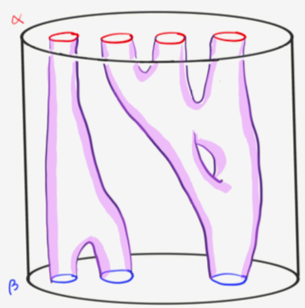
ECH is a gauge theory for (Y^3, λ) and $\Gamma \in H_1(Y; \mathbb{Z})$ due to Hutchings.

$ECC_*(Y, \lambda, \Gamma, J)$ is a $\mathbb{Z}_2$ vector space generated by **Reeb currents** $\alpha = \{(\alpha_i, m_i)\}$:

- α_i is an embedded Reeb orbit, $m_i \in \mathbb{Z}_{>0}$,
- if α_i is hyperbolic, $m_i = 1$,
- $\sum_i m_i[\alpha_i] = \Gamma$.

$*$ is given by the **ECH index**, a topological index defined via c_1, CZ, and relative self-intersection pairing, wrt $Z \in H_2(Y, \alpha, \beta)$.
Get a relative $\mathbb{Z}_d$-grading, d is divisibility of $c_1(\xi) + 2PD(\Gamma)$ in $H^2(Y; \mathbb{Z})$ mod torsion.

$\langle \partial^{ECH} \alpha, \beta \rangle$ counts **currents**, realized by unions of **holomorphic curves**



Partition writhe fun,
index inequality,
(yay for adjunction!)

-Hutchings' 02 Haiku

Dee squared is zero;
obstruction bundle gluing
is complicated.

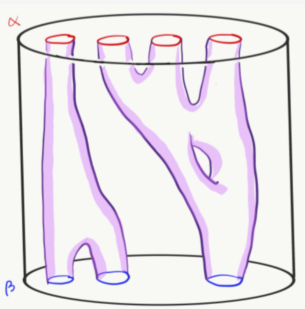
Hutchings-Taubes' 07 & 09 Haiku

Invariance of ECH

$ECC_*(Y,\lambda,\Gamma,J)$ is generated by **Reeb currents** $\alpha=\{(\alpha_i,m_i)\}$ over $\mathbb{Z}_2$

Grading is given by the **ECH index**, a topological index defined via c_1, CZ, and relative self-intersection pairing, wrt $Z\in H_2(Y,\alpha,\beta)$.

$\langle\partial^{ECH}\alpha,\beta\rangle$ counts **currents**, realized by unions of **holomorphic curves**



For generic J,
ECH index one yields
somewhere injective.

-Hutchings' 02 Haiku

Dee squared is zero;
obstruction bundle gluing
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Hutchings-Taubes' 07 & 09 Haiku

Jason Hise

Theorem (Taubes G&T (2010), no. 5, 2497-3000)

If Y is connected, there is a canonical isomorphism of relatively graded modules

$$ECH_*(Y,\lambda,\Gamma,J)=\widehat{HM}^{-*}(Y,\mathfrak{s}_\xi+\mathrm{PD}(\Gamma)),$$

which sends the ECH contact invariant to the SWF contact invariant

ECH is a topological invariant of Y !

(shift Γ when changing choice of ξ)

$\boxed{\text{“free” contact invariant } c(\xi)=[\emptyset]}$

$c(\xi) := [\emptyset] \in ECH(Y, \xi, 0)$

$\rightsquigarrow$ SWF contact invariant defined (implicitly) in Kronheimer-Mrowka '97

$\rightsquigarrow$ work by Echeverria '20 to establish naturality
SWF contact inv.

① If (Y, ξ) is OT then $c(\xi)$ vanishes by the existence of an embedded hyperbolic Reeb orbit which has smallest action and bounds a unique Fredholm + ECH index 1 plane

② If (Y, ξ) is exactly symp. fillable then $c(\xi) \neq 0$
Hutchings 11, Hutchings-Taubes 13 (and naturality)
$\rightarrow$ expected: strongly symp fillable $\Rightarrow c(\xi) \neq 0$

③ ECH has only been computed for
T^3, S^3, $S^2 \times S^1$, some tight lens spaces, S^1 bundles over Σ_g
so don't have much in the way of meaningful ex.

Algebraic Torsion in ECH

In the beginning, Hofer found a plane in $\mathbb{R} \times Y^3$ (1993)
(proved existence by "failure" of compactness)

⇝ then came Hofer-Wysocki-Zehnder, Eliashberg-Givental-Hofer } enter symplectic field theory (SFT)

The order of **algebraic torsion** is a non-neg integer derived from a filtration on the topological complexity of the J-hol curves of Fredholm and ECH index 1 with no negative ends.

⇝ inspired by Latschev-Wendl's construction of algebraic torsion in SFT

⇝ no one is bold enough to claim algebraic torsion is actually an invariant of ξ

But we have a $\boxed{\text{weak}}$ naturality property which allows us to use algebraic torsion to obstruct exact symplectic fillings and cobordisms

⇝ Don't have much in the way of any computations or examples in SFT and fewer in ECH

We use a friend of the ECH index to define $\boxed{AT}$:

$$J_0(\alpha, \beta, Z) := -c_\tau(Z) + Q_\tau(Z) + CZ_\tau^J(\alpha, \beta)$$

⇝ more directly encodes topology
⇝ slicker reinterpretation of adjunction

ECH $\quad I(\alpha, \beta, Z) := c_\tau(Z) + Q_\tau(Z) + CZ_\tau^I(\alpha, \beta)$

Fred $\quad \mathrm{ind}(u) = 2c_\tau(u) - \chi(u) + CZ_\tau^{ind}(\alpha, \beta)$

$$CZ_\tau^I(\alpha) := \sum_i \sum_{k=1}^{m_i} CZ_\tau(\alpha_i^k) \qquad CZ_\tau^J(\alpha) := \sum_i \sum_{k=1}^{m_i - 1} CZ_\tau(\alpha_i^k)$$

use

$$J_+(\alpha,\beta,Z) := J_0(\alpha,\beta,Z) + |\alpha| - |\beta|$$

Given Reeb current $\gamma = \{(\gamma_i, m_i)\}$, $|\gamma| := \sum_i \begin{cases} 1 & \gamma_i \text{ elliptic} \\ m_i & \gamma_i \text{ pos hyper} \\ \lceil \frac{m_i}{2} \rceil & \gamma_i \text{ neg hyper} \end{cases}$

If $u \in \mathcal{M}(\alpha,\beta,J)$ is somewhere injective and connected then $J_+(u) \geq 0$ and $\boxed{J_+(u) \geq 2\left(g - 1 + \delta(u) + \sum_i \begin{cases} n_{\alpha_i} & \text{elliptic} \\ 1 & \text{hyper} \end{cases} - \sum_j \begin{cases} n_{\beta_j} & \text{ellipic} \\ 1 & \text{hyper} \end{cases}\right)}$

<u>TLDR:</u> (Hutchings '09, appendix to Lotscher-Wendl '11)

- parity of $I(u) - J_+(u)$ agrees with parity of # pos hyper orbits agrees with parity of $I(u)$
- so if a J-hol curve contributes to ∂^{ECH} then J_+ is even (need even # of pos hypers for odd ECH index)
- J_+ is additive under gluing
- Get a J_+-spectral sequence

$$\partial^{ECH} = \partial_0 + \partial_1 + \ldots$$

where ∂_ℓ is contribution from J-curves w/ $J_+(u) = 2\ell$

- Since $(\partial^{ECH})^2 = 0 \Rightarrow \partial_0^2 = 0$, $\partial_0\partial_1 + \partial_1\partial_0 = 0$, etc

so can define <u>J_+ - spectral sequence</u>

$E^*(Y, \lambda, J)$ where E^1 is homology wrt ∂_0

E^2 of ∂_1 acting on E^1

$\vdots$

$\boxed{at(Y, \lambda, J)}$ is the smallest nonneg k s.t. ϕ becomes 0 on E^{k+1} page

⚠ This spectral sequence is <u>not</u> invariant under deformation of the contact form b/c we do not have control over J_+ index of multiply covered curves in exact symp cobordisms

$\rightsquigarrow$ neg J_+ curves could contribute to chain map...

- Can still restrict to a subcomplex "simple" ECH to obstruct existence of these multiply covered curves

$\rightsquigarrow$ provides a weaker naturality property

if $\exists$ exact symp cobordism from (Y_+, λ_+) to (Y_-, λ_-)

then $at_{simp}(Y_+, \lambda_+) \geq at(Y_-, \lambda_-)$ (we don't know if true more generally)

Q: How hard can $\mathfrak{at}(Y, \lambda, J)$ be to compute?

LOL!!

And so -

Theorem 8.1. *Let (Y^3, ξ) be a compact connected contact toric manifold characterized by two real numbers t_1, t_2, which define the corresponding moment cone*

(a) *If $t_2 - t_1 < \pi$ then $c(\xi) \neq 0$ and $\mathfrak{at}_{\mathrm{simp}}(Y, \lambda, J) = \infty$;* (irrational ellipsoid lurking nearby)

(b) *if $t_2 - t_1 > \pi$ then $c(\xi) = 0$ and $\mathfrak{at}(Y, \lambda, J) = 0$;*

(c) *if $t_2 - t_1 = \pi$ then $\mathfrak{at}_{\mathrm{simp}}(Y, \lambda, J) > 0$,*

for all ECH data (λ, J).

Step 1: Sort out MB Reeb dynamics, play w/ curves defining toric contact form

→ Do a nondeg pert up to large action L_ε

→ associated Reeb VF has exactly 2 orbits in each positive MB torus

- elliptic e
- hyperbolic h

Step 2: Do some homology computations to figure out which hyperbolic orbits could be killers of ϕ

Step 3: Construct a J_ε-hol plane positively asymptotic to h (it has ind $= I = 1$)

uses automatic transversality in MB setting

then we show it persists under pert. of λ, J

step 4: Show h does not have an alibi and that the plane is unique

"and more"

- ⤳ action/homology constraints
- ⤳ adjunction constraints
- ⤳ intersection positivity (much trickier than in MB land)
- ⤳ asymptotic expansion of curve a la Siefring

Step 5: Check that h is a generator of the simple subcomplex of ECH, e.g. for any $\beta = \{(\beta_j, n_j)\}$ at the negative end of a possibly broken J-hol curve w/ pos end at h, $n_j = 0$

⤳ minimizes cobordism map woes so as to establish

$$\Phi : ECC^L_{simp}(Y_+, \lambda_+, J_+) \longrightarrow ECC^L(Y_-, \lambda_-, J_-)$$

s.t. $\Phi(\emptyset) = \emptyset$

$\Phi = \Phi_0 + \Phi_1 + \dots$ and $\sum_{i+j=k} (\partial_i \Phi_j - \Phi_i \partial_j) = 0$

Definition 6.15. A Reeb current $\alpha = \{(\alpha_i, m_i)\}$ is *simple* with respect to J when the following conditions hold:

- $m_i = 1$ for all i;
- For any $\beta = \{(\beta_j, n_j)\}$ at the negative end of a (possibly broken) J-holomorphic curve with positive end at α, all $n_j = 1$.

Given $L \in (0, \infty]$, define $ECC^L_{\text{simp}}(Y, \lambda, J)$ to be the subcomplex $ECC(Y, \lambda, J)$ generated by simple **ECH generators**of action $< L$.

It is important to note that this notion of simple does not agree with the simple terminology in symplectic field theory literature. The second condition in Definition 6.15, ensures that the usual cobordism map woes are minimized in the following sense, as needed in the proof of [LW11, Lemma A.14] that establishes the existence of a chain map induced by an exact symplectic cobordism from (Y_+, λ_+) to (Y_-, λ_-),

$$\Phi : ECC^L_{\text{simp}}(Y_+, \lambda_+, J_+) \to ECC(Y_-, \lambda_-, J_-)$$

such that

(1) $\Phi(\emptyset) = \emptyset$;
(2) there is a decomposition $\Phi = \Phi_0 + \Phi_1 + ...$ such that $\sum_{i+j=k}(\partial_i \Phi_j - \Phi_i \partial_j) = 0$ for each $k \in \mathbb{Z}_{\geq 0}$.

In particular:

> If the orbit set α_+ is simple, no holomorphic curve in $\mathcal{M}^J(\alpha_+, \alpha_-)$ admits a multiply covered component and no broken holomorphic curve arising as a limit of a sequence of curves in $\mathcal{M}^J(\alpha_+, \alpha_-)$ admits a multiply covered component in the cobordism level.

This ensures that

- If α_+ is simple and if $u \in \mathcal{M}^J(\alpha_+, \alpha_-)$ has $I(u) = 0$ then u is cut out transversely.
- Φ is well-defined because if α_+ is simple then there are only finitely many curves with $I(u) = 0$ and $J_+(u) \geq 0$ for all $u \in \mathcal{M}^J(\alpha_+, \alpha_-)$.

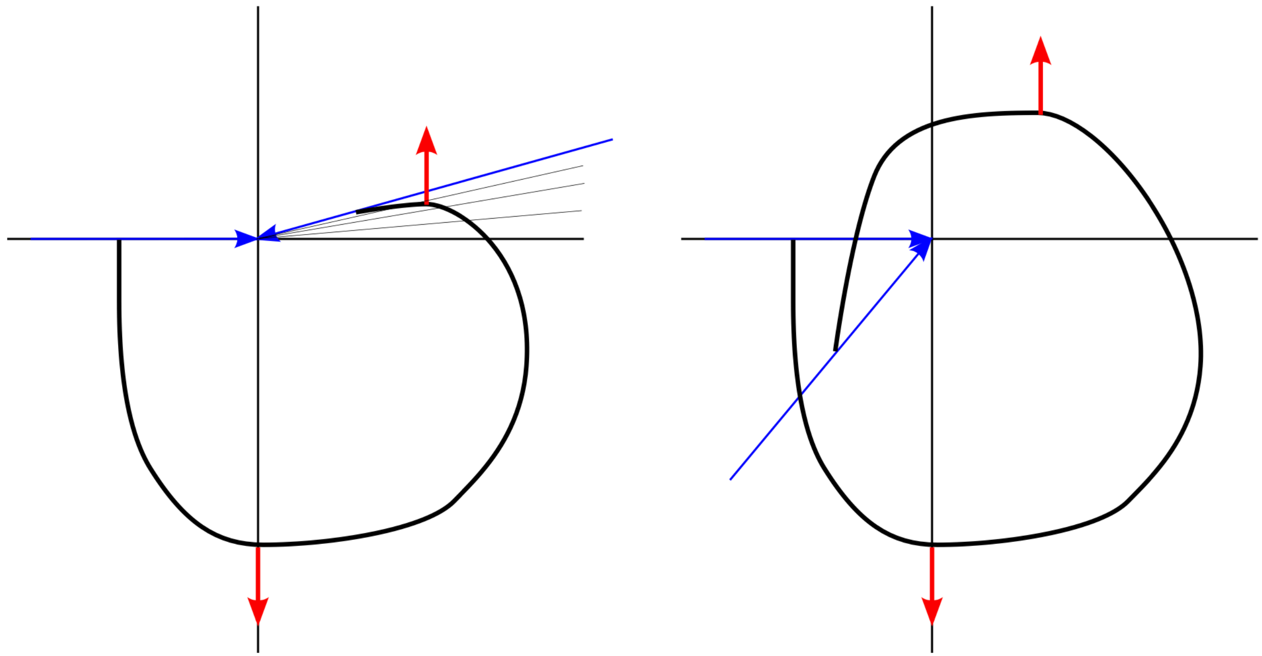

FIGURE 14. Curve $P(x)$ defining the toric contact form λ used to compute algebraic torsion in ECH when the angle between the rays is $> \pi$. The curve must have points where the normal vector is $(0, -1)$ and $(0, 1)$, and the radial rays from the origin must be everywhere transverse. Observe (left) that when the angle is close to π, we have just enough space to satisfy both conditions. On the right is a case where the angle is greater than 2π.